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Note

Pointwise universal trigonometric series

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ABSTRACT

A series $S_a = \sum_{n=-\infty}^{\infty} a_n z^n$ is called a *pointwise universal trigonometric series* if for any $f \in C(\mathbb{T})$, there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $\sum_{j=-n_k}^{n_k} a_j z^j$ converges to $f(z)$ pointwise on $\mathbb{T}$. We find growth conditions on coefficients allowing and forbidding the existence of a pointwise universal trigonometric series. For instance, if $|a_n| = O(e^{|n|\ln^{-1-\varepsilon}|n|})$ as $|n| \to \infty$ for some $\varepsilon > 0$, then the series S_a cannot be pointwise universal. On the other hand, there exists a pointwise universal trigonometric series S_a with $|a_n| = O(e^{|n|\ln^{-1}|n|})$ as $|n| \to \infty$.

1. Introduction

Throughout this article $\mathbb{C}$ is the field of complex numbers, $\mathbb{T} = \{z \in \mathbb{C}:\ |z| = 1\}$, $\mathbb{R}$ is the field of real numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+$ is the set of non-negative integers and $\mathbb{N}$ is the set of positive integers. For $z \in \mathbb{C}$, $\operatorname{Re} z$ is the real part of z. For a compact metric space K, $C(K)$ stands for the space of continuous complex-valued functions on K. An *interval* in $\mathbb{T}$ is a closed connected subset of $\mathbb{T}$ of positive length. Symbol μ stands for the normalized Lebesgue measure on $\mathbb{T}$. Recall that *Fourier coefficients* of $f \in L_1(\mathbb{T})$ or of a finite Borel σ-additive complex valued measure ν on $\mathbb{T}$ are given by the formula

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n}\,\mu(dz) \quad \text{and} \quad \widehat{\nu}(n) = \int_{\mathbb{T}} z^{-n}\,\nu(dz) \quad \text{for } n \in \mathbb{Z}.$$

In particular, $\widehat{f}(n) = \widehat{\nu}(n)$ if ν is the measure with the density f with respect to μ. Recall also that $\{z^n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{T}) = L_2(\mathbb{T},\mu)$, which yields

$$\langle f, g\rangle_{L_2(\mathbb{T})} = \int_{\mathbb{T}} f(z)\overline{g(z)}\,\mu(dz) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} \quad \text{for any } f, g \in L_2(\mathbb{T}). \tag{1.1}$$

Definition 1.1. Let X be a topological vector space and let $\{x_n\}_{n\in\mathbb{Z}_+}$ be a sequence in X. We say that $\sum_{n=0}^{\infty} x_n$ is a *universal series* if for every $x \in X$, there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers such that $\sum_{j=0}^{n_k} x_j \to x$ as $k \to \infty$. Similarly if $\{x_n\}_{n\in\mathbb{Z}}$ is a bilateral sequence in X, then we say that $\sum_{n=-\infty}^{\infty} x_n$ is a *universal series* if for every $x \in X$, there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers such that $\sum_{j=-n_k}^{n_k} x_j \to x$ as $k \to \infty$.

Remark 1.2. Note that if X is metrizable, then universality of $\sum_{n=0}^{\infty} x_n$ (respectively, of $\sum_{n=-\infty}^{\infty} x_n$) is equivalent to density of $\{S_n:\ n \in \mathbb{Z}_+\}$ in X, where $S_n = \sum_{j=0}^{n} x_j$ (respectively, $S_n = \sum_{j=-n}^{n} x_j$). If X is non-metrizable, the latter may fail due to the fact that a closure of a set can differ from its sequential closure.

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Universal series have been studied by many authors. Those interested in the subject ought to look into [1], which together with a systematic approach to the theory of universal series provides a large survey part and extensive list of references on the subject. We would also like to mention papers [2,4–6,14,17,18], which are not in the list of references in [1] and the papers [3,7–10] dealing specifically with universal trigonometric series. We start by mentioning two old results on universal series. Seleznev [16,1] proved the following theorem on universal power series.

Theorem S. *There exists a sequence $\{a_n\}_{n\in\mathbb{Z}_+}$ of complex numbers such that for each entire function f and any compact subset K of $\mathbb{C}\setminus\{0\}$ with connected $\mathbb{C}\setminus K$, there is a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers for which $\sum_{j=0}^{n_k} a_j z^j$ converges to $f(z)$ uniformly on K.*

Menshov [13,1] constructed a sequence $c\in c_0(\mathbb{Z})$ such that the series $\sum_{n=-\infty}^{\infty} c_n z^n$ is universal in the space $L_0(\mathbb{T})$ of (equivalence classes of) μ-measurable functions $f:\mathbb{T}\to\mathbb{C}$ with the measure convergence topology. Such series are referred to as universal trigonometric series. Since $L_0(\mathbb{T})$ is metrizable, $C(\mathbb{T})$ is dense in $L_0(\mathbb{T})$ and every measure convergent sequence of functions has a subsequence, which is almost everywhere convergent, the result of Menshov is equivalent to the following statement.

Theorem M. *There exists $a\in c_0(\mathbb{Z})$ such that for each $f\in C(\mathbb{T})$, there is a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers for which $\sum_{j=-n_k}^{n_k} a_j z^j$ converges to $f(z)$ for almost all $z\in\mathbb{T}$ with respect to the Lebesgue measure.*

We study the natural question whether one can replace almost everywhere convergence by pointwise convergence. In other words, we deal with universal trigonometric series in the space $C_p(\mathbb{T})$, being $C(\mathbb{T})$ endowed with the pointwise convergence topology. Our first observation is an easy consequence of Theorem S.

Proposition 1.3. *There exists a sequence $\{a_n\}_{n\in\mathbb{Z}_+}$ in $\mathbb{C}$ such that for each $f\in C(\mathbb{T})$, there is a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers for which $\sum_{j=0}^{n_k} a_j z^j$ converges to $f(z)$ for all $z\in\mathbb{T}$.*

Thus there is a universal power series in $C_p(\mathbb{T})$. The next natural step is to try to figure out whether one can make coefficients of a universal power or at least a trigonometric series in $C_p(\mathbb{T})$ small. It turns out in considerable contrast with Theorem M that passing from almost everywhere convergence to pointwise convergence forces the coefficients of a universal trigonometric series to grow almost exponentially. In order to formulate the explicit result, we introduce the following notation.

Definition 1.4. We say that a sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of positive real numbers belongs to the class $\mathcal{X}_-$ if

$$\lim_{n\to\infty} n(c_{n+1}-c_n)=+\infty \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{c_n}{n^2+1}<\infty.$$

We say that a sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of positive real numbers belongs to the class $\mathcal{X}_+$ if

$$\lim_{n\to\infty} n(c_{n+1}-c_n)=+\infty \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{c_n}{n^2+1}=\infty.$$

Remark 1.5. It is easy to see that for any $\varepsilon>0$, the sequence $\{(n+1)\ln^{-1-\varepsilon}(n+2)\}_{n\in\mathbb{Z}_+}$ belongs to $\mathcal{X}_-$. On the other hand, the sequence $\{(n+1)\ln^{-1}(n+2)\}_{n\in\mathbb{Z}_+}$ belongs to $\mathcal{X}_+$.

For a bilateral sequence $a=\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers and $m\in\mathbb{Z}_+$ we denote

$$S_m^a(z)=\sum_{k=-m}^{m} a_k z^k.$$

In other words, $S_m^a:\mathbb{T}\to\mathbb{C}$ are partial sums of the trigonometric series with coefficients a_n. For an interval J in $\mathbb{T}$, symbol $\mathcal{R}_a(J)$ stands for the set of $g\in C(J)$ such that there is a strictly increasing sequence $\{m_k\}_{k\in\mathbb{Z}_+}$ of positive integers for which $S_{m_k}^a(z)\to g(z)$ for any $z\in J$.

Theorem 1.6. *Let $\{a_n\}_{n\in\mathbb{Z}}$ be a bilateral sequence of complex numbers such that there exists $c\in\mathcal{X}_-$ for which $|a_n|\leqslant e^{c_{|n|}}$ for any $n\in\mathbb{Z}$. Then for any interval J in $\mathbb{T}$, the set $\mathcal{R}_a(J)$ contains at most one element. In particular, the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is not universal in $C_p(\mathbb{T})$.*

The following theorem shows that Theorem 1.6 is sharp.

Theorem 1.7. *Let $c \in \mathcal{X}_+$. Then there exists a sequence $a = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that $|a_n| \leqslant e^{c_{|n|}}$ for each $n \in \mathbb{Z}$ and the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is universal in $C_p(\mathbb{T})$. That is, $\mathcal{R}_a(\mathbb{T}) = C(\mathbb{T})$.*

Theorems 1.6, 1.7 and Remark 1.5 imply the following corollary.

Corollary 1.8. *There exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ such that the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is pointwise universal and $a_n = O(e^{|n|\ln^{-1}|n|})$. On the other hand, for each $\varepsilon > 0$ and any sequence $\{a_n\}_{n\in\mathbb{Z}}$ satisfying $a_n = O(e^{|n|\ln^{-1-\varepsilon}|n|})$ for some $\varepsilon > 0$, the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is not pointwise universal.*

The following question remains open.

Question 1.9. Does the conclusion of Theorem 1.7 remain true under the extra condition that $a_n = 0$ for $n < 0$?

2. Proof of Proposition 1.3

Let $a = \{a_n\}_{n\in\mathbb{Z}_+}$ be the sequence provided by Theorem S. We shall simply show that it satisfies all requirements of Proposition 1.3. Let $f \in C(\mathbb{T})$ and $J_k = \{e^{it}\colon\ 0 \leqslant t \leqslant 2\pi - k^{-1}\}$ for $k \in \mathbb{N}$. Since each J_k is compact, has empty interior in $\mathbb{C}$ and has connected complement in $\mathbb{C}$, we can use the classical Mergelyan theorem to pick a sequence $\{p_k\}_{k\in\mathbb{N}}$ of polynomials such that $|f(z) - p_k(z)| < 2^{-k-1}$ for any $k \in \mathbb{N}$ and $z \in J_k$. By Theorem S, we can choose a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $|p_k(z) - S_k(z)| < 2^{-k-1}$ for any $k \in \mathbb{N}$ and $z \in J_k$, where $S_k(z) = \sum_{j=0}^{n_k} a_j z^j$. Hence $|f(z) - S_k(z)| < 2^{-k}$ for any $k \in \mathbb{N}$ and $z \in J_k$. Since each $z \in \mathbb{T}$ belongs to J_k for each sufficiently large k, we see that $S_k(z) \to f(z)$ as $k \to \infty$ for any $z \in \mathbb{T}$. The proof of Proposition 1.3 is complete.

3. Proof of Theorem 1.6

We need the following result due to Mandelbrojt [11], which is closely related to the Denjoy–Carleman theorem characterizing quasi-analytic classes of infinitely differentiable functions. A formally slightly weaker statement can be found in an earlier work of Mandelbrojt [12], although one can easily see that this weaker form is actually equivalent to the stronger one. See also [15] for a different proof.

Theorem Mb.

I. *If $c \in \mathcal{X}_-$, then for any interval J of $\mathbb{T}$ there exists a non-zero infinitely differentiable function $f : \mathbb{T} \to [0, \infty)$ such that $f(z) = 0$ for $z \in \mathbb{T} \setminus J$ and $|\widehat{f}(n)| \leqslant e^{-c_{|n|}}$ for each $n \in \mathbb{Z}$.*
II. *If $c \in \mathcal{X}_+$, $f : \mathbb{T} \to \mathbb{C}$ is infinitely differentiable, $|\widehat{f}(n)| \leqslant e^{-c_{|n|}}$ for each $n \in \mathbb{Z}$ and there is $z_0 \in \mathbb{T}$ such that $f^{(j)}(z_0) = 0$ for each $j \in \mathbb{Z}_+$, then f is identically 0.*

We are ready to prove Theorem 1.6. Assume the contrary. Then there exists an interval J in $\mathbb{T}$, two different functions $f, g \in C(J)$ and two strictly increasing sequences $\{m_n\}_{n\in\mathbb{Z}_+}$ and $\{k_n\}_{n\in\mathbb{Z}_+}$ of positive integers such that $S^a_{k_n}(z)$ converges $f(z)$ and $S^a_{m_n}(z)$ converges $g(z)$ as $n \to \infty$ for each $z \in J$. Passing to subsequences, if necessary, we can without loss of generality assume that $m_n < k_n < m_{n+1}$ for each $n \in \mathbb{Z}_+$. Since $f \neq g$, there is $z_0 \in J$ such that $f(z_0) \neq g(z_0)$. Denote $p_n = C(S^a_{k_n} - S^a_{m_n})$, where $C = 2(f(z_0) - g(z_0))^{-1}$. Then $p_n(z)$ converges to $h(z)$ for each $z \in J$, where $h(z) = C(f(z) - g(z))$. Clearly $h \in C(J)$ and $h(z_0) = 2$. Since h is continuous, we can pick an interval $J_0 \subseteq J$ such that $\operatorname{Re} h(z) \geqslant 1$ for each $z \in J_0$. We shall prove the following statement:

$$\text{for any interval } I \subset J_0 \text{ there is } n_0 \in \mathbb{N} \text{ such that } \min_{z\in I} \operatorname{Re} p_n(z) < 0 \text{ for any } n \geqslant n_0. \tag{3.1}$$

Suppose that (3.1) is not satisfied. Then there exist an interval $I \subset J_0$ and a strictly increasing sequence $\{j_n\}_{n\in\mathbb{Z}_+}$ of positive integers satisfying $\min_{z\in I} \operatorname{Re} p_{j_n}(z) \geqslant 0$ for any $n \in \mathbb{Z}_+$. Take a sequence $c \in \mathcal{X}_-$ such that $|a_n| \leqslant e^{c_{|n|}}$ for any $n \in \mathbb{Z}$. It is easy to see that the sequence $\{d_n\}_{n\in\mathbb{Z}_+}$ also belongs $\mathcal{X}_-$, where $d_n = c_n + \ln(n^2+1)$. By Theorem Mb there is a non-zero infinitely differentiable function $\rho : \mathbb{T} \to [0, \infty)$ such that $\rho(z) = 0$ for $z \in \mathbb{T} \setminus I$ and $|\widehat{\rho}(n)| \leqslant e^{-d_{|n|}} = (n^2+1)^{-1} e^{-c_{|n|}}$ for each $n \in \mathbb{Z}$. Since ρ vanishes outside I, $\operatorname{Re} p_{j_n} \geqslant 0$ on I and p_{j_n} converge to h pointwise on $J \supseteq I$, we see that $\rho \operatorname{Re} p_{j_n}$ is a sequence of non-negative continuous functions pointwise convergent to $\rho \operatorname{Re} h$. By the Fatou theorem

$$\varliminf_{n\to\infty} \int_{\mathbb{T}} \rho(z) \operatorname{Re} p_{j_n}(z)\, \mu(dz) \geqslant \int_{\mathbb{T}} \rho(z) \operatorname{Re} h(z)\, \mu(dz) \geqslant \int_{I} \rho(z)\, \mu(dz) > 0 \tag{3.2}$$

since $\mathrm{Re}\, h(z) \geqslant 1$ for each $z \in J_0 \supseteq I$. On the other hand, using (1.1), we obtain

$$\int_{\mathbb{T}} \rho(z)\,\mathrm{Re}\, p_n(z)\,\mu(dz) = \mathrm{Re}\langle \rho, p_n\rangle_{L_2(\mathbb{T})} \leqslant \left|\langle \rho, p_n\rangle_{L_2(\mathbb{T})}\right| = \left|\sum_{l\in\mathbb{Z}} \widehat{p_n}(l)\overline{\widehat{\rho}(l)}\right| = \left|\sum_{m_n<|l|\leqslant k_n} Ca_l\overline{\widehat{\rho}(l)}\right|$$
$$\leqslant \sum_{m_n<|l|\leqslant k_n} \frac{|C||a_l|e^{-c_{|l|}}}{l^2+1} \leqslant \sum_{m_n<|l|\leqslant k_n} \frac{|C|}{l^2+1} = O\left(m_n^{-1}\right) = o(1) \quad \text{as } n\to\infty.$$

The above display contradicts (3.2), which completes the proof of (3.1).

Using (3.1) and continuity of p_n, we can pick a sequence $\{J_n\}_{n\in\mathbb{Z}_+}$ of intervals in $\mathbb{T}$ starting from J_0 and a strictly increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive integers such that $J_{n+1} \subseteq J_n$ for each $n \in \mathbb{Z}_+$ and $\mathrm{Re}\, p_{r_n}(z) < 0$ for each $z \in J_n$, $n \in \mathbb{N}$. Since any decreasing sequence of compact sets has non-empty intersection, there is $w \in \bigcap_{n=0}^{\infty} J_n$. Then $\mathrm{Re}\, p_{r_n}(w) < 0$ for any $n \in \mathbb{N}$. Since $p_n(w) \to h(w)$, we have $\mathrm{Re}\, h(w) \leqslant 0$. On the other hand, $\mathrm{Re}\, h(w) \geqslant 1$ since $w \in J_0$. This contradiction completes the proof of Theorem 1.6.

4. Proof of Theorem 1.7

First, we introduce some notation. Symbol $\omega(\mathbb{Z})$ stands for the space $\mathbb{C}^{\mathbb{Z}}$ of all complex bilateral sequences with the coordinatewise convergence topology, while $\varphi(\mathbb{Z})$ is the subspace of $\omega(\mathbb{Z})$ of sequences with finite support. That is, $x \in \omega(\mathbb{Z})$ belongs to $\varphi(\mathbb{Z})$ if and only if there is $n \in \mathbb{N}$ such that $x_j = 0$ if $|j| > n$. Let also $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical linear basis of $\varphi(\mathbb{Z})$. The next result is a part of Theorem 1 from [1], formulated in an equivalent form and adapted to sequences labeled by $\mathbb{Z}$ rather than $\mathbb{Z}_+$. For sake of completeness we sketch its proof.

Lemma 4.1. *Let X be a metrizable topological vector space, A be a linear subspace of $\omega(\mathbb{Z})$ endowed with its own topology, which turns A into a complete metrizable topological vector space such that the topology of A is stronger than the one inherited from $\omega(\mathbb{Z})$ and $\varphi(\mathbb{Z})$ is a dense linear subspace of A. Assume also that $S : \varphi(\mathbb{Z}) \to X$ is a linear map and $S(U \cap \varphi(\mathbb{Z}))$ is dense in X for any neighborhood U of zero in A. Then there exists $a \in A$ such that the series $\sum_{n=-\infty}^{\infty} a_n Se_n$ is universal in X.*

Proof. For each $n \in \mathbb{N}$, let $P_n : A \to A$ be the linear map defined by the formula $P_n a = \sum_{j=-n}^{n} a_j e_j$. Since the topology of A is stronger than the one inherited from $\omega(\mathbb{Z})$, P_n are continuous linear projections on A and SP_n are continuous linear operators from A to X. We use symbol $\mathcal{U}$ to denote the set of $a \in A$ such that the series $\sum_{n=-\infty}^{\infty} a_n Se_n$ is universal in X. Since the space $S(\varphi(\mathbb{Z}))$ of countable algebraic dimension is dense in X, X is separable. Since X is also metrizable, there is a countable base $\{W_k\colon k \in \mathbb{N}\}$ of topology of X. It is straightforward to see that

$$\mathcal{U} = \left\{a \in A\colon \{SP_n a\colon n \in \mathbb{Z}_+\} \text{ is dense in } X\right\} = \bigcap_{k=1}^{\infty} \Omega_k, \quad \text{where } \Omega_k = \bigcup_{n=1}^{\infty} (SP_n)^{-1}(W_k). \tag{4.1}$$

Since the operators $SP_n : A \to X$ are continuous, the sets Ω_k are open in A.

Linearity of S and density of $S(\varphi(\mathbb{Z}) \cap U)$ in X for each neighborhood of 0 in A imply that $S(\varphi(\mathbb{Z}) \cap U)$ is dense in A for any open $U \subseteq A$ such that $U \cap \varphi(\mathbb{Z}) \neq \varnothing$. Since $\varphi(\mathbb{Z})$ is dense in A, we see that $S(U \cap \varphi(\mathbb{Z}))$ is dense in X for any non-empty open subset U of A. Hence for each $k \in \mathbb{N}$ and a non-empty open subset U of A, there is $a \in \varphi(\mathbb{Z}) \cap U$ such that $Sa \in W_k$. Since $a \in \varphi(\mathbb{Z})$, $P_n a = a$ for each sufficiently large $n \in \mathbb{N}$. It follows that $a \in \Omega_k$. Thus $\Omega_k \cap U \neq \varnothing$ for any non-empty open subset U of A. That is, Ω_k is a dense open subset of A for each $k \in \mathbb{N}$. Since A is complete and metrizable, the Baire theorem and (4.1) imply that $\mathcal{U}$ is a dense G_δ-set in A. In particular, $\mathcal{U} \neq \varnothing$ as required. □

We are ready to prove Theorem 1.7. For $k \in \mathbb{N}$ let

$$J_k = \left\{e^{it}\colon 0 \leqslant t \leqslant 2\pi - k^{-1}\right\} \quad \text{and} \quad p_k : C(\mathbb{T}) \to \mathbb{R}, \qquad p_k(f) = \sup_{z\in J_k}\left|f(z)\right|. \tag{4.2}$$

Consider the topological vector space $C_\tau(\mathbb{T})$ being $C(\mathbb{T})$ with the topology τ defined by the sequence $\{p_k\}_{k\in\mathbb{N}}$ of seminorms. Then $C_\tau(\mathbb{T})$ is metrizable and locally convex. Moreover, a sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $C_\tau(\mathbb{T})$ if and only if f_n converges to f uniformly on J_k for each $k \in \mathbb{N}$. Since the union of J_k is $\mathbb{T}$, τ-convergence implies pointwise convergence. Hence τ is stronger than the topology of $C_p(\mathbb{T})$. Thus any series universal in $C_\tau(\mathbb{T})$ is also universal in $C_p(\mathbb{T})$. Let $c \in \mathcal{X}_+$. Consider the space A_c of complex bilateral sequences $a = \{a_n\}_{n\in\mathbb{Z}}$ such that $a_n = o(e^{c_{|n|}})$ as $n \to \infty$. The space A_c with the norm $\|a\| = \sup_{n\in\mathbb{Z}} |a_n| e^{-c_{|n|}}$ is a Banach space isometric to c_0. It is also straightforward to see that $\varphi(\mathbb{Z})$ is dense in A_c and that the topology of A_c is stronger than the one inherited from $\omega(\mathbb{Z})$. In order to prove Theorem 1.7 it is enough to find $a \in A_c$ such that the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is universal in $C_\tau(\mathbb{T})$. Consider the linear map $S : \varphi(\mathbb{Z}) \to C(\mathbb{T})$ such that $Se_n(z) = z^n$ for $n \in \mathbb{Z}$ and $z \in \mathbb{T}$. By Lemma 4.1, it suffices to demonstrate that for any neighborhood U of zero in A_c, $S(U \cap \varphi(\mathbb{Z}))$ is dense in $C_\tau(\mathbb{T})$. Since A_c is a normed space, it is enough to show that $\Omega = S(U \cap \varphi(\mathbb{Z}))$ is dense in $C_\tau(\mathbb{T})$, where $U = \{a \in A_c\colon \|a\| \leqslant 1\}$. Assume the contrary. Then Ω is not dense in $C_\tau(\mathbb{T})$. Since Ω is convex and balanced

(= stable under multiplication by $z \in \mathbb{C}$ with $|z| \leqslant 1$) and the topological vector space $C_\tau(\mathbb{T})$ is locally convex, the Hahn–Banach theorem implies that there exists a non-zero continuous linear functional $F : C_\tau(\mathbb{T}) \to \mathbb{C}$ such that $|F(f)| \leqslant 1$ for any $f \in \Omega$. Since the increasing sequence $\{p_k\}_{k\in\mathbb{N}}$ of seminorms defines the topology τ and F is τ-continuous, there exists $k \in \mathbb{N}$ such that F is bounded with respect to the seminorm p_k. The definition of p_k and the Riesz theorem on the shape of continuous linear functionals on Banach spaces $C(K)$ imply that there exists a non-zero finite Borel σ-additive complex-valued measure ν on $\mathbb{T}$ supported on J_k and such that

$$F(f) = \int_{\mathbb{T}} f(z)\,\nu(dz) = \int_{J_k} f(z)\,\nu(dz) \quad \text{for any } f \in C(\mathbb{T}).$$

Since $\|e^{c_{|n|}} e_n\| = 1$ for each $n \in \mathbb{Z}$, we see that $f_n \in \Omega$ for any $n \in \mathbb{Z}$, where $f_n(z) = e^{c_{|n|}} z^{-n}$. Since $|F(f)| \leqslant 1$ for any $f \in \Omega$, we have $|F(f_n)| \leqslant 1$ for $n \in \mathbb{Z}$. Hence

$$1 \geqslant \bigl|F(f_n)\bigr| = e^{c_{|n|}} \left| \int_{\mathbb{T}} z^{-n}\,\nu(dz) \right| = e^{c_{|n|}} \bigl|\widehat{\nu}(n)\bigr| \quad \text{for any } n \in \mathbb{Z}.$$

Thus $|\widehat{\nu}(n)| \leqslant e^{-c_{|n|}}$ for any $n \in \mathbb{Z}$. Since $c \in \mathcal{X}_+$, we have $\lim_{k\to\infty} k(c_{k+1} - c_k) = +\infty$. It immediately follows that $\lim_{k\to\infty} \frac{c_k}{\ln k} = \infty$. Hence $e^{-c_k} = o(k^{-j})$ as $k \to \infty$ for every $j \in \mathbb{N}$. Since $|\widehat{\nu}(n)| \leqslant e^{-c_{|n|}}$ for any $n \in \mathbb{Z}$, we get $|\widehat{\nu}(n)| = o(|n|^{-j})$ as $|n| \to \infty$ for each $j \in \mathbb{N}$. Hence ν is absolutely continuous with respect to μ and the density (= Radon–Nikodym derivative) $\rho = \frac{d\nu}{d\mu}$ is an infinitely differentiable complex valued function on $\mathbb{T}$. Since $\widehat{\rho}(n) = \widehat{\nu}(n)$ for any $n \in \mathbb{Z}$, we obtain $|\widehat{\rho}(n)| \leqslant e^{-c_{|n|}}$ for any $n \in \mathbb{Z}$. Since ν is supported on J_k, ρ vanishes on $\mathbb{T} \setminus J_k$. By Theorem Mb, $\rho = 0$ and therefore $\nu = 0$. We have arrived to a contradiction, which completes the proof of Theorem 1.7.

Remark 4.2. We have actually proven a slightly stronger result than the one stated in Theorem 1.7. Namely, we have shown that for each $c \in \mathcal{X}_+$, there is a sequence $a = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that $|a_n| \leqslant e^{c_{|n|}}$ for every $n \in \mathbb{Z}$ and the trigonometric series $\sum_{n=-\infty}^{\infty} a_n z^n$ is universal in $C(\mathbb{T})$ endowed with the metrizable locally convex topology τ (defined by the sequence $\{p_k\}_{k\in\mathbb{N}}$ of seminorms from (4.2)) stronger than the pointwise convergence topology.

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