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On the iterates of positive linear operators preserving the affine functions

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ABSTRACT

In this note we study the limit behavior of the iterates of a large class of positive linear operators preserving the affine functions and, as a byproduct of our result, we obtain the limit of the iterates of Meyer-König and Zeller operators.

1. Introduction and the main result

In 1967 Kelisky and Rivlin, see [12], studied the limit behavior of the iterates of Bernstein operators and in 1970 Karlin and Ziegler, see [11], generalized the results in [12] to a class of positive linear approximation operators. Their results have attracted much attention lately and several new proofs and generalizations have been given (see [4,9,1,7,3,15,8,2] and the references therein).

Recently, in [3,15], the authors used the contraction principle to study the over-iterates of a class of positive linear operators preserving the affine functions. Essentially, the results in [3,15] can be applied to finitely defined operators. However, the technique used for the discrete case fails to work for other classical positive linear operators such as, the Meyer-König and Zeller (in short MKZ), or the May operators.

In this note we study the limit behavior for the iterates of a class of positive linear operators $U\colon C[0,1]\to C[0,1]$ and, as a consequence of our result, we obtain the limit of the iterates of the MKZ operators. For the sake of simplicity we restrict ourself to $C[0,1]$, and mention that our results apply to operators defined on $C[a,b]$.

The following notations will be used throughout this paper: $e_i\colon[0,1]\to\mathbb{R}$, for the monomial functions $e_i(x)=x^i$, $i=0,1,2$, and

$$\omega(f;\delta):=\sup\bigl\{\bigl|f(x)-f(y)\bigr|\colon |x-y|\leqslant\delta,\ x,y\in[0,1]\bigr\},$$

for the classical modulus of continuity. The main result of this note is the following theorem.

Theorem 1. *Let $U\colon C[0,1]\to C[0,1]$ be a positive linear operator preserving the affine functions. If there exist $a\geqslant 1$, $m>0$ such that*

$$Ue_2-e_2\geqslant m(e_1-e_2)^a, \tag{1}$$

then

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$$\lim_{k\to\infty} U^k f = Lf, \quad \textit{uniformly on } [0,1], \textit{ for all } f \in C[0,1],$$

where $Lf = f(0)e_0 + (f(1) - f(0))e_1$ *denotes the Lagrange interpolating polynomial of degree one associated to* f *at the endpoints.*

Remark 1. Our result specializes to some classes of special operators. It is worth mentioning that, condition (1) is satisfied by the quasi-totality of classical positive linear approximation operators defined on $C[0,1]$. For example, the Bernstein operators B_n satisfy

$$B_n e_2 - e_2 = (1 - 1/n)(e_1 - e_2), \quad n \geqslant 1.$$

In Section 2 we prove that the MKZ operators M_n satisfy the inequality

$$M_n e_2 - e_2 \geqslant (n+1)^{-1}(e_1 - e_2)^2, \quad n \geqslant 1.$$

The May operators, see [13], defined by

$$S_n f(x) := \int_0^1 f(t)\rho_n(x,t)\,\mathrm{d}t, \quad n \in \mathbb{N},$$

where ρ_n denotes the kernel function, satisfy

$$S_n e_2 - e_2 = \lambda n^{-1}(e_1 - e_2), \quad n \geqslant 1,$$

for some $\lambda > 0$.

Before we give the proof of Theorem 1 we need a lemma that will be used in our analysis.

Lemma 2. *If* $U: C[0,1] \to C[0,1]$ *is a positive linear operator preserving the affine functions then*

$$\left|U^k f - Lf\right| \leqslant \left(1 + \frac{e_1 - U^k e_2}{\delta^2}\right)\omega(f;\delta), \quad k = 1, 2, \ldots, \tag{2}$$

for all $\delta > 0$ *and* $f \in C[0,1]$.

Proof. Let $\delta > 0$. Obviously, we have that

$$\left|f(t) - f(x)\right| \leqslant \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta),$$

for all $t, x \in [0,1]$. It follows, based on the positivity of L, that

$$|f - Lf| \leqslant \left(1 + \frac{e_1 - e_2}{\delta^2}\right)\omega(f;\delta).$$

By applying the operator U^k to the preceding inequality we obtain that inequality (2) is satisfied, and the lemma is proved. $\square$

Now we are ready to give the proof of the main result of the paper.

Proof of Theorem 1. We have, based on (1), that

$$U^{k+1}e_2 - U^k e_2 \geqslant m U^k (e_1 - e_2)^a, \quad k = 1, 2, \ldots. \tag{3}$$

Since the function $t \mapsto t^a$ is convex on $[0,1]$, we can apply Jessen's inequality, see [10], to the positive linear operator U^k, and we obtain that

$$U^k(e_1 - e_2)^a \geqslant \left(U^k(e_1 - e_2)\right)^a = \left(e_1 - U^k e_2\right)^a. \tag{4}$$

Combining (3) and (4) we obtain that

$$U^{k+1}e_2 - U^k e_2 \geqslant m\left(e_1 - U^k e_2\right)^a \geqslant 0. \tag{5}$$

By applying again Jessen's inequality to the positive operator U and the convex function e_2, we obtain that $e_2 \leqslant Ue_2 \leqslant$

$Ue_1 = e_1$, and hence,

$$U^k e_2 \leqslant U^{k+1} e_2 \leqslant e_1, \quad k = 0, 1, \ldots.$$

It follows, as an application of Dini's Theorem, that the sequence $(U^k e_2)_{k \geqslant 0}$ is uniformly convergent, and we have, based on (5), that $\lim_{k\to\infty} U^k e_2 = e_1$, uniformly. An application of Lemma 2 completes the proof. □

2. The iterates of Meyer-König and Zeller operators

In this section we apply the main result of the paper to discuss the limit of the iterates of a special class of operators. In 1960 Meyer-König and Zeller [14] introduced a sequence of positive linear operators which were studied, modified, and generalized by several authors.

The classical MKZ operators, in the modified version of Cheney and Sharma [6], $M_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, are defined by

$$M_n f(x) = \begin{cases} \sum_{k=0}^{\infty} \binom{n+k}{k} (1-x)^{n+1} x^k f\left(\frac{k}{n+k}\right), & x \in [0,1), \\ f(1), & x = 1. \end{cases}$$

Recently, Adell, Badía and de la Cal, see [1], established functional-type identities between the iterates of the MKZ operator and those of the celebrated Baskakov operator.

It is known that M_n are positive linear operators preserving the affine functions and that

$$M_1 e_2(x) = \frac{2x^2 - x - (1-x)^2 \log(1-x)}{x}, \quad x \in (0,1),$$
$$M_2 e_2(x) = \frac{4x^3 - 5x^2 + 2x + 2(1-x)^3 \log(1-x)}{x^2}, \quad x \in (0,1).$$

It is worth mentioning that, for a general n, the second moment of the MKZ operators cannot be expressed as a finite combination of elementary functions since this moment turns out to be a generalized hypergeometric function. This was a major obstacle in calculating the limit of the iterates of the MKZ operators. To the best of our knowledge, all the attempts to calculate the iterate limit failed. We were unable to find any mention of these limits in the literature and we believe that this line of investigation is completely new. In what follows we give an answer to this problem.

From [5, Eq. (2.4)], see also [16], we have that

$$M_n e_2 - e_2 \geqslant (n+1)^{-1} e_1 (1-e_1)^2 \geqslant (n+1)^{-1} (e_1 - e_2)^2, \quad n \geqslant 1.$$

Thus, for this class of operators, condition (1) is satisfied with $a = 2$. We have, as a consequence of Theorem 1, that the following corollary holds.

Corollary 3. *The sequence $(M_n^k)_{k\in\mathbb{N}}$ of the iterates of Meyer-König and Zeller operators converges strongly to L.*

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