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Remarks on Hardy spaces defined by non-smooth approximate identity

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ABSTRACT

We study in this paper some relations between Hardy spaces H^1_ϕ which are defined by non-smooth approximate identity $\phi(x)$, and the end-point Triebel–Lizorkin spaces $\dot{F}^{0,q}_1$ $(1 \leqslant q \leqslant \infty)$. First, we prove that $H^1(\mathbb{R}^n) \subset H^1_\phi(\mathbb{R}^n)$ for compact ϕ which satisfies a slightly weaker condition than Fefferman and Stein's condition. Then we prove that non-trivial Hardy space $H^1_\phi(\mathbb{R})$ defined by approximate identity ϕ must contain Besov space $\dot{B}^{0,1}_1(\mathbb{R})$. Thirdly, we construct certain functions $\phi(x) \in B^{0,1}_1 \cap \mathrm{Log}^{\frac{1}{2}}_0([-1,1])$ and a function $b(x) \in \bigcap_{q>1} \dot{F}^{0,q}_1$ such that Daubechies wavelet function $\psi \in H^1_\phi$ but $b^*_\phi \notin L^1$.

1. Introduction

Let $\mathbb{E} = \{\phi(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$ and $\forall x \in \mathbb{R}^n,\ \phi(x) \in \mathbb{R}\}$ and let $\mathbb{E}_c = \{\phi \in \mathbb{E};\ \phi$ has compact support$\}$. For $t > 0$, we denote $\phi_t(x) = t^{-n}\phi(xt^{-1})$. For $f(x) \in L^1(\mathbb{R}^n)$, define

$$f^*_\phi(x) = \sup_{|x-y|<t} \left| f * \phi_t(y) \right|. \tag{1.1}$$

We study in this paper the Hardy space

$$H^1_\phi\left(\mathbb{R}^n\right) = \left\{ f(x) \in L^1\left(\mathbb{R}^n\right);\ f^*_\phi(x) \in L^1\left(\mathbb{R}^n\right) \right\}.$$

In 1972, Fefferman and Stein [2] proved that (1) $H^1_\phi(\mathbb{R}^n) = H^1(R^n)$ for any smooth ϕ which decays quickly at infinity; and (2) $H^1 \subset H^1_\phi$ for non-smooth ϕ associated with additional regular conditions. In 1979, Weiss [5] asked whether there was an H^1_ϕ that was neither $\{0\}$ nor H^1. In 1983, Uchiyama and Wilson [4] proved that there exists $\phi(x) \in \mathbb{E}$ such that $H^1_\phi(\mathbb{R}) \neq H^1(\mathbb{R})$ and $h(x) \in H^1_\phi(\mathbb{R})$ where $h(x)$ is Haar function. For $r > 0$, we say that $f(x)$ is an r-logarithm regular function, if $|f(x) - f(y)| \leqslant C|\log|x-y||^{-r}$ and denote $f(x) \in \log^r$; we say that $f(x) \in \mathrm{Log}^r_0(B)$, if $f(x) \in \log^r$ and $\operatorname{supp} f \subset B$. Let $B^{0,1}_1$ be the corresponding non-homogeneous space to homogeneous Besov space $\dot{B}^{0,1}_1$. It is easy to see that

$$B^{0,1}_1 \subset L^1, \quad \mathrm{Log}^r_0\left([-1,1]\right) \in L^\infty \quad \text{and} \quad B^{0,1}_1 \cap \mathrm{Log}^r_0\left([-1,1]\right) \subset \mathbb{E}.$$

We will consider some more delicate results for non-smooth ϕ.

Throughout this paper, we denote by $\psi(x)$ the regular Daubechies wavelets with $\psi(x) \in C^2_0([-2^M, 2^M])$ where $M \in \mathbb{N}$ and $M \geqslant 3$; and denote by $\tilde{}$ the reflection operator that maps $f(x)$ to $\tilde{f}(x) = f(-x)$.

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For end point Triebel–Lizorkin space $\dot{F}_1^{0,q}$ $(1 \leqslant q \leqslant \infty)$ (which are atomic spaces), it is known that

$$\dot{B}_1^{0,1} = \dot{F}_1^{0,1} \subsetneqq \dot{F}_1^{0,q}(1<q<2) \subsetneqq \dot{F}_1^{0,2} = H^1.$$

We are interested in the relations between $H_\phi^1(\mathbb{R}^n)$ and the end point Triebel–Lizorkin space $\dot{F}_1^{0,q}(\mathbb{R}^n)$. More precisely, we will study the following three items:

1. If ϕ satisfies a slightly weaker condition than Fefferman and Stein's, then $H^1 \subset H_\phi^1$.
2. For arbitrary approximate identity ϕ even lacking regularity, if $H_\phi^1(\mathbb{R}) \neq \{0\}$, then $\dot{B}_1^{0,1}(\mathbb{R}) \subset H_\phi^1(\mathbb{R})$.
3. There is a function $\phi \in B_1^{0,1} \cap \mathrm{Log}_0^r([-1,1])$ and a function $b \in \bigcap_{q>1} \dot{F}_1^{0,q}$ such that regular Daubechies wavelet function $\psi \in H_\phi^1$ but $b \notin H_\phi^1$. This result is more delicate than Uchiyama and Stein's.

2. A weaker sufficient condition for Hardy space

We first prove a simple lemma.

Lemma 2.1. *For any $j \in \mathbb{R}$ and $k \in \mathbb{R}^n$, we have*

$$\left\|2^{jn} f\left(2^j x - k\right)\right\|_{H_\phi^1} = \|f\|_{H_\phi^1}.$$

Proof. It can be verified that

$$\left\|2^{jn} f\left(2^j x\right)\right\|_{H_\phi^1} = \|f\|_{H_\phi^1} \quad \text{and} \quad \left\|f(x-k)\right\|_{H_\phi^1} = \|f\|_{H_\phi^1}. \tag{2.2}$$

Fefferman and Stein [2] have proved that $H^1 \subset H_\phi^1$ under the condition

$$\sum_{j \geqslant 1} \tau(j) < \infty \tag{2.3}$$

where $\tau(j) = \sup_{|x-y| \leqslant 2^{-j}} |\phi(x) - \phi(y)|$, $\forall j \in \mathbb{N}$.

$\forall \phi \in \mathbb{E}_c$, let A be the smallest positive real number such that $\operatorname{supp} \phi(x)$ is contained in the ball $\mathbf{B}(0, A)$, and denote $B = 2 + [\log_2(A+1)]$. We define

$$\omega(j) = \sup_{x, t \geqslant 2^j} 2^{jn} t^{-n} \int_{|z| \leqslant 1} \left|\phi\left(x - t^{-1} z\right) - \phi(x)\right| dz. \tag{2.4}$$

It is easy to see that $\omega(j) \leqslant C\tau(j-B)$, $\forall j \geqslant B+1$. Now we introduce a slightly weaker condition:

$$\sum_{j \geqslant 1} \omega(j) < \infty. \tag{2.5}$$

The following Theorem 1 establishes Fefferman and Stein's result under a slightly weaker condition (2.5), where we also give an example to show such a condition cannot be improved in some sense. Note the proof here is more concise. □

Theorem 1.

(i) *If $\phi \in \mathbb{E}_c$ satisfying the condition* (2.5), *then $H^1 \subset H_\phi^1$.*
(ii) *There exists $\phi \in \mathbb{E}_c$ such that $\tau(j) \sim j^{-1}$, $\omega(j) \sim j^{-1}$ and $H_\phi^1 = \{0\}$.*

Proof. (i) The Hardy space H^1 can be characterized by its L^∞ atom (cf. [1,6]):

$$H^1 = \left\{ f(x) = \sum_s \lambda_s a_s(x)\text{: where } \lambda_s \in l^1 \text{ and } a_s(x) \text{ are } L^\infty \text{ atoms} \right\}.$$

Applying the above atom property of Hardy space, it is sufficient to prove that all L^∞ atoms $a(x)$ belong to H_ϕ^1. By Lemma 2.1, it is sufficient to consider the L^∞ atoms $a(x)$ in unit ball.

It follows from $a * \phi_t(y) = \int a(tz)\phi(yt^{-1} - z)\,dz = t^{-n} \int a(z)\phi(\frac{y-z}{t})\,dz$ that $|a * \phi_t(y)| \leqslant C$. If $|x| \leqslant 8A+8$, then $a_\phi^*(x) \leqslant C$. On the other hand, $\operatorname{supp} a * \phi_t(y) \subset B(0, At+1)$, which yields that $\sup_{|x-y|<t} |a * \phi_t(y)| = 0$ when $|x| > (A+1)t+1$.

If $2^j \leqslant |x| < 2^{j+1}$ and $|x| \leqslant (A+1)t+1$, we have $t \geqslant \frac{2^j-1}{A+1} \geqslant 2^{j-B}$ and

$$a * \phi_t(y) = t^{-n} \int a(z)\phi\left(\frac{y-z}{t}\right)dz = t^{-n} \int a(z)\left[\phi\left(\frac{y-z}{t}\right) - \phi\left(\frac{y}{t}\right)\right]dz,$$

which results that

$$\left|a * \phi_t(y)\right| \leqslant C \sup_{t \geqslant \frac{2^j-1}{A+1}} t^{-n} \int_{|z| \leqslant 1} \left|\phi\left(\frac{y-z}{t}\right) - \phi\left(\frac{y}{t}\right)\right| dz.$$

If $j \geqslant B+1$ and $2^j \leqslant |x| < 2^{j+1}$, then $a^*_\phi(x) \leqslant C2^{-nj}\omega(j-B)$. Henceforth $a^*_\phi(x) \in L^1$.

(ii) Define

$$\phi_1(x) = \sum_{j \geqslant 1, 2^{j-1} \leqslant k < 2^j} j^{-1}\tilde{\psi}\left(2^{2j}x - k\right),$$

then $\phi_1(x) \in L^\infty([B(0, 2^M)])$ and $\psi^*_{\phi_1}(x) \notin L^1$. We may choose $\phi_0(x) \in C_0^\infty(R^n)$ such that $\phi(x) = \phi_1(x) + \phi_0(x) > 0$ on $B([0, 2^M])$, thus we see that $\phi(x) \in F$ and $\psi(x)$ is not in H^1_ϕ. Applying Theorem 2 in the following Section 3, we know that $H^1_\phi = \{0\}$. □

3. Meyer's wavelets and $\dot{B}^{0,1}_1$

In this section, we will prove the following Theorem 2 for one dimension case. However, we don't know whether it is true for higher dimension cases.

Theorem 2. *If $H^1_\phi(\mathbb{R}) \neq \{0\}$, then $\dot{B}^{0,1}_1(\mathbb{R}) \subset H^1_\phi(\mathbb{R})$.*

To prove this theorem, we need a lemma belonging to Wiener (cf. [4,6]).

Lemma 3.1. *Let $f_1(x), f_2(x) \in L^1(\mathbb{R})$. If there exist an $\epsilon > 0$ and an interval $I \subset \mathbb{R}$ for which $|\hat{f}_1(\xi)| > \epsilon$, $\xi \in I$, and $\operatorname{supp} \hat{f}_2 \subset I$, then there exists an $h(x) \in L^1(\mathbb{R})$ such that $\hat{f}_2(\xi) = \hat{h}(\xi)\hat{f}_1(\xi)$.*

We also need an estimation for convolution function:

Lemma 3.2. *If $f \in H^1_\phi$ and $g \in L^1$, then*

$$\|f * g\|_{H^1_\phi} \leqslant \|f\|_{H^1_\phi}\|g\|_{L^1}.$$

Proof. In fact, we have

$$\sup_{|x-y|<t} \int \left|g(z)\right|\left|f * \phi_t(y-z)\right| dz \leqslant \int \left|g(z)\right| f^*_\phi(x-z)\, dz. \qquad \square$$

Proof of Theorem 2. If $f \not\equiv 0$ and $f \in H^1_\phi$, we may assume that f is real-valued. Since that $\hat{f}$ is continuous, then there exists a neighborhood V_{x_0} and $\epsilon > 0$ such that $\hat{f}(\xi) > \epsilon$, $\forall \xi \in V_{x_0}$. Since that f is real-valued, then we have also $\hat{f}(\xi) > \epsilon$, $\forall \xi \in V_{-x_0}$. Considering the dilation $x \to x_0 x$, we may assume that there exists $r > 1$ such that

$$\left|\hat{f}(\xi)\right| > \epsilon, \quad \forall \xi \in \left[-r, -r^{-1}\right] \cup \left[r^{-1}, r\right].$$

Let $\Psi(x) \in S(\mathbb{R})$ be a real-valued even function such that

$$\operatorname{supp} \hat{\Psi} \subset \left[-r, -r^{-1}\right] \cup \left[r^{-1}, r\right] \quad \text{and} \quad \sum_k \hat{\Psi}\left(r^k \xi\right) \equiv 1 \quad \text{for any } \xi \neq 0.$$

According to Wiener's Lemma 3.1, there exists $h_1(x) \in L^1(\mathbb{R})$ such that $\hat{\Psi}(\xi) = \widehat{h_1}(\xi)\hat{f}(\xi)$. If we define $\hat{h}(\xi) = \widehat{h_1}(\xi) + \widehat{h_1}(-\xi)$, then $\hat{\Psi}(\xi) = \hat{h}(\xi)\hat{f}(\xi)$. By Lemma 3.2, we have

$$\|\Psi\|_{H^1_\phi} \leqslant \|h\|_{L^1}\|f\|_{H^1_\phi}.$$

Thus when Φ^M is a Meyer's wavelet, we may write

$$\widehat{\Phi^M}(\xi) = \widehat{\Phi^M}(\xi)\sum_k \hat{\Psi}\left(r^k\xi\right) = \sum_{|k|\leqslant C} \widehat{\Phi^M}(\xi)\hat{\Psi}\left(r^k\xi\right).$$

Observing each term in the right-hand side, we see $\Phi^M * \Psi(r^k x) \in H^1_\phi$, which leads to $\Phi^M \in H^1_\phi$. Applying Lemma 2.1, we have $2^j\Phi^M(2^jx-k) \in H^1_\phi$.

Now for any $f(x) \in \dot{B}^{0,1}_1$, we may write $f(x) = \sum_{j,k} a_{j,k}2^j\Phi^M(2^jx-k)$ in which $\sum_{j,k}|a_{j,k}| < \infty$. Let $\tau_{j,k} = 2^j\Phi^M(2^jx-k)$, then $f * \phi_t(x) = \sum_{j,k} a_{j,k}(\tau_{j,k}) * \phi_t(x)$, which yields

$$f^*_\phi(x) \leqslant \sum_{j,k}|a_{j,k}|(\tau_{j,k})^*_\phi(x).$$

Thus we see $f \in H^1_\phi$. $\square$

4. End point Triebel–Lizorkin spaces

In 1983, Uchiyama and Wilson [4] proved that there exists $\phi(x) \in \mathbb{E}$ such that $H^1_\phi(R) \neq H^1(\mathbb{R})$ and $h(x) \in H^1_\phi(\mathbb{R})$ where $h(x)$ is Haar function. Noting that $\bigcap_{q>1}\dot{F}^{0,q}_1$ is close to $\dot{B}^{0,1}_1$, we can construct a group of more delicate examples in this section. In fact, we have:

Theorem 3. *There exist a group of function $\phi(x) \in B^{0,1}_1 \cap \mathrm{Log}^{\frac{1}{2}}_0([-1,1])$ and a group of functions $b(x) \in \bigcap_{q>1}\dot{F}^{0,q}_1 \bigcap_{0<p\leqslant 1} H^p$ such that Daubechies wavelet function $\psi(x) \in H^1_\phi$ but $b(x) \notin H^1_\phi$.*

Next, we construct two functions $\eta(x)$ and $b(x)$, whose properties are required in the proof of Theorem 3. For $j \in \mathbb{N}$, let $\tau_j \in \mathbb{N}$ be a monotonically increasing function with j such that

$$j \leqslant \tau_j \leqslant Cj^2 \quad \text{and} \quad \tau_{j+1} \geqslant \frac{j}{2} + \tau_j, \tag{4.6}$$

then we denote $\widetilde{v_{j,\tau_j}}(x) = \sum_{2^j\leqslant k<2^{j+1}} a_{j,k}\psi(2^{j+\tau_j}x-k)$ and $b_j(x) = \sum_{0\leqslant k<2^j} b_{j,k}\psi(2^jx-k)$, where $0 < C_1 \leqslant a_{j,k}, b_{j,k} \leqslant C_2$. Finally we define

$$\eta(x) = \sum_{j\geqslant 2M} j^{-1}\log^{-2} j v_{j,\tau_j}(x),$$

and

$$b(x) = \sum_{j\geqslant 2M} j^{-1}\log j b_j(x).$$

Using these two functions we can prove the following results.

Lemma 4.1. *Let the functions $\eta(x)$ and $b(x)$ be defined as above, we have*

(i) $\eta(x) \in \mathrm{Log}^{\frac{1}{2}}_0([0,\frac{1}{2}])$.
(ii) $b(x) \in \bigcap_{q>1}\dot{F}^{0,q}_1 \bigcap_{p>0} H^p$.
(iii) $b^*_\eta(x) \notin L^1$.
(iv) $\psi^*_\eta(x) \in L^1$.

Proof. (i) It is easy to verify that $\operatorname{supp}\eta(x) \subset [0,\frac{1}{2}]$ and

$$\|\eta(x)\|_{B^{0,1}_1} = \|\tilde{\eta}(-x)\|_{B^{0,1}_1} \leqslant \sum_{j\geqslant 2M} j^{-1}\log^{-2} j 2^{-\tau_j} < +\infty.$$

For $j, j' \geqslant 2M$ and $j \neq j'$, we have $\operatorname{supp} v_{j,\tau_j}(x) \cap \operatorname{supp} v_{j',\tau_{j'}}(x) = \emptyset$. Combine these properties with $\tau_j \leqslant Cj^2$ (according to (4.6)), we have $\eta(x) \in \mathrm{Log}^{\frac{1}{2}}_0([0,\frac{1}{2}])$.

(ii) Using the wavelet characterization theory of Triebel–Lizorkin spaces in [7], $\forall 1 < q \leqslant \infty$, we have

$$\|b(x)\|_{\dot{F}_1^{0,q}} = \left(\int \left(\sum_{\substack{j \geqslant 2M \\ 0 \leqslant k < 2^j}} j^{-q} (\log j)^q \chi\left(2^j x - k\right) \right)^{\frac{1}{q}} dx \right)^{\frac{1}{q}} < \infty.$$

Furthermore, using the wavelet characterization of Hardy spaces in [3], $\forall 0 < p \leqslant \infty$, we have

$$\|b(x)\|_{H^p} = \left(\int \left(\sum_{\substack{j \geqslant 2M \\ 0 \leqslant k < 2^j}} j^{-2} (\log j)^2 \chi\left(2^j x - k\right) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} < \infty.$$

(iii) We begin with computing $b * \eta_t(x)$.

$$\begin{aligned} b * \eta_t(x) = \frac{1}{t} \int &\left(\sum_{s \geqslant 2M} s^{-1} \log s \sum_{0 \leqslant k < 2^s} b_{s,k} \psi\left(2^s y - k\right) \right) \\ &\times \left(\sum_{s \geqslant 2M} s^{-1} \log^{-2} s \sum_{0 \leqslant k < 2^s} a_{s,k} \psi\left(2^{s+\tau_s} t^{-1} y - 2^{s+\tau_s} t^{-1} x - k\right) \right) dy. \end{aligned}$$

Note that

$$\operatorname{supp} v_{s,\tau_s}\left(\frac{\cdot - x}{t}\right) \subset \left[x + 2^{-(s+\tau_s)}\left(2^s - 2^M\right)t, x + 2^{-(s+\tau_s)}\left(2^{s+1} + 2^M - 1\right)t\right],$$

and

$$\operatorname{supp} b_s(y) \subset \left[-2^{M-s}, 2^{M-s} - 2^{-s} + 1\right].$$

For $1 \leqslant i \leqslant \frac{j}{2}$, we consider the case where $t = 2^{i+\tau_j}$ and $x = -2^i$. When $s \neq j$ we have

$$\operatorname{supp} v_{s,\tau_s}\left(\frac{\cdot - x}{t}\right) \cap \operatorname{supp} b(\cdot) = \emptyset.$$

Hence we see

$$\begin{aligned} b * \eta_{2^{i+\tau_j}}\left(-2^i\right) &= j^{-1} \log^{-2} j \cdot b * (v_{j,\tau_j})_{2^{i+\tau_j}}\left(-2^i\right) \\ &= j^{-1} \log^{-2} j \cdot t^{-1} \int b(y) \sum_{0 \leqslant k < 2^j} a_{j,k+2^j} \psi\left(2^{j-i} y - k\right) dy. \end{aligned}$$

Applying the orthogonality of wavelets basis, we have

$$\begin{aligned} b * \eta_{2^{i+\tau_j}}\left(-2^i\right) &= j^{-1} \log^{-2} j 2^{-(i+\tau_j)} (j-i)^{-1} \log(j-i) 2^{i-j} \sum_{0 \leqslant k < 2^{j-i}} b_{j-i,k} a_{j,k+2^j} \\ &\sim j^{-2} \log^{-1} j 2^{-(i+\tau_j)}. \end{aligned}$$

But $|x + 2^i| < 2^{i+\tau_j}$, thus we see

$$b_\eta^*(x) = \sup_{|x-y|<t} \left| b * \eta_t(y) \right| \geqslant \left| b * \eta_{2^{i+\tau_j}}\left(-2^i\right) \right| \geqslant C j^{-2} \log^{-1} j 2^{-(i+\tau_j)}.$$

To estimate the L^1-norm of $b_\eta^*(x)$, we need define the sets $E_{i,j} = \{2^{i-1+\tau_j} < x < 2^{i+\tau_j} - 2^i\}$ and we have the estimate

$$\int_{E_{i,j}} b_\eta^*(x)\, dx \geqslant C j^{-2} \log^{-1} j.$$

Note that if $1 \leqslant i < i' \leqslant \frac{j}{2}$, then $E_{i,j} \cap E_{i',j} = \emptyset$. Let $U_j = \{x\colon\ 2^{\tau_j} < x < 2^{\frac{j}{2}+\tau_j}\}$, then $\bigcup_{1 \leqslant i \leqslant \frac{j}{2}} E_{i,j} \subset U_j$. Summing up on $1 \leqslant i \leqslant \frac{j}{2}$, we get

$$\int_{U_j} b_\eta^*\, dx \geqslant \sum_i \int_{E_{i,j}} b_\eta^*\, dx \geqslant C j^{-1} \log^{-1} j.$$

Since $\tau_{j+1} \geqslant \frac{j}{2} + \tau_j$, hence for $j \neq s$, we have $U_j \cap U_s =$. These lead to

$$\left\| b_\eta^*(x) \right\|_{L^1} \geqslant \sum_{j \geqslant 2M} \int_{U_j} b_\eta^*(x)\,dx \geqslant \sum_{j \geqslant 2M} j^{-1} \log^{-1} j = \infty.$$

(iv) It is easy to see that $\psi_\eta^*(x)$ cannot be maximized by a monotone function in L^1. Let $F_{j,t}(x) \equiv \psi * (v_{j,\tau_j})_t(x) = t^{-1} \int \psi(x-y) v_{j,\tau_j}(yt^{-1})\,dy$, and take the transformation $y \to -y$, we get

$$\begin{aligned} F_{j,t}(x) &= t^{-1} \int \psi(x+y) \widetilde{v_{j,\tau_j}}\left(-yt^{-1}\right) dy \\ &= t^{-1} \int \psi(x+y) \sum_{2^j \leqslant k < 2^{j+1}} a_{j,k} \psi\left(t^{-1} 2^{j+\tau_j} y - k\right) dy. \end{aligned}$$

It is easy to see that

$$\operatorname{supp} F_{j,t}(x) \subset Q_{t,j} = \left[-2^M + t\left(2^{-\tau_j} - 2^{M-j-\tau_j}\right), 2^M + t\left(2^{1-\tau_j} + 2^{M-j-\tau_j} - 2^{-j-\tau_j}\right)\right].$$

According to the regularity of $\psi(x)$, we have the following points.

1. If $t \geqslant 2^{j+\tau_j}$, then $|F_{j,t}(x)| \leqslant Ct^{-3} 2^{2(j+\tau_j)}$.
2. If $2^{\tau_j} \leqslant t \leqslant 2^{j+\tau_j}$, then $|F_{j,t}(x)| \leqslant Ct 2^{-2(j+\tau_j)}$.
3. If $0 < t \leqslant 2^{\tau_j}$, then $|F_{j,t}(x)| \leqslant Ct^2 2^{-2j-3\tau_j}$.

Let $G_j(x) = \sup_{|x-y|<t} |F_{j,t}(y)|$, we have:

1. If $|x| \geqslant 2^{j+\tau_j}$, then $G_j(x) \leqslant C|x|^{-3} 2^{2(j+\tau_j)}$.
2. If $2^{\tau_j} \leqslant |x| \leqslant 2^{j+\tau_j}$, then $|F_{j,t}(x)| \leqslant C|x| 2^{-2(j+\tau_j)}$.
3. If $|x| \leqslant 2^{\tau_j}$, then $|F_{j,t}(x)| \leqslant C(1+|x|)^2 2^{-2j-3\tau_j}$.

It is easy to see that $\int G_j(x)\,dx < \infty$. Finally we have

$$\int \psi_\eta^*(x)\,dx \leqslant \sum_{j \geqslant 2M} j^{-1} \log^{-2} j \int G_j(x)\,dx < \infty. \qquad \square$$

Proof of Theorem 3. We may construct a $\phi_1(x) \in C_0^\infty([-1,1])$ such that $\phi = \phi_1 + \eta \geqslant 0$. According to [2], we have $b_{\phi_1}^*(x) \in L^1$ and $\psi_{\phi_1}^*(x) \in L^1$. Henceforth finally we have $\phi(x) \in B_1^{0,1} \cap \operatorname{Log}_0^{\frac{1}{2}}([-1,1])$, $b_\phi^*(x) \notin L^1$ and $\psi_\phi^*(x) \in L^1$. $\square$

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