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Localization results for generalized Baskakov/Mastroianni and composite operators ☆

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ABSTRACT

In this paper we study localization results for classical sequences of linear positive operators that are particular cases of the generalized Baskakov/Mastroianni operators and also for certain class of composite operators that can be derived from them by means of a suitable transformation. Amongst these composite operators we can find classical sequences like the Meyer–König and Zeller operators and the Bleimann, Butzer and Hahn ones. We extend in different senses the traditional form of the localization results that we find in the classical literature and we show several examples of sequences with different behavior to this respect.

1. Introduction and notation

A classical method to approximate a function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ consists in considering a sequence of linear operators $\{L_n : W \subseteq C(I) \to C(I)\}_{n\in\mathbb{N}}$ defined on certain subspace W to obtain a sequence of approximants $\{L_n f\}_{n\in\mathbb{N}}$ for the initial function f. We can find many examples of such type of approximation processes. Probably, the sequence of the Bernstein operators on $[0,1]$, $\{B_n : C[0,1] \to C[0,1]\}_{n\in\mathbb{N}}$, represents the best known case. However it is possible to find in the literature many other instances of similar sequences of linear positive operators as the Baskakov operators, Baskakov–Schurer operators, Mirakjan or generalized Mirakjan operators, etc. With the purpose of obtaining results for a wide class of linear positive operators we are going to consider here the generalized sequence of Baskakov [5] or the similar sequence of the Mastroianni operators [12] that, for $x \in [0,\infty)$ and a suitable function $f : I_\Phi \to \mathbb{R}$, are defined by

$$L_n f(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!} D^i \phi_n(x) f\left(\frac{i}{n}\right),$$

where $\Phi = \{\phi_n : [0,\infty) \to \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of analytic functions and $I_\Phi \subseteq [0,\infty)$ a subinterval which have to meet the following conditions:

(A) $\phi_n(0) = 1$ for every $n \in \mathbb{N}$.

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(B) We have $I_\Phi = [0,\infty)$ or $I_\Phi = [0, A]$ with $A > 0$ and

$$(-1)^k D^k \phi_n(x) \geqslant 0$$

for every $n \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $x \in I_\Phi$.

(C1) (Original Baskakov operators) There exists $c \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$D^{k+1}\phi_n(x) = -nD^k\phi_{n-c}(x), \tag{1}$$

for all $n \geqslant c$, $k \in \mathbb{N}$ and $x \in [0,\infty)$.

(C2) (Mastroianni operators) For every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ there exist $p(n,k) \in \mathbb{N}$ and $\alpha_{n,k} : [0,\infty) \to \mathbb{R}$ such that

$$D^{i+k}\phi_n(x) = (-1)^k \alpha_{n,k}(x) D^i \phi_{p(n,k)}(x) \tag{2}$$

for every $i \in \mathbb{N}_0$ and $x \in [0,\infty)$ in such a way that $\lim_{n\to\infty} \frac{n}{p(n,k)} = \lim_{n\to\infty} \frac{\alpha_{n,k}(x)}{n^k} = 1$.

The operators L_n are linear and positive. As a matter of fact, depending on f, $L_n f$ could be defined in the whole real axis but condition (B) guarantees the positivity only on I_Φ. It is also well known that the operators L_n preserve the degree of the polynomials and all the convexities and they also present good simultaneous approximation properties since they approximate not only the function but also all its derivatives (see [4, p. 344]).

From the generalized Baskakov operators we can derive many of the classical sequences of linear positive operators by making an adequate selection of Φ as we show in the following table.

Baskakov operators	$\phi_n(x) = (1+x)^{-n}$	$I_\Phi = [0,\infty)$
Baskakov–Schurer operators	$\phi_n(x) = (1+x)^{-(n+p)}$	
Szász operators	$\phi_n(x) = e^{-nx}$	
Szász–Mirakjan operators	$\phi_n(x) = e^{-(n+p)x}$	
Bernstein operators	$\phi_n(x) = (1-x)^n$	$I_\Phi = [0,1]$

We can find a large number of papers devoted to the study of the properties of convergence of these operators. In particular, the 'conservative properties' of the approximation operators are of special interest. That is to say, it is important to determine whether the operators reproduce the properties of the functions that we are trying to approximate. To this respect the preservation of the shape properties like the positivity or convexities are key points in the analysis of the Bernstein type operators like the Baskakov/Mastroianni sequences.

This paper is devoted to the study of a particular type of preserving properties usually known as 'localization results'. Consider the Bernstein operators and the functions $f_1, f_2 : [0,1] \to \mathbb{R}$ such that $f_1|_J = f_2|_J$ for certain open subinterval $J \subseteq [0,1]$. In this situation it is well known that for $x \in J$ we cannot deduce that $B_n f_1(x) = B_n f_2(x)$. However we have a special behavior for such a point x expressed by means of the infinitesimal relation

$$B_n f_1(x) = B_n f_2(x) + o\big(n^{-1}\big). \tag{3}$$

From the outstanding book on Bernstein operators by Lorentz [11, p. 7 and Theorem 4.1.3], similar localization results for Bernstein and other operators appear in many monographs (see also for instance [9, identity (3.3), p. 308]). It is immediate that, in general, a localization result can be written in the following form too: given a function $f : I_\Phi \to \mathbb{R}$ such that $f|_J = 0$ for certain open subinterval $J \subseteq I_\Phi$, for all $x \in J$ we have

$$L_n f(x) = o\big(n^{-1}\big).$$

Our purpose is to extend this type of localization results in various senses and to show that several classical sequences of linear positive operators present different behaviors to this respect.

Given $m \in \mathbb{N}_0$, consider a polynomial $p : I_\Phi \to \mathbb{R}$ of degree m, where we say that p is of degree m whenever $D^m p$ is a non-vanishing constant. Since the operators L_n preserve the degree of the polynomials, for any $r > m$ it is immediate that

$$D^r L_n f = 0.$$

To extend the localization results to the study of the convergence for the derivatives, suppose now that a function $f : I_\Phi \to \mathbb{R}$ behaves as a polynomial of degree m locally. Then we cannot assert that $D^r L_n$ vanishes but we can try to obtain conclusions on the local convergence for such derivative. Here we understand the local polynomial behavior in two senses:

(a) Local behavior: we say that the function f is a polynomial of degree m locally on the subinterval $J \subseteq I_\Phi$ open with respect to the topology of I_Φ whenever $f|_J = p$ for certain polynomial p of degree m.

(b) Pointwise behavior: we say that f is a polynomial of degree m punctually at $x \in I_\Phi$ whenever f is differentiable of any order at x and $D^i f(x) = 0$ for all $i > m$ and $D^m f(x) \neq 0$.

It is straightforward that condition (a) implies condition (b) for any point $x \in J$. In both cases we want to check whether it is obtained a special order of convergence towards zero for the derivative $D^r L_n f$ on J or at x respectively.

In Section 2 we study this problem for the generalized Baskakov operators. Moreover, since in general these operators are defined for functions of exponential growth, we study exponential type moments with the purpose of obtaining localization results for the class of functions for which the operators yield convergence. In Section 3 we prove results for composite operators that can be obtained from the generalized Baskakov operators. In this last section we show several examples of sequences with different behaviors for the localization results.

In order to deal with functions with pointwise polynomial behavior we present the following notation.

Definition 1. Given a function $f : I_\Phi \to \mathbb{R}$ differentiable of any order at $x \in I_\Phi$ we define

$$\deg_x(f) = \min\{s \in \mathbb{N}_0\colon\ D^i f(x) = 0,\ \forall i \geqslant s\} - 1,$$

where we assume the convention that $\min(\emptyset) = \infty$.

It is immediate to prove the properties that we include in the following lemma.

Lemma 2. *Given the functions $f, g : I_\Phi \to \mathbb{R}$ differentiable of any order at $x \in I_\Phi$,*

(i) *if* $\deg_x(f), \deg_x(g) \geqslant 0$ *then* $\deg_x(f \cdot g) = \deg_x(f) + \deg_x(g)$,
(ii) *if* $\deg_x(f) = -1$ *then* $\deg_x(f \cdot g) = -1$,
(iii) $\deg_x(f + g) \leqslant \max\{\deg_x(f), \deg_x(g)\}$,
(iv) *for* $s \in \mathbb{N}_0$, $\deg_x(D^s f) = \max\{\deg_x(f) - s, -1\}$,
(v) *if f is a polynomial,* $\deg(f) = \deg_x(f)$.

Let us fix some notation. Throughout the paper $t : \mathbb{R} \ni z \mapsto t(z) = z \in \mathbb{R}$ stands for the identity map on $\mathbb{R}$. We also denote by t the restrictions of the identity map to any subinterval. For all $i \in \mathbb{N}_0$, D^i is the ith differential operator and whenever it is necessary we will use brackets, $D^i[\cdot]$, to mark the scope of a differential operator. We will make an extensive use of the functional notation and only when an expression may be misunderstood we use the notation d^i/dx^i to indicate the variable for which we are differentiating. We will say that $f : I_\Phi \to \mathbb{R}$ is of exponential growth whenever it is possible to find $K, \alpha \in \mathbb{R}^+$ such that $|f| \leqslant K e^{\alpha t}$ on I_Φ. Given $x \in \mathbb{R}$ and $i, j \in \mathbb{N}_0$, $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$, with $x^{\underline{0}} = 1$, is the falling factorial and σ_i^j denotes the second kind Stirling numbers. Finally, we use the standard notation o and $\mathcal{O}$ for infinitesimal expressions and in addition we write

$$a_n = o\big(n^{-\infty}\big)$$

to denote the fact that $a_n = o(n^{-i})$, for all $i \in \mathbb{N}$.

2. Localization results for generalized Baskakov operators

With the aim of obtaining results for the derivatives of the operators L_n we introduce the following modified sequence: given $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, $x \in [0, \infty)$ and suitable $f : I_\Phi \to \mathbb{R}$, we define the operators

$$L_{r,n} f(x) = \sum_{i=0}^{\infty} (-1)^{i+r} \frac{x^i}{i!} f\left(\frac{i}{n}\right) \frac{D^{i+r}\phi_n(x)}{n^r}.$$

These operators are linear and positive. They already appear in [10] and by means of the classical differentiation formulas for L_n (see [4, p. 345], though we would like to point up that in Eq. (5.3.80) of [4] the factor $i!/n^i$ should be removed) we can establish the following relation between $L_{r,n}$ and the derivatives of L_n,

$$D^r L_n f = n^r L_{r,n}\big(\Delta^r_{\frac{1}{n}} f\big), \tag{4}$$

where $\Delta^r_{\frac{1}{n}}$ is the forward difference operator of order r. Along the paper we adopt the convention that $\Delta^r_{\frac{1}{n}} f(x) = 0$ whenever some of the knots of the forward difference ($x + \frac{j}{n}$, $j = 0, \ldots, r$) falls outside I_Φ. With the above definition it is obvious that $L_{0,n} = L_n$.

Let us study first the moments for the operators $L_{r,n}$. In particular in the following result we find a sufficient condition for the central moments of exponential type to have a good convergence behavior.

Proposition 3. *Given $r \in \mathbb{N}_0$ let us suppose that*

$$D^r\phi_n = (-1)^r q_r(n) e^{(an+b)g}, \tag{5}$$

where $a, b \in \mathbb{R}$, $a \neq 0$, $q_r(n)$ is a polynomial on n of degree r such that $q_r(n) > 0$ for n large enough and g is a function analytic at 0 with $g(0) = 0$ and $Dg(0) = -1/a$. Then for any $x \in [0, \infty)$ and $h \in \mathbb{N}_0$,

$$L_{r,n}\big(\big(e^t - e^x\big)^h\big)(x) = \mathcal{O}\big(n^{-[\frac{h+1}{2}]}\big).$$

Proof. Let us take $k > 0$. Then for any given $x \in I_\Phi$, a Taylor series expansion of the function $D^r\phi_n(x - e^{\frac{k}{n}}t)$ at the origin, yields

$$D^r\phi_n\big(x - e^{\frac{k}{n}}t\big) = \sum_{i=0}^{\infty} \frac{d^i}{dt^i}\big(D^r\phi_n\big(x - e^{\frac{k}{n}}t\big)\big)\Big|_{t=0}\frac{t^i}{i!} = \sum_{i=0}^{\infty} D^{r+i}\phi_n(x)(-1)^i e^{i\frac{k}{n}}\frac{t^i}{i!}.$$

Now if we evaluate at x, we have

$$D^r\phi_n\big(x - e^{\frac{k}{n}}x\big) = (-1)^r n^r \sum_{i=0}^{\infty}(-1)^{r+i}\frac{x^i e^{k\frac{i}{n}}}{i!}\frac{D^{i+r}\phi_n(x)}{n^r} = (-1)^r n^r L_{r,n}\big(e^{kt}\big)(x).$$

Therefore

$$L_{r,n}\big(e^{kt}\big)(x) = (-1)^r n^{-r} D^r\phi_n\big(x\big(1 - e^{\frac{k}{n}}\big)\big). \tag{6}$$

Notice that if we admit ϕ_n to be analytic on I_Φ for all n then this last expression holds true at least for n large enough even for $x \in \mathbb{R}$.

From the hypotheses, it follows that $g = \frac{-1}{a}t + \sum_{i=2}^{\infty} a_i t^i$ and it is also immediate that $1 - e^{k/n} = -\sum_{j=1}^{\infty}\frac{(k/n)^j}{j!}$ then

$$\begin{aligned}(an+b)g\big(x\big(1 - e^{k/n}\big)\big) &= -xn\big(1 - e^{k/n}\big) - \frac{b}{a}x\big(1 - e^{k/n}\big) + (an+b)\sum_{i=2}^{\infty} a_i x^i \big(1 - e^{k/n}\big)^i \\ &= xk + \underbrace{\frac{b}{a}\frac{k}{n}x + \sum_{j=2}^{\infty}\left(n + \frac{b}{a}\right)x\frac{(k/n)^j}{j!} + (an+b)\sum_{i=2}^{\infty} a_i x^i (-1)^i \left(\sum_{j=1}^{\infty}\frac{(k/n)^j}{j!}\right)^i}_{=H}.\end{aligned}$$

Take into account that H is an expansion on $\frac{1}{n}$ where all the monomials are of the form $k^s(\frac{1}{n})^\nu$ with $s \leqslant 2\nu$ and hence, since all the series that appear in the formula are absolutely convergent, we can rewrite it as an expansion on $\frac{1}{n}$ of the form

$$(an+b)g\big(x\big(1 - e^{k/n}\big)\big) = kx + \sum_{j=1}^{\infty} a_j(k,x)n^{-j}$$

with $a_j(x,k)$ a polynomial on k of degree at most $2j$ for all j. If we use the power expansion of e^t we can also deduce that

$$e^{(an+b)g(x(1-e^{k/n}))} = e^{kx}\sum_{j=1}^{\infty}\tilde{a}_j(x,k)n^{-j},$$

where again $\tilde{a}_j(x,k)$ is a polynomial on k of degree at most $2j$. Therefore with (5) we have

$$D^r\phi_n\big(x\big(1 - e^{\frac{k}{n}}\big)\big) = (-1)^r q_r(n) e^{kx}\sum_{j=1}^{\infty}\tilde{a}_j(x,k)n^{-j},$$

and together with (6), finally

$$L_{r,n}\big(e^{kt}\big)(x) = \frac{q_r(n)}{n^r}e^{kx}\sum_{j=1}^{\infty}\tilde{a}_j(x,k)n^{-j}.$$

Now, by means of Newton's binomial formula,

$$\begin{aligned}
L_{r,n}\big(\big(e^t-e^x\big)^h\big)(x) &= \sum_{k=0}^{h}\binom{h}{k}L_{r,n}\big(e^{kt}(-1)^{h-k}e^{(h-k)x}\big)(x)\\
&= \frac{q_r(n)}{n^r}\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}e^{(h-k)x}e^{kx}\sum_{j=1}^{\infty}\tilde{a}_j(x,k)n^{-j}\\
&= \frac{q_r(n)}{n^r}e^{hx}\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}\sum_{j=1}^{\infty}\tilde{a}_j(k,x)n^{-j}\\
&= \frac{q_r(n)}{n^r}e^{hx}\sum_{j=1}^{\infty}n^{-j}\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}\tilde{a}_j(k,x)\\
&= \frac{q_r(n)}{n^r}e^{hx}\sum_{j=1}^{\infty}n^{-j}\Delta_1^h\big[\tilde{a}_j(t,x)\big](0).
\end{aligned}$$

Since $\tilde{a}_j(k,x)$ is a polynomial on k of degree at most $2j$ we have that $\Delta_1^h[\tilde{a}_j(t,x)](0)=0$ for all $h>2j$ and therefore all summands for $j<[\frac{h+1}{2}]$ vanish. Then, as $q_r(n)$ is a polynomial of degree r, the last expression is $\mathcal{O}(n^{-[\frac{h+1}{2}]})$ and we end the proof. □

For the classical sequences that appear in the table of page 426 we have that for n large enough,

Baskakov operators	$D^r\phi_n(x)=(-1)^r(n+r-1)^{\underline{r}}e^{-(n+r)\log(1+x)}$
Baskakov–Schurer operators	$D^r\phi_n(x)=(-1)^r(n+r+p-1)^{\underline{r}}e^{-(n+r+p)\log(1+x)}$
Szász operators	$D^r\phi_n(x)=(-1)^r n^r e^{-nx}$
Szász–Mirakjan operators	$D^r\phi_n(x)=(-1)^r(n+p)^r e^{-(n+p)x}$
Bernstein operators	$D^r\phi_n(x)=(-1)^r n^{\underline{r}} e^{(n-r)\log(1-x)}$

and therefore all of them are under the assumptions of the preceding result and then Proposition 3 holds for the corresponding operators $L_{r,n}$.

From now on we assume that Φ is chosen in such a way that the conditions of Proposition 3 hold. In this way, the theses of the proposition are true for the operators $L_{r,n}$ that we are going to handle.

Indeed, once we have proved the preceding result for exponential type moments, it is posible to reach a similar conclusion for moments defined in terms of non-exponential functions as it can be seen in the following corollary that we will partially extend in the following section.

Corollary 4. *Let $\eta: I_\Phi \to \mathbb{R}$ be a function of class C^1 on I_Φ such that $0\leqslant D\eta\leqslant e^t$. Then, for all $h\in\mathbb{N}_0$ and $x\in I_\Phi$,*

$$L_{r,n}\big(\big(\eta-\eta(x)\big)^h\big)(x)=\mathcal{O}\big(n^{-\frac{h}{2}}\big).$$

Proof. For any $x_1,x_2\in I_\Phi$ with $x_1\leqslant x_2$ we have

$$0\leqslant \eta(x_2)-\eta(x_1)=\int_{x_1}^{x_2}D\eta(z)\,dz\leqslant\int_{x_1}^{x_2}e^z dz=e^{x_2}-e^{x_1}.$$

In this case, if h is an even number it is straightforward that we have $0\leqslant(\eta-\eta(x))^h\leqslant(e^t-e^x)^h$ and then

$$0\leqslant L_{r,n}\big(\big(\eta-\eta(x)\big)^h\big)(x)\leqslant L_{r,n}\big(\big(e^t-e^x\big)^h\big)(x)=\mathcal{O}\big(n^{-\frac{h}{2}}\big).$$

On the other hand, if $h=2p+1$, $p\in\mathbb{N}_0$, we can use a Schwartz type inequality to obtain

$$\begin{aligned}
\big|L_{r,n}\big(\big(\eta-\eta(x)\big)^h\big)(x)\big| &= \big|L_{r,n}\big(\big(\eta-\eta(x)\big)^{2p}\big(\eta-\eta(x)\big)\big)(x)\big|\\
&\leqslant \big(L_{r,n}\big(\big(\eta-\eta(x)\big)^{4p}\big)(x)\big)^{\frac{1}{2}}\big(L_{r,n}\big(\big(\eta-\eta(x)\big)^{2}\big)(x)\big)^{\frac{1}{2}}\\
&= \big(\mathcal{O}\big(n^{-2p}\big)\big)^{\frac{1}{2}}\big(\mathcal{O}\big(n^{-1}\big)\big)^{\frac{1}{2}}=\mathcal{O}\big(n^{-\frac{h}{2}}\big). \qquad \square
\end{aligned}$$

In particular if we take $\eta = t$, for the usual polynomial moments the last corollary yields

$$L_{r,n}\big((t-x)^h\big)(x) = \mathcal{O}\big(n^{-\frac{h}{2}}\big), \quad \forall x \in I_\Phi,\ h \in \mathbb{N}. \tag{7}$$

Since the polynomial moments will play an important role throughout the paper, we will denote

$$T_{r,h,n}(x) = L_{r,n}\big((t-x)^h\big)(x) \tag{8}$$

and also, for short, $T_{h,n} = T_{0,h,n}$. Notice that $T_{r,h,n}$ is a function on I_Φ so that $D^s T_{r,h,n}(x)$ stands for the derivative $\frac{d^s}{dz^s}|_{z=x} T_{r,h,n}(z)$.

In the following proposition we prove several facts about the moments $T_{r,h,n}$ that in part are a consequence of Corollary 4 and improve (7).

Corollary 5. *For any $s, r, h \in \mathbb{N}_0$ and $x \in I_\Phi$:*

(i) *$D^s T_{r,h,n}(x)$ is a polynomial on n^{-1} of degree $r+h$ and order $[\frac{h+1}{2}]$. As a consequence*

$$D^s T_{r,h,n}(x) = \mathcal{O}\big(n^{-[\frac{h+1}{2}]}\big).$$

(ii)
$$D^s T_{r,h,n}(x) = \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} h^{\underline{s-j}} D^j\big[L_{r,n}\big((t-x)^{h-s+j}\big)\big](x).$$

Proof. Let us prove claim (i). In [4, Eq. (5.3.81)] by means of a McLaurin series it is obtained an alternative representation for the operators L_n. Similar arguments apply to prove the following expression for $L_{r,n}$,

$$L_{r,n} f(x) = \sum_{i=0}^{\infty} (-1)^{i+r} \frac{x^i}{i!} \frac{D^{i+r}\phi_n(0)}{n^r} \Delta^i_{\frac{1}{n}} f(0), \tag{9}$$

which for the case $r = 0$ corresponds to the mentioned equation that appears in [4]. From the assumptions of Proposition 3, that are supposed to be satisfied, it is immediate that $D^{i+r}\phi_n(0)$ is a polynomial of degree $i + r$ on n. Moreover, for every $j \in \mathbb{N}_0$ and $f = t^j$, we can use the second kind Stirling numbers, σ^i_j, to write the forward difference in (9) as $\Delta^i_{\frac{1}{n}}(t^j)(0) = i!\sigma^i_j n^{-j}$. In this way, since $\Delta^i_{\frac{1}{n}}(t^j) = 0$ for $i > j$, from (9) we deduce that $L_{r,n}(t^j)(x)$ is a polynomial on n^{-1} of degree $r + j$. Since $T_{r,h,n}(x)$ is a linear combination of $L_{r,n}(t^j)(x)$, $i = 0, \ldots, h$, we have that $T_{r,h,n}(x)$ is again a polynomial on n^{-1} of degree $r + h$. Finally, with (7) we easily deduce that $T_{r,h,n}(x)$ is a polynomial on n^{-1} with exponents between $[\frac{h+1}{2}]$ and $r + h$. Accordingly, as $T_{r,h,n}(x)$ is a finite polynomial on n^{-1}, it is immediate that the same conclusion is satisfied for any derivative $D^s T_{r,h,n}(x)$ too.

For claim (ii), as a consequence of the definition of $L_{r,n}$ and Leibnitz's formula

$$\begin{aligned}
D^s T_{r,h,n}(x) &= \frac{d^s}{dz^s}\bigg|_{z=x} \big[L_{r,n}\big((t-z)^h\big)(z)\big] \\
&= \sum_{i=0}^{\infty} (-1)^{i+r} \frac{1}{i!n^r} \frac{d^s}{dz^s}\bigg|_{z=x} \left[z^i D^{i+r}\phi_n(z)\left(\frac{i}{n} - z\right)^h\right] \\
&= \sum_{i=0}^{\infty} (-1)^{i+r} \frac{1}{i!n^r} \sum_{j=0}^{s} \binom{s}{j} D^j\big[t^i D^{i+r}\phi_n\big](x)(-1)^{s-j} h^{\underline{s-j}} \left(\frac{i}{n} - x\right)^{h-s+j} \\
&= \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} h^{\underline{s-j}} \sum_{i=0}^{\infty} (-1)^{i+r} D^j\left[\frac{t^i}{i!} \frac{D^{i+r}\phi_n}{n^r}\right](x) \left(\frac{i}{n} - x\right)^{h-s+j} \\
&= \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} h^{\underline{s-j}} D^j\big[L_{r,n}\big((t-x)^{h-s+j}\big)\big](x). \qquad \square
\end{aligned}$$

Let us prove now our first localization result. The following proposition is valid for a function that vanishes locally in the sense that we explain in point (a) of page 426. This result is a preliminary step to prove the main theorem of this section.

Proposition 6. *Let $f : I_\Phi \to \mathbb{R}$ of exponential growth and a subinterval $J \subseteq I_\Phi$ open with respect to the topology of I_Φ be such that $f|_J = 0$. Then, for all $r \in \mathbb{N}_0$ and $x \in J$,*

$$D^r L_n f(x) = o\big(n^{-\infty}\big).$$

Proof. It is possible to find a subinterval J_1 open with respect to the topology of J with $x \in J_1 \subseteq J$ and $n_0 \in \mathbb{N}$ large enough such that for any $n \geqslant n_0$,

$$\Delta^r_{\frac{1}{n}} f|_{J_1} = 0. \tag{10}$$

Moreover, since f is of exponential growth, there exits $K, \alpha > 0$ such that $|f| \leqslant Ke^{\alpha t}$ and then for any $z \in I_\Phi$,

$$\left|\Delta^r_{\frac{1}{n}} f(z)\right| \leqslant \sum_{i=0}^{r} \binom{r}{i} \left| f\left(z + \frac{i}{n}\right)\right| \leqslant \sum_{i=0}^{r} \binom{r}{i} K e^{\alpha(z+\frac{i}{n})} = K\left(1 + e^{\frac{\alpha}{n}}\right)^r e^{\alpha z} \leqslant K_1 e^{\alpha z},$$

with $K_1 = K(1 + e^{\alpha})^r$. Therefore $\Delta^m_{\frac{1}{n}} f$ is also a function of exponential growth for the constants K_1 and α which do not depend on n. This fact together with (10) implies that for any even number $h \in \mathbb{N}$ big enough, we can find $K_2 > 0$ such that for all $n \geqslant n_0$,

$$\left|\Delta^r_{\frac{1}{n}} f\right| \leqslant K_2\left(e^t - e^x\right)^h.$$

Since the operators $L_{r,n}$ are positive, we obtain

$$\left|L_{r,n}\left(\Delta^r_{\frac{1}{n}} f\right)(x)\right| \leqslant K_2 L_{r,n}\left(\left(e^t - e^x\right)^h\right)(x) = \mathcal{O}\left(n^{-\frac{h}{2}}\right).$$

But h is arbitrarily large and we can assert that $L_{r,n}(\Delta^r_{\frac{1}{n}} f)(x) = o(n^{-\infty})$. Now, identity (4) yields

$$D^r L_n f(x) = n^r L_{r,n}\left(\Delta^r_{\frac{1}{n}} f\right)(x) = n^r o\left(n^{-\infty}\right) = o\left(n^{-\infty}\right). \qquad \square$$

We are at this point in a position to show a localization result for the derivatives of the L_n operators with pointwise conditions. We will obtain such a result as an immediate consequence of the proposition below. Following the terminology of page 426 we analyze in this proposition and the next theorem the localization results for the rth derivative and functions with polynomial behavior of degree less than r. The preceding proposition studies the case $r = -1$ and now we are going to extend the result for $r \geqslant 0$.

Proposition 7. *Let $f : I_\Phi \to \mathbb{R}$ be a function of exponential growth and let $r \in \mathbb{N}_0$, $h \in \mathbb{N}$ and $x \in I_\Phi$ be such that f is differentiable at x of order $r + h + 2$. Let us suppose that $D^i f(x) = 0$, for $r \leqslant i \leqslant r + h$. Then*

$$D^r L_n f(x) = \mathcal{O}\left(n^{-[\frac{h+2}{2}]}\right).$$

Proof. Consider the sequence of linear positive operators given by

$$\tilde{L}_n(g) = D^r L_n I^r(g),$$

where $Ig(z) = \int_0^z g(u)\,du$ and $I^r = I \circ \overset{r)}{\cdots} \circ I$ is the rth composition of I.

It is straightforward that for all $\alpha \in \mathbb{N}$,

$$I^r(t - x)^\alpha = \frac{1}{(\alpha + r)^{\underline{r}}}(t - x)^{\alpha+r} + p_{r-1}$$

with p_{r-1} a polynomial of degree $r - 1$. We know that L_n preserves the degree of the polynomials so that $D^r L_n(p_{r-1}) = 0$ and then by (4)

$$\tilde{L}_n\left((t - x)^\alpha\right) = \frac{1}{(\alpha + r)^{\underline{r}}} D^r L_n\left((t - x)^{\alpha+r} + p_{r-1}\right) = \frac{n^r}{(\alpha + r)^{\underline{r}}} L_{r,n}\left(\Delta^r_{\frac{1}{n}}(t - x)^{\alpha+r}\right). \tag{11}$$

By using the definition of forward difference we can write

$$\begin{aligned}
\Delta^r_{\frac{1}{n}}(t - x)^{\alpha+r} &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(t + \frac{i}{n} - x\right)^{\alpha+r} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \sum_{j=0}^{\alpha+r} \binom{\alpha + r}{j} (t - x)^j \left(\frac{i}{n}\right)^{\alpha+r-j}
\end{aligned}$$

$$= \sum_{j=0}^{\alpha+r} \binom{\alpha+r}{j} \frac{1}{n^{\alpha+r-j}} (t-x)^j \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} i^{\alpha+r-j}$$
$$= \sum_{j=0}^{\alpha+r} \binom{\alpha+r}{j} \frac{1}{n^{\alpha+r-j}} (t-x)^j \Delta_1^r t^{\alpha+r-j}(0).$$

If we take into account that $\Delta_1^r t^{\alpha+r-j}(0) = r!\sigma_{\alpha+r-j}^r = 0$ whenever $\alpha < j$, with (11) and claim (i) of Corollary 5 we obtain

$$\begin{aligned}\tilde{L}_n\big((t-x)^\alpha\big)(x) &= \frac{r!}{(\alpha+r)^{\underline{r}}} \sum_{j=0}^{\alpha} \binom{\alpha+r}{j} \sigma_{\alpha+r-j}^r \frac{1}{n^{\alpha-j}} L_{r,n}\big((t-x)^j\big)(x) \\ &= \frac{r!}{(\alpha+r)^{\underline{r}}} \sum_{j=0}^{\alpha} \binom{\alpha+r}{j} \sigma_{\alpha+r-j}^r \frac{1}{n^{\alpha-j}} \mathcal{O}\big(n^{-[\frac{j+1}{2}]}\big) \\ &= \mathcal{O}\big(n^{-[\frac{\alpha+1}{2}]}\big).\end{aligned} \tag{12}$$

Since f is differentiable of order $r+h+2$ at x, there exists a bounded subinterval $J \subseteq I_\Phi$ open with respect to the topology of I_Φ with $x \in J$ such that $f \in C^{r+h+1}(J)$. Consider any subinterval $J_1 \subseteq J$ open with respect to I_Φ with $x \in J_1$ and $d(J_1, I_\Phi - J) > 0$. It is possible to define a function $g \in C^{r+h+1}(I_\Phi)$ in such a way that

$$\begin{cases} g|_{I_\Phi - J} = 0, \\ g|_{J_1} = f|_{J_1}. \end{cases}$$

Moreover, it is clear that g is differentiable of order $r+h+2$ at x.

We will prove only for even h since for odd h the same arguments can be used. Then, the hypotheses on the derivatives of f at x, the last identity and a Taylor series for $D^r g$ guarantee that for all $z \in I_\Phi$,

$$\begin{aligned} D^r g(z) &= \sum_{i=0}^{h+2} \frac{D^{r+i} g(x)}{i!} (z-x)^i + \mu(z)(z-x)^{h+2} \\ &= \frac{D^{r+h+1} g(x)}{(h+1)!} (z-x)^{h+1} + \frac{D^{r+h+2} g(x)}{(h+2)!} (z-x)^{h+2} + \mu(z)(z-x)^{h+2}, \end{aligned}$$

for certain bounded function $\mu : I_\Phi \to \mathbb{R}$ such that $\lim_{z\to x} \mu(z) = \mu(x) = 0$. Therefore, since $\tilde{L}_n$ is linear and positive if we apply the operator and evaluate at x we have

$$\left| \tilde{L}_n\big(D^r g\big)(x) - \frac{D^{r+h+1} g(x)}{(h+1)!} \tilde{L}_n\big((t-x)^{h+1}\big)(x) - \frac{D^{r+h+2} g(x)}{(h+2)!} \tilde{L}_n\big((t-x)^{h+2}\big)(x) \right| \leqslant \|\mu\|_{I_\Phi} \cdot \tilde{L}_n\big((t-x)^{h+2}\big)(x),$$

where $\|\cdot\|_{I_\Phi}$ stands for the sup norm on I_Φ. Hence, on account of (12), it follows that $\tilde{L}_n(D^r g)(x) = \mathcal{O}(n^{-[\frac{h+2}{2}]})$.

We know that $(f-g)|_{J_1} = 0$ and then from Proposition 6 it follows that $D^r L_n(f-g)(x) = o(n^{-\infty})$. Finally, it is immediate that $\tilde{L}_n(D^r g) = D^r L_n g$ and then

$$D^r L_n f(x) = D^r L_n g(x) + D^r L_n (f-g)(x) = \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) + o\big(n^{-\infty}\big) = \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \qquad \square$$

The immediate conclusion of the last proposition is the announced localization theorem for the derivatives of the L_n operators and functions with pointwise polynomial behavior in the sense of point (b) of page 426.

Theorem 8. *Let $f : I_\Phi \to \mathbb{R}$ be of exponential growth and let $r \in \mathbb{N}_0$ and $x \in I_\Phi$ be such that f is differentiable at x of any order with $\deg_x(f) < r$. Then*

$$D^r L_n f(x) = o\big(n^{-\infty}\big).$$

From this theorem we can also derive a localization result in the line of point (a) of page 426. In fact, it is a simple corollary that, in the conditions of the theorem, if we also have that $f|_J = p$ for a polynomial p of degree $m < r$ and $J \subseteq I_\Phi$ open with respect to I_Φ, we deduce that $D^r L_n f(x) = o(n^{-\infty})$ for all $x \in J$.

Of course, the preceding theorem is valid for Baskakov, Baskakov–Schurer, Szász, Szász–Mirakjan operators for functions of exponential growth and for Bernstein operators with bounded functions.

Remark 9. Theorem 8 can be formulated in a way similar to that of the classical localization results presented in the introduction of the paper (see identity (3)). That is to say, given two functions $f_1, f_2 : I_\Phi \to \mathbb{R}$ of exponential growth and differentiable of any order at $x \in I_\Phi$ with $D^i f_1(x) = D^i f_2(x)$ for all $i \geqslant r$, we have

$$D^r L_n f_1(x) - D^r L_n f_2(x) = o\big(n^{-\infty}\big).$$

As a consequence, if we have $D^r f_1|_J = D^r f_2|_J$ for certain $J \subseteq I_\Phi$ open with respect to I_Φ then we have that

$$D^r L_n f_1(x) - D^r L_n f_2(x) = o\big(n^{-\infty}\big), \quad \forall x \in J.$$

3. Composite operators

We can find in the literature several classical sequences that are obtained by composition from the generalized Baskakov operators L_n. With the purpose of studying such composite operators we are going to analyze a general class of linear operators that are obtained from L_n through a transformation of the following type: given the subintervals $I \subseteq \mathbb{R}$ and $I_1 \subseteq I_\Phi$, a C^∞ diffeomorphism $\varphi : I \to I_1 \subseteq I_\Phi$ and a C^∞ function $q : I_1 \to \mathbb{R}$ with $q > 0$, we consider any transformation

$$\Psi : \mathbb{R}^I \to \mathbb{R}^{I_\Phi}$$

such that

$$\Psi(f)|_{I_1} = q \cdot \big(f \circ \varphi^{-1}\big).$$

Since φ and q are of class C^∞, it is immediate that the fact that f is differentiable of order k at certain $x \in I$ implies that Ψf is also differentiable of order k at $\varphi(x)$. On the other hand it is also clear that the inverse transformation is given by

$$\Psi^{-1} : \mathbb{R}^{I_1} \to \mathbb{R}^I,$$
$$\Psi^{-1}(f) = \frac{1}{q \circ \varphi} \cdot (f \circ \varphi).$$

For every such a transformation Ψ we consider the composite operator

$$L_n^\Psi : \mathbb{R}^I \to \mathbb{R}^I,$$
$$L_n^\Psi = \Psi^{-1} \circ L_n \circ \Psi.$$

Such kind of transformations and composite operators, Ψ and L_n^Ψ, can also be found in [7] where they are introduced in order to obtain estimates in weighted approximation.

In this section we are going to extend our localization results to this kind of composite operators. At the end of the section we will show several examples of classical sequences that can be obtained from the generalized Baskakov operators by means of a suitable transformation Ψ. It is of interest the fact that the behavior for such composite sequences with respect to the localization results may differ considerably from the case of the Baskakov operators and the examples that we give later illustrate this aspect.

In what follows Ψ is one of the transformations described above, given for certain fixed functions φ, q, and L_n^Ψ is the corresponding composite operator.

We can transfer easily some of the properties of the Baskakov operators to the L_n^Ψ operators. For instance, L_n^Ψ are linear and positive and we will see that L_n^Ψ reproduces the behavior of L_n for the convergence of moments. For this purpose, in the next lemma we first give a representation for the derivatives of the transformations of a function in terms of the derivatives of the function that we will also use later at several points.

Lemma 10. *There exist the sequences of functions $\psi_{s,k} : I \to \mathbb{R}$ and $\tilde{\psi}_{s,k} : I_1 \to \mathbb{R}$, $s, k \in \mathbb{N}_0$, $s \leqslant k$, such that for all $f_1 : I_1 \to \mathbb{R}$ and $f_2 : I \to \mathbb{R}$ differentiable enough we have*

$$\text{(i)} \qquad D^k \Psi^{-1}(f_1) = \sum_{s=0}^{k} \big(D^s[f_1] \circ \varphi\big) \cdot (\tilde{\psi}_{s,k} \circ \varphi),$$

$$\text{(ii)} \qquad D^k \Psi(f_2) = \sum_{s=0}^{k} \big(D^s[f_2] \circ \varphi^{-1}\big) \cdot \big(\psi_{s,k} \circ \varphi^{-1}\big).$$

Proof. Let us prove (i). For $k = 0$ it is clear that we can take $\tilde{\psi}_{0,0} = 1/q$. Let us proceed now by induction on k. If we suppose that the result holds for certain value of k, by simply differentiation we have

$$
\begin{aligned}
D^{k+1}\Psi^{-1}(f_1) &= \sum_{s=0}^{k}\bigl[\bigl(D^{s+1}[f_1]\circ\varphi\bigr)\cdot D\varphi\cdot(\tilde{\psi}_{s,k}\circ\varphi)+\bigl(D^{s}[f_1]\circ\varphi\bigr)\cdot D[\tilde{\psi}_{s,k}\circ\varphi]\bigr]\\
&= \sum_{s=0}^{k+1}\bigl(D^{s}[f_1]\circ\varphi\bigr)\cdot(\tilde{\psi}_{s,k+1}\circ\varphi)
\end{aligned}
$$

with

$$
\tilde{\psi}_{s,k+1}\circ\varphi=\begin{cases} D[\tilde{\psi}_{0,k}\circ\varphi], & \text{if } s=0,\\ D\varphi\cdot(\tilde{\psi}_{s-1,k}\circ\varphi)+D[\tilde{\psi}_{s,k}\circ\varphi], & \text{if } 1\leqslant s\leqslant k,\\ D\varphi\cdot(\tilde{\psi}_{k,k}\circ\varphi), & \text{if } s=k+1.\end{cases}
$$

Claim (ii) can be proved in much the same way. □

As a consequence of Proposition 7 we can prove the following lemma which partially extend Corollary 4.

Lemma 11. *Let $\eta : I \to \mathbb{R}$ be a function of class C^∞ on I such that $\Psi(\eta)$ is of exponential growth. Then for all $h \in \mathbb{N}_0$ and $x \in I$,*

$$L_n^{\Psi}\bigl(\bigl(\eta-\eta(x)\bigr)^h\bigr)(x)=\mathcal{O}\bigl(n^{-[\frac{h+1}{2}]}\bigr).$$

Proof. From the definition we know that

$$L_n^{\Psi}\bigl(\bigl(\eta-\eta(x)\bigr)^h\bigr)(x)=\frac{1}{q(\varphi(x))}L_n\bigl(\Psi\bigl(\bigl(\eta-\eta(x)\bigr)^h\bigr)\bigr)\bigl(\varphi(x)\bigr).$$

But claim (ii) of Lemma 10 implies that for any $k \in \mathbb{N}_0$ with $k < h$,

$$D^k\bigl[\Psi\bigl(\bigl(\eta-\eta(x)\bigr)^h\bigr)\bigr]\bigl(\varphi(x)\bigr)=\sum_{s=0}^{k}D^s\bigl[\bigl(\eta-\eta(x)\bigr)^h\bigr](x)\cdot\psi_{s,k}(x)$$

and it is immediate that $D^s[(\eta-\eta(x))^h](x)=0$ for all $s \leqslant k < h$. Then $D^k\Psi((\eta-\eta(x))^h)(\varphi(x))=0$, for all $k<h$. Thus, by Proposition 7 we have

$$L_n\bigl(\Psi\bigl(\bigl(\eta-\eta(x)\bigr)^h\bigr)\bigr)\bigl(\varphi(x)\bigr)=\mathcal{O}\bigl(n^{-[\frac{h+1}{2}]}\bigr)$$

which ends the proof. □

As we have explained before, the behavior for the composite operators L_n^{Ψ} is different from the case of the Baskakov operators studied in Section 2. Only the results that do not involve the derivatives can be transferred to L_n^{Ψ} directly. In fact, the following result is the extension for the composite operators of Proposition 6 and the case $r=0$ of Proposition 7.

Proposition 12. *Given a function $f : I \to \mathbb{R}$ such that Ψf is of exponential growth, the following claims hold*:

(i) *Let $J \subseteq I$ be a subinterval open with respect to the topology of I such that $f|_J = 0$, then for all $r \in \mathbb{N}_0$ and $x \in J$,*

$$D^r L_n^{\Psi} f(x)=o\bigl(n^{-\infty}\bigr).$$

(ii) *Let $x \in I$ and $h \in \mathbb{N}$ be such that f is differentiable at x of order $h+2$. Let us suppose that $D^i f(x)=0$ for all $i=0,\dots,h$, then*

$$L_n^{\Psi} f(x)=O\bigl(n^{-[\frac{h+2}{2}]}\bigr).$$

Proof. For a given $x \in I$ let us denote $\tilde{x}=\varphi(x)$ and $\tilde{f}=\Psi f$. By means of claims (i) and (ii) of Lemma 10 we know that, provided f is differentiable enough, for every $k \in \mathbb{N}_0$,

$$D^k L_n^{\Psi}(f)(x)=D^k\bigl[\Psi^{-1}L_n\tilde{f}\,\bigr](x)=\sum_{s=0}^{k}D^s L_n\tilde{f}(\tilde{x})\cdot\tilde{\psi}_{s,k}(\tilde{x}),\tag{13}$$

$$D^k\tilde{f}(\tilde{x})=\sum_{s=0}^{k}D^s f(x)\cdot\psi_{s,k}(x).\tag{14}$$

Let us prove claim (i). If $f|_J = 0$ and $x \in J$, we have that $\tilde{f}|_{\varphi(J)} = 0$ and $\tilde{x} \in \varphi(J)$. Besides $\varphi(J) \subseteq I_\Phi$ is an open subinterval with respect to I_Φ. Then by Proposition 6, $D^s L_n(\tilde{f})(\tilde{x}) = o(n^{-\infty})$ for any $s \in \mathbb{N}_0$. Therefore, from (13) with $k = r$, claim (i) follows.

For (ii), if x is under the assumptions of the claim, from (14) it follows that $D^k \tilde{f}(\tilde{x}) = 0$ for $k = 0, \dots, h$ and then Proposition 7 proves that

$$L_n^\Psi f(x) = \frac{1}{q(\tilde{x})} L_n \tilde{f}(\tilde{x}) = \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \qquad \square$$

We can now formulate our main result for composite operators. As it is described on page 426, we take a function with pointwise polynomial behavior of degree m and we want to establish a localization result for the rth derivative with $r > m$. We will see that for this situation the result depends on the asymptotic expression of the composite operators for the monomials t^δ (that is to say on $L_n^\Psi(t^\delta)$) which correspond for the L_n operators to the functions $\Psi(t^\delta) = q \cdot (\varphi^{-1})^\delta$. In the theorem, an asymptotic expansion of the type $a_n = \sum_{j=0}^\infty b_j n^{-1}$ must be understood in the sense that for every $s \in \mathbb{N}$ we have $a_n = \sum_{j=0}^s b_j n^{-1} + \mathcal{O}(n^{-(s+1)})$.

Theorem 13. *Let us suppose that for all $\delta \in \mathbb{N}_0$,*

$$L_n^\Psi\big(t^\delta\big) = t^\delta + \sum_{j=1}^\infty n^{-j} p_{\delta,j}, \tag{15}$$

where for all δ and j we have that $p_{\delta,j} \in C^\infty(I)$. Given $x \in I$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}_0 \cup \{-1\}$ with $r > m$, let $f : I \to \mathbb{R}$ be a function differentiable of any order at x with $\deg_x(f) = m$ and such that Ψf is of exponential growth. Then

$$D^r L_n^\Psi(f)(x) = \begin{cases} o(n^{-\infty}), & \text{if } m = -1, \\ o(n^{-\infty}), & \text{if } m = 0 \text{ and } L_n^\Psi(1) = 1, \\ o(n^{-\infty}), & \text{if } m = 1 \text{ and } L_n^\Psi(1) = 1,\ L_n^\Psi(t) = t. \end{cases} \tag{16}$$

Furthermore, given $\alpha \in \mathbb{N}$ suppose that $\alpha = 1$ or $\alpha \geqslant 2$ with

$$\deg_x(p_{\delta,j}) < \delta + r - m, \quad \forall \delta \leqslant 2\alpha - 2,\ \forall j \leqslant \alpha - 1 \tag{17}$$

then

$$D^r L_n^\Psi(f)(x) = \mathcal{O}\big(n^{-\alpha}\big).$$

Proof. Let us denote again $\tilde{f} = \Psi f$ and $\tilde{x} = \varphi(x)$. Accordingly, identities (13) and (14) remain valid.

Take an arbitrary $h \in \mathbb{N}$. Since f is differentiable of any order at x we can find a subinterval $\Gamma \subseteq I$ open with respect to I with $x \in \Gamma$ such that $f \in C^{h+r}(\Gamma)$ and therefore we also have $\tilde{f} \in C^{h+r}(\varphi(\Gamma))$. Then, for any $s = 1, \dots, r$ and any $z \in \varphi(\Gamma)$, consider the functions $g = \tilde{f} - \sum_{i=0}^h \frac{D^i \tilde{f}(z)}{i!}(t-z)^i$ and $g_s = \tilde{f} - \sum_{i=0}^h \frac{D^{s+i}\tilde{f}(z)}{(s+i)!}(t-z)^{s+i}$ defined both of them on I_Φ. Since $D^j g(z) = D^{s+j} g_s(z) = 0$ for all $j = 0, \dots, h$, by means of Proposition 7 we know that $L_n g(z) = \mathcal{O}(n^{-[\frac{h+2}{2}]})$ and $D^s L_n g_s(z) = \mathcal{O}(n^{-[\frac{h+2}{2}]})$ and therefore for all $z \in \varphi(\Gamma)$,

$$L_n(\tilde{f})(z) = \underbrace{\sum_{i=0}^h \frac{D^i \tilde{f}(z)}{i!} L_n\big((t-z)^i\big)(z)}_{=G_{h,n}(z)} + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big), \tag{18}$$

$$D^s L_n(\tilde{f})(z) = \sum_{i=0}^h \frac{D^{s+i}\tilde{f}(z)}{(s+i)!} D^s L_n\big((t-z)^{s+i}\big)(z) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \tag{19}$$

It is clear that $G_{h,n} \in C^r(\varphi(\Gamma))$ and by claim (i) of Corollary 5 we know that for every $z \in \varphi(\Gamma)$, $G_{h,n}(z)$ is a polynomial on n^{-1} of degree h and consequently we can consider the decomposition

$$G_{h,n}(z) = \tilde{G}_{h,n}(z) + \hat{G}_{h,n}(z), \tag{20}$$

where both $\tilde{G}_{h,n}(z)$ and $\hat{G}_{h,n}(z)$ are polynomials on n^{-1} with $\tilde{G}_{h,n}(z)$ of degree $[\frac{h+2}{2}] - 1$ and $\hat{G}_{h,n}(z) = \mathcal{O}(n^{-[\frac{h+2}{2}]})$. Then (18) can be also written as

$$L_n(\tilde{f})(z) = \tilde{G}_{h,n}(z) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big).$$

Notice that both identity (20) and the degrees on n^{-1} of the members of such a decomposition are preserved by differentiating.

Moreover, it is easily seen that we can also consider the functions $\Psi^{-1}G_{h,n}$, $\Psi^{-1}\tilde{G}_{h,n}$, $\Psi^{-1}\hat{G}_{h,n} \in C^r(\Gamma)$ (even if $G_{h,n}$, $\tilde{G}_{h,n}$ and $\hat{G}_{h,n}$ are not defined on the whole interval I_1) which maintain the same properties as polynomials in n^{-1} and again from (18), it is a simple matter to obtain

$$L_n^{\Psi} f(z) = \Psi^{-1}G_{h,n}(z) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) = \Psi^{-1}\tilde{G}_{h,n}(z) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big), \quad \forall z \in \Gamma. \tag{21}$$

Let us study the derivatives of $G_{h,n}$. If we make use of the definition of $G_{h,n}$ given in (18), the notation $T_{i,n}$ established in (8) and Leibnitz's formula, we have

$$D^s G_{h,n}(\tilde{x}) = \sum_{i=0}^{h}\sum_{j=0}^{s}\binom{s}{j}\frac{D^{i+s-j}\tilde{f}(\tilde{x})}{i!}D^j T_{i,n}(\tilde{x}).$$

Notice that in the last expression we can assume that $j \leqslant i$ since otherwise the derivative of the moment $T_{i,n}$ vanishes. Then, we can easily make the following changes in the order of the sums $\sum_{i=0}^{h}\sum_{j=0}^{s} = \sum_{j=0}^{s}\sum_{i=0}^{h} = (\tilde{i} = i - j) = \sum_{j=0}^{s}\sum_{\tilde{i}=0}^{h-j}$. On the other hand, for $\tilde{i} > h - j$, from claim (i) of Corollary 5, we have

$$D^j T_{\tilde{i}+j,n}(\tilde{x}) = \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big),$$

so, except an infinitesimal expression $\mathcal{O}(n^{-[\frac{h+2}{2}]})$, we can extend the sum over $\tilde{i}$ up to $\sum_{\tilde{i}=0}^{h}$. Writing again i instead of $\tilde{i}$, we obtain

$$D^s G_{h,n}(\tilde{x}) = \sum_{i=0}^{h}\frac{D^{s+i}\tilde{f}(\tilde{x})}{(s+i)!}\sum_{j=0}^{s}\frac{(s+i)!}{(i+j)!}\binom{s}{j}D^j T_{i+j,n}(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big).$$

If we apply claim (ii) of Corollary 5 to compute $D^j T_{i+j,n}(\tilde{x})$, we get

$$\begin{aligned} D^s G_{h,n}(\tilde{x}) &= \sum_{i=0}^{h}\frac{D^{s+i}\tilde{f}(\tilde{x})}{(s+i)!}\sum_{j=0}^{s}\frac{(s+i)!}{(i+j)!}\binom{s}{j}\sum_{\alpha=0}^{j}\binom{j}{\alpha}(-1)^{j-\alpha}(i+j)^{\underline{j-\alpha}}D^{\alpha}L_n\big((t-\tilde{x})^{i+\alpha}\big)(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) \\ &= \sum_{i=0}^{h}\frac{D^{s+i}\tilde{f}(\tilde{x})}{(s+i)!}\sum_{\alpha=0}^{s}\frac{(s+i)!}{(i+\alpha)!}\binom{s}{\alpha}D^{\alpha}L_n\big((t-\tilde{x})^{i+\alpha}\big)(\tilde{x})\sum_{j=\alpha}^{s}\binom{s-\alpha}{s-j}(-1)^{j-\alpha} + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) \end{aligned}$$

and it is immediate that the sum on j in the last expression is equal to 1 whenever $\alpha = s$ and vanishes otherwise. So we finally have

$$D^s G_{h,n}(\tilde{x}) = \sum_{i=0}^{h}\frac{D^{s+i}\tilde{f}(\tilde{x})}{(s+i)!}D^s L_n\big((t-\tilde{x})^{s+i}\big)(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big).$$

Therefore, with (19) and (20) we can assert that for all $s = 1, \ldots, r$,

$$D^s L_n(\tilde{f})(\tilde{x}) = D^s G_{h,n}(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) = D^s \tilde{G}_{h,n}(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big).$$

Then, if we use (13) we deduce that

$$\begin{aligned} D^r L_n^{\Psi}(f)(x) &= D^r\big[\Psi^{-1}L_n(\tilde{f})\big](x) \\ &= \sum_{s=0}^{r} D^s L_n(\tilde{f})(\tilde{x})\cdot\tilde{\psi}_{s,r}(\tilde{x}) \\ &= \sum_{s=0}^{r} D^s \tilde{G}_{h,n}(\tilde{x})\cdot\tilde{\psi}_{s,r}(\tilde{x}) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) \\ &= D^r\big[\Psi^{-1}(\tilde{G}_{h,n})\big](x) + \mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \end{aligned} \tag{22}$$

Let us obtain now an expression for $\Psi^{-1}(\tilde{G}_{h,n})$. For this purpose, consider, for every $z \in \Gamma$, the function $g^* = f - \sum_{i=0}^{h}\frac{D^i f(z)}{i!}(t-z)^i$ for which it is clear that $D^i g^*(z) = 0$ for all $i = 0, \ldots, h$. Then, Proposition 12 implies that

$$L_n^\Psi f(z)=\sum_{i=0}^{h}\frac{D^i f(z)}{i!}L_n^\Psi\big((t-z)^i\big)(z)+\mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big)$$

and hence with (21) we deduce that

$$\Psi^{-1}(\tilde{G}_{h,n})(z)=\sum_{i=0}^{h}\frac{D^i f(z)}{i!}L_n^\Psi\big((t-z)^i\big)(z)+\mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \tag{23}$$

But, from hypotheses (15) of the theorem, with the aid of Newton's binomial formula, we have for every $i\in\mathbb{N}$ that

$$L_n^\Psi\big((t-z)^i\big)(z)=\sum_{j=1}^{[\frac{h+2}{2}]-1}n^{-j}q_{i,j}(z)+\mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big) \tag{24}$$

with

$$q_{i,j}=\sum_{\delta=0}^{i}\binom{i}{\delta}(-1)^{i-\delta}t^{i-\delta}p_{\delta,j}.$$

Notice also that, in particular for $i=0$, if we take $q_{0,j}=p_{0,j}$ from (15) it follows that

$$L_n^\Psi\big((t-z)^0\big)(z)=L_n^\Psi(1)(z)=1+\sum_{j=1}^{[\frac{h+2}{2}]-1}n^{-j}q_{0,j}(z)+\mathcal{O}\big(n^{-[\frac{h+2}{2}]}\big). \tag{25}$$

Take into account that in the case that (17) holds we have

$$\deg_x\big(t^{i-\delta}p_{\delta,j}\big)=i-\delta+\deg_x(p_{\delta,j})<i-\delta+\delta+r-m=i+r-m$$

and therefore, in this case, the central moments at x also meet condition (17). That is to say,

$$\deg_x(q_{i,j})<i+r-m. \tag{26}$$

From (23), (24) and (25), since $\Psi^{-1}(\tilde{G}_{h,n})$ is a polynomial on n^{-1} of degree $[\frac{h+2}{2}]-1$, it is immediate that

$$\Psi^{-1}(\tilde{G}_{h,n})=f+\sum_{j=1}^{[\frac{h+2}{2}]-1}n^{-j}\sum_{i=a}^{h}\frac{D^i f}{i!}q_{i,j}, \tag{27}$$

where

$$a=\begin{cases}0, & \text{in the general case,}\\ 1, & \text{if } L_n^\Psi(1)=1,\\ 2, & \text{if } L_n^\Psi(1)=1 \text{ and } L_n^\Psi(t)=t.\end{cases}$$

Let us see that the assertions of the theorem follows from (22) and (27).

For the three cases of (16), it is immediate that whenever $i\geqslant a$ we have $\deg_x(D^i f)=-1$ and therefore $\deg_x(D^i f\cdot q_{i,j})=-1$ and accordingly $D^r[D^i f\cdot q_{i,j}](x)=0$. Moreover, since $\deg_x(f)=m<r$ we always have $D^r f(x)=0$. Then the rth derivative of all the summands in (27) vanishes and therefore $D^r[\Psi^{-1}(\tilde{G}_{h,n})](x)=0$. Hence by means of (22), since h is arbitrary, we obtain (16).

For $m>-1$ and $\alpha\geqslant 2$, let us take $h=2\alpha-2$. If condition (17) is satisfied then we have seen that (26) also holds and hence

$$\deg_x\big(D^i f\cdot q_{i,j}\big)=m-i+\deg_x(q_{i,j})<m-i+i+r-m=r.$$

Accordingly, we have again that all the rth derivatives of the summands in (27) vanish and it follows that

$$D^r\big[\Psi^{-1}(\tilde{G}_{h,n})\big](x)=0$$

and (22) leads us again to the result. Finally, for $\alpha=1$ the result is straightforward from (27). □

Remark 14. The result of the preceding theorem is sharp in the following sense. If condition (17) of Theorem 13 does not hold for certain values of α, r and m, then it is not possible to guarantee that $D^r L_n^\Psi f(x) = \mathcal{O}(n^{-\alpha})$ for every function with pointwise polynomial behavior of degree m at x.

For instance if, for $\alpha, m \in \mathbb{N}_0$ with $m \leqslant 2\alpha - 1$, $\delta = m$ and certain $j_0 \leqslant \alpha - 1$, we have that (17) fails at $x \in I_\Phi$ this means that

$$\deg_x(p_{m,j_0}) \geqslant m + r - m = r. \tag{28}$$

Consider now the function t^m for which, from the hypotheses of Theorem 13, we know that $L_n^\Psi(t^m) = t^m + \sum_{j=1}^\infty n^{-j} p_{m,j}$. In the proof of the theorem it is justified (see (22)) that

$$D^r L_n^\Psi\big(t^m\big)(x) = \sum_{j=1}^{\alpha-1} n^{-j} D^r p_{m,j}(x) + \mathcal{O}\big(n^{-\alpha}\big).$$

But because of (28) we cannot assure that $D^r p_{m,j_0}(x) = 0$. In particular, in many cases we have that the functions $p_{\delta,j}$ are polynomials on the whole I (not only pointwise at x) and then $\deg_x(p_{m,j_0}) \geqslant r$ implies that $\deg(p_{m,j_0}) \geqslant r$ and we will have $D^r p_{m,j_0}(x) = 0$ only for a finite set of points x. For the rest of points we conclude that

$$D^r L_n^\Psi\big(t^{m-1}\big)(x) \neq \mathcal{O}\big(n^{-\alpha}\big).$$

In what follows let us use the results of this section to analyze several cases of composite linear positive operators.

3.1. The Meyer–König and Zeller operators

The Meyer–König and Zeller operators in the slight variation by Cheney and Sharma [8] are defined for $f : [0,1) \to \mathbb{R}$ by

$$M_n f(x) = (1-x)^{n+1} \sum_{i=0}^\infty \binom{n+i}{i} x^i f\left(\frac{i}{n+i}\right).$$

Let us take $\phi_n(x) = (1+x)^{-(n+1)}$ in the definition of the generalized Baskakov operators and let us consider the transformation Ψ given by $\varphi = \frac{t}{1-t}$ and $q = 1$ (that is to say, $\Psi f = f \circ \varphi^{-1}$). Then $M_n = L_n^\Psi$ so that the Meyer–König and Zeller operators can be obtained as composite operators from the generalized Baskakov operators. It is also well known that $M_n(t^i) = t^i$ for $i = 0, 1$. On the other hand, in [1, Theorem 2] Abel proves that for these operators the functions $p_{\delta,j}$ are polynomials on $[0,1)$ with

$$\deg(p_{\delta,j}) = \delta + j.$$

Then Theorem 13 yields the following result.

Theorem 15. *Let $f : [0,1) \to \mathbb{R}$ a function such that $f \leqslant Ce^{\alpha \frac{t}{1-t}}$ for certain constants $C, \alpha \in \mathbb{R}^+$, differentiable of any order at $x \in [0,1)$. Given $m \in \mathbb{N}_0 \cup \{-1\}$ such that* $\deg_x(f) = m$ *and $r \in \mathbb{N}_0$, $r > m$, we have*

$$D^r M_n(f)(x) = \begin{cases} o(n^{-\infty}), & \text{if } m = -1, 0, 1, \\ \mathcal{O}(n^{-(r-m)}), & \text{if } m > 1. \end{cases}$$

3.2. The Bleimann, Butzer and Hahn operators

For a function $f : [0,\infty) \to \mathbb{R}$ and $x \in [0,\infty)$, the Bleimann, Butzer and Hahn operators [6] are defined by

$$\mathrm{BBH}_n(f)(x) = \frac{1}{(1+x)^n} \sum_{i=0}^n \binom{n}{i} x^i f\left(\frac{i}{n+1-i}\right).$$

It is well known that the BBH operators are linear and positive and also that $\mathrm{BBH}_n(t^i) = t^i$ for $i = 0, 1$. We have already seen on page 426 that the generalized Baskakov operators yield the Bernstein operators by taking $\phi_n = (1-t)^n$ and $I_\Phi = [0,1]$. We can derive the BBH operators from the Bernstein operators (see [3]) as $\mathrm{BBH}_n = L_{n+1}^\Psi$ if we take $I_1 = [0,1)$, $\varphi = \frac{t}{1+t}$, $q = 1 - t$ and the transformation

$$\Psi(f)(x) = \begin{cases} q(x) \cdot (f \circ \varphi^{-1})(x), & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1. \end{cases}$$

Moreover, in [2, Proposition 3] it is proved that

$$\mathrm{BBH}_n\big((1+t)^\delta\big) = (1+t)^\delta + \sum_{j=1}^{\infty} \frac{1}{(n+1)^j}\tilde{p}_{\delta,j}(t+1),$$

with $\tilde{p}_{\delta,j}$ a polynomial of degree $\delta + j$. From this expression it is easy to deduce, for $n > 1$, that

$$L_n^{\Psi}\big(t^\delta\big) = \mathrm{BBH}_{n-1}\big(t^\delta\big) = t^\delta + \sum_{j=1}^{\infty} \frac{1}{n^j} p_{\delta,j},$$

again with $p_{\delta,j}$ a polynomial of degree $\delta + j$. Therefore we can use Theorem 13 to obtain the following result.

Theorem 16. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $(t+1)f$ is bounded and differentiable of any order at $x \in [0,\infty)$. Given $m \in \mathbb{N}_0 \cup \{-1\}$ such that $\deg_x(f) = m$ and $r \in \mathbb{N}_0$ with $r > m$,*

$$D^r\mathrm{BBH}_n(f)(x) = \begin{cases} o(n^{-\infty}), & \text{if } m = -1, 0, 1, \\ \mathcal{O}(n^{-(r-m)}), & \text{if } m > 1. \end{cases}$$

3.3. An example with different behavior

Consider the sequence of linear positive operators L_n^{Ψ} given for $\phi_n(x) = e^{-nx}$, $\varphi = -\log(1-t) : I = [0,1) \to I_1 = I_\Phi = [0,\infty)$ and $\Psi f = f \circ \varphi^{-1}$. For any $f : [0,1) \to \mathbb{R}$ and $x \in [0,1)$, from the definition of the Baskakov operators, it is a simple matter to find the explicit expression

$$L_n^{\Psi} f(x) = (1-x)^n \sum_{i=0}^{\infty} \frac{\log^i((1-x)^{-n})}{i!} f\big(1 - e^{-\frac{i}{n}}\big).$$

It is immediate that $L_n^{\Psi}(1) = 1$. Moreover, with the aid of (6), for every $z \in [0,1)$, we have

$$L_n^{\Psi}(t)(z) = L_n(\Psi t)\big(\varphi(z)\big) = L_n\big(\varphi^{-1}\big)\big(\varphi(z)\big) = L_n\big(1 - e^{-t}\big)\big(\varphi(z)\big) = 1 - e^{-n\varphi(z)(1-e^{\frac{-1}{n}})}.$$

It can be checked that

$$1 - e^{-n\varphi(z)(1-e^{\frac{-1}{n}})} = 1 - e^{-\varphi(z)} - \frac{1}{2n}\varphi(z)e^{-\varphi(z)} + \mathcal{O}\big(n^{-2}\big)$$

so we finally obtain

$$L_n^{\Psi}(t) = t + \frac{1}{2n}(1-t)\log(1-t) + \mathcal{O}\big(n^{-2}\big).$$

That is to say $p_{1,1} = \frac{1}{2}(1-t)\log(1-t)$ and then $D^k p_{1,1}(x) \neq 0$ (except for $Dp_{1,1}(\frac{e-1}{e}) = 0$) for all $k \in \mathbb{N}_0$ and $x \in [0,1)$ or, what is the same, $\deg_x(p_{1,1}) = \infty$. With this all at hand, Theorem 13 and Remark 14 lead us to the following result.

Theorem 17. *Let $f : [0,1) \to \mathbb{R}$ be a function such that $f(1-e^{-t})$ is of exponential growth and differentiable of any order at $x \in [0,1)$. Given $m \in \mathbb{N}_0 \cup \{-1\}$ such that $\deg_x(f) = m$ and $r \in \mathbb{N}_0$ with $r > m$,*

$$D^r L_n^{\Psi}(f)(x) = \begin{cases} o(n^{-\infty}), & \text{if } m = -1, 0, \\ \mathcal{O}(n^{-1}), & \text{if } m > 0. \end{cases}$$

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