



On a system involving a critically growing nonlinearity [☆]

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ABSTRACT

This paper deals with the system

$$\begin{cases} -\Delta u = \lambda u + q|u|^3 u \phi & \text{in } B_R, \\ -\Delta \phi = q|u|^5 & \text{in } B_R, \\ u = \phi = 0 & \text{on } \partial B_R. \end{cases}$$

We prove existence and nonexistence results depending on the value of λ .

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1. Introduction

Recently, in the paper [2], it has been studied the following system

$$\begin{cases} -\Delta u = \eta|u|^{p-1}u + \varepsilon q \phi f(u) & \text{in } \Omega, \\ -\Delta \phi = 2qF(u) & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 5$, $q > 0$, $\varepsilon, \eta = \pm 1$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $F(s) = \int_0^s f(t) dt$.

If $f(s) = s$, (1) becomes the well-known Schrödinger–Poisson system in a bounded domain which has been investigated by many authors (see e.g. [4,6,10,11,13,14]). In [2] it has been showed that if $f(s)$ grows at infinity as s^4 , then a variational approach based on the reduction method (e.g. as in [4]) becomes more difficult because of a loss of compactness in the coupling term. In this case problem (1) recalls, at least formally, the more known Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^p + u^5 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \quad (2)$$

which has been studied and solved for $p \in [1, 5[$ by Brezis and Nirenberg in the very celebrated paper [5]. In that paper, by means of a deep investigation of the compactness property of minimizing sequences of a suitable constrained functional, it was showed that, if $p = 1$ and Ω is a ball, the Palais–Smale condition holds at the level of the infimum if and only if the

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parameter λ lies into an interval depending on the first eigenvalue of the operator $-\Delta$. In the same spirit of [5] and [2], in this paper we are interested in studying the following problem

$$\begin{cases} -\Delta u = \lambda u + q|u|^3 u \phi & \text{in } B_R, \\ -\Delta \phi = q|u|^5 & \text{in } B_R, \\ u = \phi = 0 & \text{on } \partial B_R, \end{cases} \tag{P}$$

where $\lambda \in \mathbb{R}$ and B_R is the ball in \mathbb{R}^3 centered in 0 with radius R .

As it is well known, problem (P) is equivalent to that of finding critical points of a functional depending only on the variable u and which includes a nonlocal nonlinear term. Many papers treated functionals presenting both a critically growing nonlinearity and a nonlocal nonlinearity (see [3,7,8,16]), but, up to our knowledge, it has been never considered the case when the term presenting a critical growth corresponds with the one containing the nonlocal nonlinearity.

From a technical point of view, the use of an approach similar to that of Brezis and Nirenberg requires different estimates with respect to those used in the above mentioned papers. Indeed, since it is just the nonlocal term of the functional the cause of the lack of compactness, it seems natural to compare it with the critical Lebesgue norm.

The main result we present is the following.

Theorem 1.1. *Set λ_1 the first eigenvalue of $-\Delta$ in B_R . If $\lambda \in]\frac{3}{10}\lambda_1, \lambda_1[$, then problem (P) has a positive ground state solution for any $q > 0$.*

The analogy with the problem (2) applies also to some nonexistence results. Indeed a classical Pohožaev obstruction holds for (P) according to the following result.

Theorem 1.2. *Problem (P) has no nontrivial solution if $\lambda \leq 0$.*

Actually, Theorem 1.2 holds also if the domain is a general smooth and star shaped open bounded set. Moreover, a standard argument allows us also to prove that there exists no solution to (P) if $\lambda \geq \lambda_1$ (see [5, Remark 1.1]).

It remains an open problem what happens if $\lambda \in]0, \frac{3}{10}\lambda_1]$.

The paper is so organized: Section 2 is devoted to prove the nonexistence result which does not require any variational argument; in Section 3 we introduce our variational approach and prove the existence of a positive ground state solution.

2. Nonexistence result

In this section, following [15], we adapt the Pohožaev arguments in [12] to our situation and we obtain the nonexistence result as a consequence of the following identities (for a similar problem see also [9]).

Let $\Omega \subset \mathbb{R}^3$ be a star shaped domain and $u, \phi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a nontrivial solution of (P). If we multiply the first equation of (P) by $x \cdot \nabla u$ and the second one by $x \cdot \nabla \phi$ we have that

$$\begin{aligned} 0 &= (\Delta u + \lambda u + q\phi|u|^3 u)(x \cdot \nabla u) \\ &= \operatorname{div}[(\nabla u)(x \cdot \nabla u)] - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2} \right) + \frac{\lambda}{2} x \cdot \nabla (u^2) + \frac{q}{5} x \cdot \nabla (\phi|u|^5) - \frac{q}{5} (x \cdot \nabla \phi)|u|^5 \\ &= \operatorname{div} \left[(\nabla u)(x \cdot \nabla u) - x \frac{|\nabla u|^2}{2} + \frac{\lambda}{2} x u^2 + \frac{q}{5} x \phi |u|^5 \right] + \frac{1}{2} |\nabla u|^2 - \frac{3}{2} \lambda u^2 - \frac{3}{5} q \phi |u|^5 - \frac{q}{5} (x \cdot \nabla \phi) |u|^5 \end{aligned}$$

and

$$\begin{aligned} 0 &= (\Delta \phi + q|u|^5)(x \cdot \nabla \phi) = \operatorname{div}[(\nabla \phi)(x \cdot \nabla \phi)] - |\nabla \phi|^2 - x \cdot \nabla \left(\frac{|\nabla \phi|^2}{2} \right) + q(x \cdot \nabla \phi)|u|^5 \\ &= \operatorname{div} \left[(\nabla \phi)(x \cdot \nabla \phi) - x \frac{|\nabla \phi|^2}{2} \right] + \frac{1}{2} |\nabla \phi|^2 + q(x \cdot \nabla \phi)|u|^5. \end{aligned}$$

Let \mathbf{n} be the unit exterior normal to $\partial\Omega$. Integrating on Ω , since by boundary conditions $\nabla u = \frac{\partial u}{\partial \mathbf{n}} \mathbf{n}$ and $\nabla \phi = \frac{\partial \phi}{\partial \mathbf{n}} \mathbf{n}$ on $\partial\Omega$, we obtain

$$-\frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \mathbf{n}} \right|^2 x \cdot \mathbf{n} = -\frac{3}{2} \lambda \|u\|_2^2 - \frac{3}{5} q \int_{\Omega} \phi |u|^5 - \frac{q}{5} \int_{\Omega} (x \cdot \nabla \phi) |u|^5 \tag{3}$$

and

$$-\frac{1}{2} \|\nabla\phi\|_2^2 - \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial\phi}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n} = q \int_{\Omega} (x \cdot \nabla\phi) |u|^5. \tag{4}$$

Substituting (4) into (3) we have

$$-\frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n} = -\frac{3}{2} \lambda \|u\|_2^2 - \frac{3}{5} q \int_{\Omega} \phi |u|^5 + \frac{1}{10} \|\nabla\phi\|_2^2 + \frac{1}{10} \int_{\partial\Omega} \left| \frac{\partial\phi}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n}. \tag{5}$$

Moreover, multiplying the first equation of (\mathcal{P}) by u and the second one by ϕ we get

$$\|\nabla u\|_2^2 = \lambda \|u\|_2^2 + q \int_{\Omega} \phi |u|^5 \tag{6}$$

and

$$\|\nabla\phi\|_2^2 = q \int_{\Omega} \phi |u|^5. \tag{7}$$

Hence, combining (5), (6) and (7), we have

$$-\lambda \|u\|_2^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n} + \frac{1}{10} \int_{\partial\Omega} \left| \frac{\partial\phi}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n} = 0.$$

Then, if $\lambda < 0$ we get a contradiction.

If $\lambda = 0$, then

$$\int_{\partial\Omega} \left| \frac{\partial\phi}{\partial\mathbf{n}} \right|^2 x \cdot \mathbf{n} = 0$$

and so by the second equation of (\mathcal{P}) we get $\|u\|_5 = 0$. Therefore $(u, \phi) = (0, 0)$ which is a contradiction.

3. Proof of Theorem 1.1

Problem (\mathcal{P}) is variational and the related C^1 functional $F : H_0^1(B_R) \times H_0^1(B_R) \rightarrow \mathbb{R}$ is given by

$$F(u, \phi) = \frac{1}{2} \int_{B_R} |\nabla u|^2 - \frac{\lambda}{2} \int_{B_R} u^2 - \frac{q}{5} \int_{B_R} |u|^5 \phi + \frac{1}{10} \int_{B_R} |\nabla\phi|^2.$$

The functional F is strongly indefinite. To avoid this indefiniteness, we apply the following reduction argument. First of all we give the following result.

Lemma 3.1. *For every $u \in H_0^1(B_R)$ there exists a unique $\phi_u \in H_0^1(B_R)$ solution of*

$$\begin{cases} -\Delta\phi = q|u|^5 & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R. \end{cases}$$

Moreover, for any $u \in H_0^1(B_R)$, $\phi_u \geq 0$ and the map

$$u \in H_0^1(B_R) \mapsto \phi_u \in H_0^1(B_R)$$

is continuously differentiable. Finally we have

$$\|\nabla\phi_u\|_2^2 = q \int_{B_R} |u|^5 \phi_u \tag{8}$$

and

$$\|\nabla\phi_u\|_2 \leq \frac{q}{5^3} \|\nabla u\|_2^5 \tag{9}$$

where

$$S = \inf_{v \in H_0^1(B_R) \setminus \{0\}} \frac{\|\nabla v\|_2^2}{\|v\|_6^2}.$$

Proof. To prove the first part we can proceed reasoning as in [4].

To show (9), we argue in the following way. By applying Hölder and Sobolev inequality to (8), we get

$$\|\nabla \phi_u\|_2^2 \leq q \|\phi_u\|_6 \|u\|_6^5 \leq \frac{q}{\sqrt{S}} \|\nabla \phi_u\|_2 \|u\|_6^5.$$

Then

$$\|\nabla \phi_u\|_2 \leq \frac{q}{\sqrt{S}} \|u\|_6^5 \leq \frac{q}{S^3} \|\nabla u\|_2^5. \quad \square$$

So, using Lemma 3.1, we can consider on $H_0^1(B_R)$ the C^1 one variable functional

$$I(u) := F(u, \phi_u) = \frac{1}{2} \int_{B_R} |\nabla u|^2 - \frac{\lambda}{2} \int_{B_R} u^2 - \frac{1}{10} \int_{B_R} |\nabla \phi_u|^2.$$

By standard variational arguments as those in [4], the following result can be easily proved.

Proposition 3.2. *Let $(u, \phi) \in H_0^1(B_R) \times H_0^1(B_R)$, then the following propositions are equivalent:*

- (a) (u, ϕ) is a critical point of functional F ;
- (b) u is a critical point of functional I and $\phi = \phi_u$.

To find solutions of (\mathcal{P}) , we look for critical points of I .

The functional I satisfies the geometrical assumptions of the Mountain Pass Theorem (see [1]).

So, we set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(B_R)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}$.

Now we proceed as follows:

Step 1. We prove that there exists a nontrivial solution to the problem (\mathcal{P}) .

Step 2. We show that such a solution is a ground state.

Remark 3.3. Observe that standard elliptic arguments based on the maximum principle work, so that we are allowed to assume that u and ϕ_u , solutions of (\mathcal{P}) , are both positive.

Proof of Step 1. (There exists a solution of (\mathcal{P}) .) Let $(u_n)_n$ be a Palais–Smale sequence at the mountain pass level c . It is easy to verify that $(u_n)_n$ is bounded so, up to a subsequence, we can suppose it is weakly convergent.

Suppose by contradiction that $u_n \rightarrow 0$ in $H_0^1(B_R)$. Then $u_n \rightarrow 0$ in $L^2(B_R)$.

Since $I(u_n) \rightarrow c$ and $\langle I'(u_n), u_n \rangle \rightarrow 0$ we have

$$\frac{1}{2} \|\nabla u_n\|_2^2 - \frac{1}{10} \|\nabla \phi_n\|^2 = c + o_n(1) \tag{10}$$

and

$$\|\nabla u_n\|_2^2 - \|\nabla \phi_n\|^2 = o_n(1) \tag{11}$$

where we have set $\phi_n = \phi_{u_n}$. Combining (10) and (11) we have

$$\|\nabla u_n\|_2^2 = \frac{5}{2}c + o_n(1)$$

and

$$\|\nabla \phi_n\|_2 = \frac{5}{2}c + o_n(1).$$

Then, since (u_n, ϕ_n) satisfies (9), passing to the limit we get

$$c \geq \frac{2}{5} \sqrt{\frac{S^3}{q}}. \tag{12}$$

Now consider a fixed smooth function $\varphi = \varphi(r)$ such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(R) = 0$. Following [5, Lemma 1.3], we set $r = |x|$ and

$$u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}.$$

The following estimates can be found in [5]

$$\|\nabla u_\varepsilon\|_2^2 = S \frac{K}{\varepsilon^{\frac{1}{2}}} + \omega \int_0^R |\varphi'(r)|^2 dr + O(\varepsilon^{\frac{1}{2}}), \quad \|u_\varepsilon\|_6^2 = \frac{K}{\varepsilon^{\frac{1}{2}}} + O(\varepsilon^{\frac{1}{2}}), \quad \|u_\varepsilon\|_2^2 = \omega \int_0^R \varphi^2(r) dr + O(\varepsilon^{\frac{1}{2}}),$$

where K is a positive constant and ω is the area of the unitary sphere in \mathbb{R}^3 .

We are going to give an estimate of the value c . Observe that, multiplying the second equation of (\mathcal{P}) by $|u|$ and integrating, we have that

$$q \|u\|_6^6 = \int_{B_R} (\nabla \phi_u |\nabla u|) \leq \frac{1}{2} \|\nabla \phi_u\|_2^2 + \frac{1}{2} \|\nabla |u|\|_2^2. \tag{13}$$

So, if we introduce the new functional $J : H^1(B_R) \rightarrow \mathbb{R}$ defined in the following way

$$J(u) := \frac{3}{5} \int_{B_R} |\nabla u|^2 - \frac{\lambda}{2} \int_{B_R} u^2 - \frac{q}{5} \int_{B_R} |u|^6,$$

by (13) we have that $I(u) \leq J(u)$, for any $u \in H_0^1(B_R)$, and $c \leq \inf_{u \in H_0^1(B_R) \setminus \{0\}} \sup_{t>0} J(tu)$.

Now we compute $\sup_{t>0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon)$, where t_ε is the unique positive solution of the equation

$$\frac{d}{dt} J(tu_\varepsilon) = 0.$$

Since

$$\frac{d}{dt} J(tu_\varepsilon) = \frac{6}{5} t \int_{B_R} |\nabla u_\varepsilon|^2 - \lambda t \int_{B_R} u_\varepsilon^2 - \frac{6}{5} t^5 q \int_{B_R} |u_\varepsilon|^6,$$

we have that

$$t_\varepsilon = \frac{1}{\|u_\varepsilon\|_6} \sqrt[4]{\frac{\frac{6}{5} \|\nabla u_\varepsilon\|_2^2 - \lambda \|u_\varepsilon\|_2^2}{\frac{6}{5} q \|u_\varepsilon\|_6^2}} = \frac{1}{\|u_\varepsilon\|_6} \sqrt[4]{\frac{S}{q} + A(\varphi) \varepsilon^{\frac{1}{2}} + O(\varepsilon)},$$

where we have set

$$A(\varphi) = \frac{\omega}{qK} \int_0^R \left(|\varphi'(r)|^2 - \frac{5}{6} \lambda \varphi^2(r) \right) dr.$$

Then

$$\sup_{t>0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon) = \frac{3}{5} t_\varepsilon^2 \int_{B_R} |\nabla u_\varepsilon|^2 - \frac{\lambda}{2} t_\varepsilon^2 \int_{B_R} u_\varepsilon^2 - \frac{q}{5} t_\varepsilon^6 \int_{B_R} |u_\varepsilon|^6 = \frac{2}{5} q \sqrt{\left(\frac{S}{q} + A(\varphi) \varepsilon^{\frac{1}{2}} + O(\varepsilon) \right)^3}. \tag{14}$$

Now, if we take $\varphi(r) = \cos(\frac{\pi r}{2R})$ as in [5], we have that

$$\int_0^R |\varphi'(r)|^2 dr = \frac{\pi^2}{4R^2} \int_0^R \varphi^2(r) dr$$

and then, if $\lambda \in]\frac{3}{10}\lambda_1, \lambda_1[$, we deduce that $A(\varphi) < 0$. Taking ε sufficiently small, from (14) we conclude that $c < \frac{2}{5}\sqrt{\frac{5^3}{q}}$, which contradicts (12).

Then we have that $u_n \rightharpoonup u$ with $u \in H_0^1(B_R) \setminus \{0\}$. We are going to prove that u is a weak solution of (\mathcal{P}) .

As in [2] it can be showed that $\phi_n \rightharpoonup \phi_u$ in $H_0^1(B_R)$. Now, set φ a test function. Since $I'(u_n) \rightarrow 0$, we have that

$$\langle I'(u_n), \varphi \rangle \rightarrow 0.$$

On the other hand,

$$\langle I'(u_n), \varphi \rangle = \int_{B_R} (\nabla u_n | \nabla \varphi) - \lambda \int_{B_R} u_n \varphi - q \int_{B_R} \phi_n |u_n|^3 u_n \varphi \rightarrow \int_{B_R} (\nabla u | \nabla \varphi) - \lambda \int_{B_R} u \varphi - q \int_{B_R} \phi_u |u|^3 u \varphi$$

so we conclude that (u, ϕ_u) is a weak solution of (\mathcal{P}) . \square

Proof of Step 2. (The solution found is a ground state.) As in Step 1, we consider a Palais–Smale sequence $(u_n)_n$ at level c . We have that $(u_n)_n$ weakly converges to a critical point u of I .

To prove that such a critical point is a ground state we proceed as follows.

First of all we prove that

$$I(u) \leq c.$$

Since $I(u_n) \rightarrow c$ and $\langle I'(u_n), u_n \rangle \rightarrow 0$, then

$$I(u_n) = \frac{2}{5} \int_{B_R} |\nabla u_n|^2 - \frac{2}{5} \lambda \int_{B_R} u_n^2 + o_n(1) \rightarrow c.$$

Moreover, being (u, ϕ_u) is a solution, we have

$$\int_{B_R} |\nabla u|^2 - \lambda \int_{B_R} u^2 - q \int_{B_R} \phi_u |u|^5 = 0.$$

Hence, by the lower semi-continuity of the H_0^1 -norm and since $u_n \rightarrow u$ in $L^2(B_R)$,

$$I(u) = \frac{2}{5} \left(\int_{B_R} |\nabla u|^2 - \lambda \int_{B_R} u^2 \right) \leq \frac{2}{5} \left(\liminf_n \int_{B_R} |\nabla u_n|^2 - \lambda \lim_n \int_{B_R} u_n^2 \right) = \frac{2}{5} \liminf_n \left(\int_{B_R} |\nabla u_n|^2 - \lambda \int_{B_R} u_n^2 \right) = c.$$

Finally, let v be a nontrivial critical point of I . Since the maximum of $I(tv)$ is achieved for $t = 1$, then

$$I(v) = \sup_{t>0} I(tv) \geq c \geq I(u). \quad \square$$

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