

Regularity properties of Schrödinger operators<sup>☆</sup>Tao Ma<sup>a</sup>, Pablo Raúl Stinga<sup>b</sup>, José L. Torrea<sup>c,\*</sup>, Chao Zhang<sup>a,1</sup><sup>a</sup> School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, China<sup>b</sup> Departamento de Matemáticas y Computación, Universidad de La Rioja, 26006 Logroño, Spain<sup>c</sup> Departamento de Matemáticas, Universidad Autónoma de Madrid and ICMAT-CSIC-UAM-UCM-UC3M, 28049 Madrid, Spain

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## ABSTRACT

Let  $\mathcal{L}$  be a Schrödinger operator of the form  $\mathcal{L} = -\Delta + V$ , where the nonnegative potential  $V$  satisfies a reverse Hölder inequality. Using the method of  $\mathcal{L}$ -harmonic extensions we study regularity estimates at the scale of adapted Hölder spaces. We give a pointwise description of  $\mathcal{L}$ -Hölder spaces and provide some characterizations in terms of the growth of fractional derivatives of any order and Carleson measures. Applications to fractional powers of  $\mathcal{L}$  and multipliers of Laplace transform type developed.

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## 1. Introduction

One of the methods applied to develop regularity estimates in the theory of partial differential equations is to consider equivalent formulations of the problems by adding a new variable. Let us give a rough description of the idea. Suppose that we want to study regularity properties of a certain function  $f(x)$  defined in some domain  $\Omega$ . Take  $f$  as the Dirichlet or initial data for some PDE  $Au = 0$  in the variables  $x \in \Omega$  and  $t$  in an interval  $I$ . The question is the following: which properties of the solution  $u$  in  $\Omega \times I$  imply regularity of  $f$ , the boundary data? The most simple and classical situation to consider is the following:

$$\begin{cases} Au \equiv \partial_{tt}u + \Delta u = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x), & \text{on } \mathbb{R}^n. \end{cases} \quad (1.1)$$

Here  $\Delta$  is the Laplacian in  $\mathbb{R}^n$ . Then  $u$  is the harmonic extension of  $f$ , namely

$$u(x, t) = e^{-t(-\Delta)^{1/2}} f(x). \quad (1.2)$$

Note that we have  $-u_t(x, 0) = (-\Delta)^{1/2} f(x)$ . Therefore, the harmonic extension  $u$  can give some information not only about  $f$  but also about the fractional Laplacian, a nonlocal operator, acting on  $f$ . It is worth to mention here that such

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a remarkable fact was applied to show that weak solutions of the critical dissipative quasi-geostrophic equation are Hölder continuous, see [3].

In general, to study the regularity properties of fractional operators like  $(-\Delta)^{1/2}$ , or more generally  $(-\Delta)^{\sigma/2}$  and  $(-\Delta)^{-\sigma/2}$ ,  $0 < \sigma < 2$ , there are essentially two possible alternatives. Either describe the operators with a pointwise integro-differential or integral formula, or characterize the Hölder classes by some norm estimate of harmonic extensions (1.1), that are in fact Poisson integrals (1.2). The first approach was taken by L. Silvestre in [12] to analyze how  $(-\Delta)^{\pm\sigma/2}$  acts on the Hölder spaces  $C^{0,\alpha}$ . Let us point out that he also needed to handle the Riesz transforms  $\partial_{x_i}(-\Delta)^{-1/2}$  as operators on  $C^{0,\alpha}$ . The second one, in the spirit of harmonic extensions, is nowadays classical. Indeed, for bounded functions  $f$  it is well known that the harmonic extension (1.2) satisfies  $\|tu_t(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^\alpha$  for all  $t > 0$  if, and only if,  $f \in C^{0,\alpha}$ ,  $0 < \alpha < 1$ , see for instance [13].

In this paper we consider the time independent Schrödinger operator in  $\mathbb{R}^n$ ,  $n \geq 3$ ,

$$\mathcal{L} := -\Delta + V, \quad (1.3)$$

where the nonnegative potential  $V$  satisfies a reverse Hölder inequality for some  $q > n/2$ , see (3.1) below. Observe that the reverse Hölder condition is just an integrability property, so no smoothness on  $V$  is assumed. Our aim is to develop the regularity theory of Hölder spaces adapted to  $\mathcal{L}$  and to study estimates of operators like *fractional integrals*  $\mathcal{L}^{-\sigma/2}$ , and *fractional powers*  $\mathcal{L}^{\sigma/2}$ . Such operators can be defined by using  $\mathcal{L}$ -harmonic extensions. The solution of the boundary value problem

$$\begin{cases} \partial_{tt}u - \mathcal{L}u = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x), & \text{on } \mathbb{R}^n, \end{cases} \quad (1.4)$$

is given by the action of the  $\mathcal{L}$ -Poisson semigroup on  $f$ :

$$u(x, t) = \mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x).$$

Let us recall that Bochner's subordination formula gives a way to express  $u$  as a mean in the time variable of the solution of the  $\mathcal{L}$ -diffusion equation, see (3.9). The powers of  $\mathcal{L}$  can be described in terms of  $u$  as in (2.1) and (2.2). Therefore, to deal with spaces and operators, we will adopt the point of view based on  $\mathcal{L}$ -harmonic extensions (1.4).

Our choice of the method turns out to be well suited for our purposes. In this Schrödinger context the pointwise description of the operators as in [12] seems to be technically difficult. In fact, even for one of the most simplest cases (the harmonic oscillator, where  $V(x) = |x|^2$ ) it is already rather involved, see [15]. On the other hand, the characterization of  $\mathcal{L}$ -Hölder spaces via  $\mathcal{L}$ -harmonic extensions does not appear to be easily obtained as a repetition of the arguments for classical Hölder spaces given in [13].

Let us begin with the definition of Hölder spaces naturally associated to  $\mathcal{L}$ . The concept is based on the *critical radii function*  $\rho(x)$  defined by Z. Shen in [11], see (3.2).

**Definition 1.1** (Hölder spaces for  $\mathcal{L}$ ). A continuous function  $f$  defined on  $\mathbb{R}^n$  belongs to the space  $C_{\mathcal{L}}^{0,\alpha}$ ,  $0 < \alpha \leq 1$ , if the quantities

$$[f]_{C^\alpha} = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \quad \text{and} \quad [f]_{M_{\mathcal{L}}^\alpha} = \sup_{x \in \mathbb{R}^n} |\rho(x)^{-\alpha} f(x)|,$$

are finite. The norm in the spaces  $C_{\mathcal{L}}^{0,\alpha}$  is  $\|f\|_{C_{\mathcal{L}}^{0,\alpha}} = [f]_{C^\alpha} + [f]_{M_{\mathcal{L}}^\alpha}$ .

The first main theorem of the paper is the following regularity result.

**Theorem 1.2.** Assume that  $q > n$ . Let  $\sigma$  be a positive number,  $0 < \alpha < 1$  and  $f \in C_{\mathcal{L}}^{0,\alpha}$ .

- (a) If  $0 < \alpha + \sigma < 1$  then  $\mathcal{L}^{-\sigma/2} f \in C_{\mathcal{L}}^{0,\alpha+\sigma}$  and  $\|\mathcal{L}^{-\sigma/2} f\|_{C_{\mathcal{L}}^{0,\alpha+\sigma}} \leq C\|f\|_{C_{\mathcal{L}}^{0,\alpha}}$ .
- (b) If  $\sigma < \alpha$  then  $\mathcal{L}^{\sigma/2} f \in C_{\mathcal{L}}^{0,\alpha-\sigma}$  and  $\|\mathcal{L}^{\sigma/2} f\|_{C_{\mathcal{L}}^{0,\alpha-\sigma}} \leq C\|f\|_{C_{\mathcal{L}}^{0,\alpha}}$ .
- (c) Let  $a$  be a bounded function on  $[0, \infty)$  and define

$$m(\lambda) = \lambda^{1/2} \int_0^\infty e^{-s\lambda^{1/2}} a(s) ds, \quad \lambda > 0.$$

Then the multiplier operator of Laplace transform type  $m(\mathcal{L})$  is bounded on  $C_{\mathcal{L}}^{0,\alpha}$ ,  $0 < \alpha < 1$ .

In order to prove Theorem 1.2 we shall need a characterization of functions  $f$  in  $C_{\mathcal{L}}^{0,\alpha}$  by means of size and integrability properties of  $\mathcal{L}$ -harmonic extensions (1.4) to the upper half space. The theory of  $BMO_{\mathcal{L}}$  spaces and Carleson measures developed in [7] will be a central tool. In fact our result provides a characterization of the  $\mathcal{L}$ -Hölder classes via Carleson measures. Moreover, our statement not only involves first order derivatives of the  $\mathcal{L}$ -Poisson semigroup but also introduces higher and fractional order derivatives. The concept of fractional derivative that we give here is of independent interest and allows us to present a more general characterization. Given a positive number  $\beta$ , let us denote by  $m$  the smallest integer which strictly exceeds  $\beta$ , that is,  $[\beta] + 1$ . Let  $F(x, t)$  be a reasonable nice function of  $x \in \mathbb{R}^n$  and  $t > 0$ . We define, following C. Segovia and R.L. Wheeden [10],

$$\partial_t^\beta F(x, t) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(x, t+r) r^{m-\beta} \frac{dr}{r}, \quad x \in \mathbb{R}^n, \quad t > 0. \quad (1.5)$$

Note that in the definition above  $\partial_t^1 = \partial_t$ . The following is the second main result.

**Theorem 1.3.** *Let  $0 < \alpha < 1$  and  $f$  be a function such that  $f(x)(1 + |x|)^{-(n+\alpha+\varepsilon)}$  is integrable for any  $\varepsilon > 0$ . Fix any  $\beta > \alpha$  and assume that  $q > n$ . The following statements are equivalent:*

- (i)  $f \in C_{\mathcal{L}}^{0,\alpha}$ .
- (ii) *There exists a constant  $c_{1,\beta}$  such that  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq c_{1,\beta} t^\alpha$ .*
- (iii) *There exists a constant  $c_{2,\beta}$  such that for all balls  $B = B(x_0, r)$  in  $\mathbb{R}^n$ ,*

$$\left( \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \frac{dx dt}{t} \right)^{1/2} \leq c_{2,\beta} |B|^{\frac{\alpha}{n}},$$

where  $\widehat{B}$  denotes the tent over  $B$  defined by  $\{(x, t): x \in B, \text{ and } 0 < t \leq r\}$ .

Moreover, the constants  $c_{1,\beta}$ ,  $c_{2,\beta}$  and  $\|f\|_{C_{\mathcal{L}}^{0,\alpha}}$  above are comparable.

Some observations are in order. The integrability condition required on  $f$  in Theorem 1.3 implies that the  $\mathcal{L}$ -harmonic extension  $\mathcal{P}_t f$  is well defined, see Proposition 3.6(a) below. Such a condition is weaker than to ask for  $f$  to be bounded (as in the classical case, see [13]) or even to have the growth  $|f(x)| \leq C\rho(x)^\alpha$  that appears in the definition of  $\mathcal{L}$ -Hölder space above, see Lemma 2.1(i). The Carleson property (iii) can be proved since there is an available Campanato-type description of  $C_{\mathcal{L}}^{0,\alpha}$ . This identification was proved by Bongioanni, Harboure and Salinas in [1], see Proposition 4.6.

Under the light of Definition 1.1 and Theorem 1.3, the natural question is how to define and characterize higher-order  $\mathcal{L}$ -Hölder spaces, that is, spaces of the type  $C_{\mathcal{L}}^{k,\alpha}$  for  $k$  a positive integer. It is already known the characterization of classical  $C^{k,\alpha}$  spaces by size properties of harmonic extensions, see [13]. In the case of the harmonic oscillator  $H = -\Delta + |x|^2$ , the definition of the Hölder spaces  $C_H^{k,\alpha}$  was given in [15]. In the case of general potentials  $V$ , because of the lack of smoothness we will not try to consider higher-order  $\mathcal{L}$ -Hölder spaces. Nevertheless, as it happens in the classical case [13], we could define higher-order spaces by using property (ii) of Theorem 1.3 in the following way. Let  $\alpha > 0$  and fix any  $\beta > \alpha$ . Then we would say that a function  $f$  belongs to the  $\mathcal{L}$ -Hölder space  $\Lambda_{\mathcal{L}}^\alpha$  if  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ . Note that this new concept depends on the choice of  $\beta$ , but in fact we can show that it does not, see Lemma 5.6 below. If  $0 < \alpha < 1$  then the definition agrees with Definition 1.1. But when  $\alpha > 1$  and  $V$  is not smooth it is not clear how to give an equivalent pointwise formulation to measure the smoothness of  $f$  as in the classical way. For the potential  $V = |x|^2$  some results in this direction can be obtained and they will appear in a forthcoming work.

The condition  $q > n$  in Theorem 1.3 seems to be natural if we expect to have some regularity for the operators involved. See Z. Shen [11] for a discussion in  $L^p$  and [2] in the  $BMO_{\mathcal{L}}^\alpha$  context.

We also consider the extreme values of  $\alpha$ . Note that the conclusion of Theorem 1.3 above is not valid in the cases  $\alpha = 1$  or  $\alpha = 0$ . In fact, we have the following results for  $\alpha = 1$ :

**Theorem 1.4** (Case  $\alpha = 1$ ). *Assume that  $q > n$ .*

- (I) *If  $f \in C_{\mathcal{L}}^{0,1}$  then for any  $\beta > 1$  there exists a constant  $c_\beta$  such that*

$$\left( \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \frac{dx dt}{t} \right)^{1/2} \leq c_\beta |B|^{\frac{1}{n}},$$

for all balls  $B$ . The converse statement is not true.

(II) Let  $\mathcal{L}_\mu = -\Delta + \mu$ , for  $\mu > 0$ . There exists a function  $f$  such that for any  $\beta > 1$  there exists a constant  $c_\beta$  that verifies  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq c_\beta t$ , for all  $t > 0$ , but  $f \notin C_{\mathcal{L}_\mu}^{0,1}$ .

It has no sense to take  $\alpha = 0$  as a Hölder exponent. By the Campanato-type description of Proposition 4.6 we see that the natural replacement in this situation is the space  $BMO_{\mathcal{L}}$ .

**Theorem 1.5** (Case  $\alpha = 0$ ). Assume that  $q > n$ .

(A) A function  $f$  is in  $BMO_{\mathcal{L}}$  if and only if for  $f$  being a function such that  $f(x)(1 + |x|)^{-(n+\varepsilon)}$  is integrable for any  $\varepsilon > 0$ , and for all  $\beta > 0$  there exists a constant  $c_\beta$  such that, for all balls  $B$ ,

$$\left( \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \frac{dx dt}{t} \right)^{1/2} \leq c_\beta.$$

(B) Let  $\mathcal{L}_\mu = -\Delta + \mu$ , for  $\mu > 0$ . There exists a function  $f \in BMO_{\mathcal{L}_\mu}$  such that, for some  $\beta > 0$ ,  $\sup_{t>0} |t^\beta \partial_t^\beta \mathcal{P}_t f(0)| = \infty$ .

We should notice that the proof of Theorem 1.2 is relatively simple and it can be presented rather quickly. This is in a big contrast with the proof given in [15] for the case of the harmonic oscillator  $H = -\Delta + |x|^2$ . In [15] pointwise formulas of  $H^{\pm\sigma}$  and Hermite–Riesz transforms must be handled. In our proof of Theorem 1.2(a) and (b) no Riesz transforms are needed. On the other hand, the results in [15] involve higher order spaces  $C_H^{k,\alpha}$ . As we pointed out before, if we would like to have higher order spaces then we should consider the spaces of the type  $\Lambda_{\mathcal{L}}^\alpha$  mentioned above. With such a description it is very simple to extend the results of Theorem 1.2 to hold for all  $\alpha, \sigma > 0$  (with the appropriate relations between them). But in this way still there is no pointwise smoothness condition on the functions  $f \in \Lambda_{\mathcal{L}}^\alpha$ , which are necessary in PDEs.

The organization of the paper is as follows. In Section 2, in order to convince the reader how useful our method is, we present the proof of Theorem 1.2. In fact for those who are just interested in regularity properties of operators, this is the most important section. In Section 3 we list a collection of estimates about Schrödinger kernels that we will need later. Some of them are known and we put them there to make the paper more readable, but there are some new (although expectable) estimates, like those of Proposition 3.6. Section 4 is a technical section about  $BMO_{\mathcal{L}}^\alpha$  spaces and Section 5 contains the proofs of Theorems 1.3, 1.4 and 1.5.

Throughout this paper, the letters  $c$  and  $C$  denote positive constants that may change in each occurrence and they will depend on the parameters involved (whenever it is necessary, we point out this dependence with subscripts). The Gamma and Beta functions will be denoted by  $\Gamma$  and  $B$ , respectively. Without mentioning it, we will repeatedly apply the inequality  $r^\eta e^{-r} \leq C_\eta e^{-r/2}$ ,  $\eta \geq 0$ ,  $r > 0$ .

## 2. Regularity of operators related to $\mathcal{L}$

In this section we prove Theorem 1.2. First we need the following technical lemma.

**Lemma 2.1.** Let  $0 < \gamma < 1$ , and  $g$  be a continuous function such that  $|g(x)| \leq C\rho(x)^\gamma$ , where  $\rho$  is the critical radii function defined in (3.2). Then

- (i) For any  $\varepsilon > 0$ , the function  $g(x)(1 + |x|)^{-(n+\gamma+\varepsilon)}$  is integrable.
- (ii) For any  $\beta > \gamma$  and any  $N > 0$  there exists a constant  $C_{\beta,N,g}$  such that

$$|s^\beta \partial_s^\beta \mathcal{P}_s g(x)| \leq C_{\beta,N,g} (\rho(x)/s)^N (\rho(x)^\gamma + s^\gamma), \quad x \in \mathbb{R}^n, s > 0.$$

- (iii) For any  $N > 0$  there exists a constant  $C_{N,g}$  such that

$$|\mathcal{P}_s g(x)| \leq C_{N,g} (\rho(x)/s)^N (\rho(x)^\gamma + s^\gamma), \quad x \in \mathbb{R}^n, s > 0.$$

**Proof.** Let us begin with (i). We have to check that the integrals

$$I = \int_{|x| < 2\rho(0)} \frac{|g(x)|}{(1 + |x|)^{n+\gamma+\varepsilon}} dx + \sum_{j=1}^{\infty} \int_{2^j \rho(0) \leq |x| < 2^{j+1} \rho(0)} \frac{|g(x)|}{(1 + |x|)^{n+\gamma+\varepsilon}} dx,$$

are finite. To that end we apply the hypothesis and some properties of the function  $\rho$  contained in Lemma 3.1 below. The inequality  $|x| = |x - 0| < 2^{j+1} \rho(0)$ ,  $j \geq 0$ , and the right inequality of (3.3) give us  $\rho(x) \leq C\rho(0)2^j$ . Therefore,

$$I \leq C \rho(0)^{\gamma+n} + C \sum_{j=1}^{\infty} \frac{(\rho(0)2^j)^{\gamma+n}}{(1+2^j \rho(0))^{n+\gamma+\varepsilon}} \leq C + C \sum_{j=1}^{\infty} 2^{-j\varepsilon} < \infty.$$

We will only prove (ii). The third statement (iii) can be proved in the same way. By (i),  $\mathcal{P}_t g(x)$  is well defined. By Proposition 3.6(b) and Lemma 3.1 below, for some constant  $C = C_{\beta, N, g}$ , we have

$$\begin{aligned} |s^\beta \partial_s^\beta \mathcal{P}_s g(x)| &\leq C \int_{\mathbb{R}^n} \frac{s^\beta \rho(x)^N}{(s + |x - y|)^{n+\beta+N}} \rho(x)^\gamma \left(1 + \frac{|x - y|}{\rho(x)}\right)^\gamma dy \\ &\leq C \rho(x)^{\gamma+N} \int_{\mathbb{R}^n} \frac{s^\beta}{(s + |x - y|)^{n+\beta+N}} dy + C \rho(x)^N \int_{\mathbb{R}^n} \frac{s^\beta}{(s + |x - y|)^{n+\beta+N-\gamma}} dy \\ &= C \rho(x)^{\gamma+N} s^{-N} + C \rho(x)^N s^{-N+\gamma}. \quad \square \end{aligned}$$

**Proof of Theorem 1.2.** We start with the proof of part (a). For  $f \in C_{\mathcal{L}}^{0, \alpha}$ , we have

$$\mathcal{L}^{-\sigma/2} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \mathcal{P}_s f(x) \frac{ds}{s^{1-\sigma}}, \quad x \in \mathbb{R}^n. \quad (2.1)$$

By Lemma 2.1(ii), since  $|f(x)| \leq C \rho(x)^\alpha$ , we get

$$\begin{aligned} \int_0^\infty |\mathcal{P}_s f(x)| \frac{ds}{s^{1-\sigma}} &\leq C \int_0^{\rho(x)} \left[ \frac{\rho(x)^{\alpha+N_1}}{s^{N_1}} + \frac{\rho(x)^{N_1}}{s^{N_1-\alpha}} \right] \frac{ds}{s^{1-\sigma}} + C \int_{\rho(x)}^\infty \left[ \frac{\rho(x)^{\alpha+N_2}}{s^{N_2}} + \frac{\rho(x)^{N_2}}{s^{N_2-\alpha}} \right] \frac{ds}{s^{1-\sigma}} \\ &\leq C_{N_1, N_2, \alpha, f} \cdot \rho(x)^{\alpha+\sigma}, \end{aligned}$$

by choosing  $0 < N_1 < \sigma$  and  $N_2 > \alpha + \sigma$ . Hence  $\mathcal{L}^{-\sigma/2} f(x)$  is well defined. Moreover, it satisfies the required growth  $|\mathcal{L}^{-\sigma/2} f(x)| \leq C \rho(x)^{\alpha+\sigma}$ . So Lemma 2.1 applies to it. Fix any  $\beta > \alpha + \sigma$ . To obtain the conclusion we apply Theorem 1.3. That is, it is enough to prove that  $\|t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{C_{\mathcal{L}}^{0, \alpha}} t^{\alpha+\sigma}$ . By using formula (2.1) and Lemma 2.1 together with Fubini's theorem, we have

$$t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x) = C t^\beta \int_0^\infty \partial_t^\beta \mathcal{P}_t(\mathcal{P}_s f)(x) \frac{ds}{s^{1-\sigma}} = C t^\beta \int_0^\infty \partial_w^\beta \mathcal{P}_w f(x) \Big|_{w=t+s} \frac{ds}{s^{1-\sigma}}.$$

Since  $\beta > \alpha + \sigma$  we can use Theorem 1.3 to get (a):

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x)| &\leq C \|f\|_{C_{\mathcal{L}}^{0, \alpha}} t^\beta \int_0^\infty (t+s)^{\alpha-\beta} \frac{ds}{s^{1-\sigma}} = C \|f\|_{C_{\mathcal{L}}^{0, \alpha}} t^{\alpha+\sigma} \int_0^\infty (1+r)^{\alpha-\beta} \frac{dr}{r^{1-\sigma}} \\ &= C B(\sigma, \beta - \alpha - \sigma) \|f\|_{C_{\mathcal{L}}^{0, \alpha}} t^{\alpha+\sigma}, \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

To prove part (b), fix any  $\beta > \alpha$ . Since  $0 < \sigma < \alpha < 1$  we can write

$$\mathcal{L}^{\sigma/2} f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (\mathcal{P}_s f(x) - f(x)) \frac{ds}{s^{1+\sigma}} = I(x, t) + II(x, t), \quad (2.2)$$

where  $I(x, t)$  is the part of the integral from 0 to  $t$ . Since  $f \in C_{\mathcal{L}}^{0, \alpha}$ ,

$$|I(x, \rho(x))| \leq \int_0^{\rho(x)} \left| \int_0^s \partial_r \mathcal{P}_r f(x) dr \right| \frac{ds}{s^{1+\sigma}} \leq C \int_0^{\rho(x)} \int_0^s r^{\alpha-1} dr \frac{ds}{s^{1+\sigma}} = C \rho(x)^{\alpha-\sigma}.$$

Taking  $N = \alpha$  in Lemma 2.1(iii) and using the growth of  $f$  we also have

$$|II(x, \rho(x))| \leq \int_{\rho(x)}^\infty (|\mathcal{P}_s f(x)| + |f(x)|) \frac{ds}{s^{1+\sigma}} \leq C \int_{\rho(x)}^\infty \left[ \frac{\rho(x)^{2\alpha}}{s^\alpha} + \rho(x)^\alpha \right] \frac{ds}{s^{1+\sigma}} = C \rho(x)^{\alpha-\sigma}.$$

The computations above say that (2.2) is well defined and that Theorem 1.3 can be applied to it. By linearity, it is enough to analyze  $t^\beta \partial_t^\beta \mathcal{P}_t I(x, t)$  and  $t^\beta \partial_t^\beta \mathcal{P}_t II(x, t)$  separately. Note that

$$t^\beta \partial_t^\beta \mathcal{P}_t I(x, t) = \frac{t^\beta}{\Gamma(-\sigma)} \int_0^t \int_0^s \partial_w^{\beta+1} \mathcal{P}_w f(x) \Big|_{w=t+r} dr \frac{ds}{s^{1+\sigma}}.$$

Apply Theorem 1.3 and the fact that  $\beta > \alpha$  to obtain

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t I(x, t)| &\leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\beta \int_0^t \int_0^s (t+r)^{\alpha-\beta-1} dr \frac{ds}{s^{1+\sigma}} \\ &= C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha \int_0^t \int_0^{s/t} (1+u)^{\alpha-\beta-1} du \frac{ds}{s^{1+\sigma}} \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha \int_0^t \frac{s}{t} \frac{ds}{s^{1+\sigma}} = C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^{\alpha-\sigma}. \end{aligned} \quad (2.3)$$

Theorem 1.3 and Fubini's theorem give us

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t II(x, t)| &\leq C \int_t^\infty (|t^\beta \partial_w^\beta \mathcal{P}_w f(x)|_{w=t+s} + |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|) \frac{ds}{s^{1+\sigma}} \\ &\leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} \int_t^\infty t^\beta (t+s)^{\alpha-\beta} + t^\alpha \frac{ds}{s^{1+\sigma}} = C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^{\alpha-\sigma}. \end{aligned} \quad (2.4)$$

Collecting estimates (2.3) and (2.4) we get the conclusion of (b).

Let us finally check (c). Fix any  $\beta > \alpha$ . Note that we have  $m(\mathcal{L})f(x) = -\int_0^\infty \partial_s \mathcal{P}_s f(x) a(s) ds$ . As  $a$  is a bounded function and  $f \in C_{\mathcal{L}}^{0,\alpha}$ ,

$$\int_0^{\rho(x)} |\partial_s \mathcal{P}_s f(x) a(s)| ds \leq C \int_0^{\rho(x)} s^{\alpha-1} ds = C \rho(x)^\alpha.$$

Moreover, by Lemma 2.1(ii) with  $\beta = 1$  and some  $N > \alpha$  at there, we obtain

$$\int_{\rho(x)}^\infty |\partial_s \mathcal{P}_s f(x) a(s)| ds \leq C \int_{\rho(x)}^\infty \left(\frac{\rho(x)}{s}\right)^N (\rho(x)^\alpha + s^\alpha) \frac{ds}{s} = C \rho(x)^\alpha.$$

Therefore,  $|m(\mathcal{L})f(x)| \leq C \rho(x)^\alpha$ , so by Lemma 2.1(i) the hypothesis of Theorem 1.3 holds for  $m(\mathcal{L})f$ . By Theorem 1.3 and Fubini's theorem we have

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t (m(\mathcal{L})f)(x)| &= t^\beta \left| \int_0^\infty \partial_w^{\beta+1} \mathcal{P}_w f(x) \Big|_{w=t+s} a(s) ds \right| \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\beta \int_0^\infty (t+s)^{\alpha-(\beta+1)} ds \\ &= C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha \int_0^\infty (1+r)^{\alpha-(\beta+1)} dr = C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha. \quad \square \end{aligned}$$

### 3. Estimates on the kernels

The nonnegative potential  $V$  in (1.3) satisfies a reverse Hölder inequality for some  $q > n/2$ :

$$\left( \frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy, \quad (3.1)$$

for all balls  $B \subset \mathbb{R}^n$ . Associated to this potential, Z. Shen defines the critical radii function in [11] as

$$\rho(x) := \sup \left\{ r > 0: \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n. \quad (3.2)$$

**Lemma 3.1.** (See [11, Lemma 1.4].) There exist  $c > 0$  and  $k_0 \geq 1$  such that for all  $x, y \in \mathbb{R}^n$

$$c^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq c \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \quad (3.3)$$

Let  $\{\mathcal{T}_t\}_{t>0}$  be the heat-diffusion semigroup associated to  $\mathcal{L}$ :

$$\mathcal{T}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} k_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t > 0. \quad (3.4)$$

**Lemma 3.2.** (See [5,9].) For every  $N > 0$  there exists a constant  $C_N$  such that

$$0 \leq k_t(x, y) \leq C_N t^{-n/2} e^{-\frac{|x-y|^2}{5t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad x, y \in \mathbb{R}^n, \quad t > 0. \quad (3.5)$$

The kernel of the classical heat semigroup  $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$  on  $\mathbb{R}^n$  is

$$h_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0. \quad (3.6)$$

**Lemma 3.3.** (See [5, Proposition 2.16].) There exists a nonnegative function  $\omega \in \mathcal{S}$ , where  $\mathcal{S}$  denotes the Schwartz's class of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^n$ , such that

$$|k_t(x, y) - h_t(x - y)| \leq \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta \omega_t(x - y), \quad x, y \in \mathbb{R}^n, \quad t > 0, \quad (3.7)$$

where  $\omega_t(x - y) := t^{-n/2} \omega((x - y)/\sqrt{t})$  and  $\delta := 2 - \frac{n}{q} > 0$ .

We define the following kernel that will be useful in the sequel. Let

$$Q_t(x, y) := t^2 \frac{\partial k_s(x, y)}{\partial s} \Big|_{s=t^2}, \quad x, y \in \mathbb{R}^n, \quad t > 0. \quad (3.8)$$

**Lemma 3.4.** (See [7, Proposition 4].) Let  $\delta$  be as in Lemma 3.3. There exists a constant  $c$  such that for every  $N$  there is a constant  $C_N$  such that

$$\begin{aligned} \text{(a)} \quad & |Q_t(x, y)| \leq C_N t^{-n} e^{-c \frac{|x-y|^2}{t^2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}; \\ \text{(b)} \quad & |Q_t(x+h, y) - Q_t(x, y)| \leq C_N \left(\frac{|h|}{t}\right)^\delta t^{-n} e^{-c \frac{|x-y|^2}{t^2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}, \quad \text{for all } |h| \leq t; \\ \text{(c)} \quad & \left| \int_{\mathbb{R}^n} Q_t(x, y) dy \right| \leq C_N \frac{(t/\rho(x))^\delta}{(1+t/\rho(x))^N}. \end{aligned}$$

**Remark 3.5.** Let  $0 < \delta' \leq \delta$ . Then we can easily deduce from Lemma 3.4(c) that for any  $N > 0$  there exists a constant  $C_N$  such that  $|\int_{\mathbb{R}^n} Q_t(x, y) dy| \leq C_N \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}$ .

Using the heat semigroup (3.4) and through Bochner's subordination formula, see [14], we have:

$$\mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{T}_{t^2/(4u)} f(x) du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} \mathcal{T}_u f(x) du, \quad (3.9)$$

for any  $x \in \mathbb{R}^n$ ,  $t > 0$ . It follows that the  $\mathcal{L}$ -Poisson kernel is given by

$$\mathcal{P}_t(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} k_{t^2/(4u)}(x, y) du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} k_u(x, y) du. \quad (3.10)$$

We will denote the classical Poisson semigroup in  $\mathbb{R}_+^{n+1}$  by  $P_t f(x) = P_t * f(x)$ , where

$$P_t(x) = c_n \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}. \quad (3.11)$$

Let us now compute the fractional derivatives (1.5) of the Poisson kernel. The formula will involve the kernel  $Q_t(x, y)$  of (3.8) and the Hermite polynomials  $H_m(r)$  defined, for  $m \in \mathbb{N}_0$  and  $r \in \mathbb{R}$ , as  $H_m(r) = (-1)^m e^{r^2} \frac{d^m}{dr^m} (e^{-r^2})$ . From the first identity in (3.10) and the definition of  $Q_t$  in (3.8), we have

$$\partial_t \mathcal{P}_t(x, y) = \frac{2}{t\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} Q_{t/(2\sqrt{u})}(x, y) du = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2/(4v^2)} Q_v(x, y) \frac{dv}{v^2}.$$

Hence, for each  $m \geq 1$ , we obtain

$$\partial_t^m \mathcal{P}_t(x, y) = \frac{2(-1)^m}{\sqrt{\pi}} \int_0^\infty H_{m-1}\left(\frac{t}{2v}\right) e^{-\frac{t^2}{4v^2}} \frac{1}{(2v)^{m-1}} Q_v(x, y) \frac{dv}{v^2}.$$

With this we can write the derivatives  $\partial_t^\beta \mathcal{P}_t(x, y)$ ,  $\beta > 0$ , as follows. For  $m = [\beta] + 1$ ,

$$\begin{aligned} \partial_t^\beta \mathcal{P}_t(x, y) &= \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(x, y) s^{m-\beta} \frac{ds}{s} \\ &= \frac{2(-1)^m e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)\sqrt{\pi}} \int_0^\infty \int_0^\infty H_{m-1}\left(\frac{t+s}{2v}\right) e^{-\frac{(t+s)^2}{4v^2}} \frac{1}{(2v)^{m-1}} Q_v(x, y) \frac{dv}{v^2} s^{m-\beta} \frac{ds}{s} \\ &= \frac{2(-1)^m e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)\sqrt{\pi}} \int_0^\infty \left[ \int_0^\infty H_{m-1}\left(\frac{t+s}{2v}\right) e^{-\frac{(t+s)^2}{4v^2}} s^{m-\beta} \frac{ds}{s} \right] \frac{1}{(2v)^{m-1}} Q_v(x, y) \frac{dv}{v^2}. \end{aligned} \quad (3.12)$$

**Proposition 3.6.** Let  $\beta > 0$ . For any  $0 < \delta' \leq \delta$  with  $0 < \delta' < \beta$ , and  $N > 0$  there exists a constant  $C = C_{N, \beta, \delta'}$  such that

- (a)  $|\mathcal{P}_t(x, y)| \leq C \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \left( 1 + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(y)} \right)^{-N};$
- (b)  $|t^\beta \partial_t^\beta \mathcal{P}_t(x, y)| \leq C \frac{t^\beta}{(|x-y|^2 + t^2)^{\frac{n+\beta}{2}}} \left( 1 + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(y)} \right)^{-N};$
- (c) For all  $|h| \leq t$ ,
 
$$|t^\beta \partial_t^\beta \mathcal{P}_t(x+h, y) - t^\beta \partial_t^\beta \mathcal{P}_t(x, y)| \leq C \left( \frac{|h|}{t} \right)^{\delta'} \frac{t^\beta}{(|x-y|^2 + t^2)^{\frac{n+\beta}{2}}} \left( 1 + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2 + t^2)^{1/2}}{\rho(y)} \right)^{-N};$$
- (d)  $\left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x, y) dy \right| \leq C \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}.$

**Proof.** Let us prove (a) first. Observe that, by the second identity of (3.10) and Lemma 3.2, we obtain

$$\begin{aligned} |\mathcal{P}_t(x, y)| &\leq Ct \int_0^\infty u^{-\frac{n+3}{2}} e^{-\frac{|x-y|^2 + t^2}{cu}} \left( 1 + \frac{\sqrt{u}}{\rho(x)} + \frac{\sqrt{u}}{\rho(y)} \right)^{-N} du \\ &\quad + Ct \int_{|x-y|^2 + t^2}^\infty u^{-\frac{n+3}{2}} e^{-\frac{|x-y|^2 + t^2}{cu}} \left( 1 + \frac{\sqrt{u}}{\rho(x)} + \frac{\sqrt{u}}{\rho(y)} \right)^{-N} du =: I + II. \end{aligned}$$

For  $I$  apply the change of variables  $r = (|x-y|^2 + t^2)/u$  to get



$$I \leq \frac{Ct}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} \left( 1 + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)} \right)^{-N} \int_1^\infty r^{\frac{n+N-1}{2}} e^{-cr} dr.$$

For  $II$ ,

$$II \leq Ct \left( 1 + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)} \right)^{-N} \int_{|x-y|^2+t^2}^\infty u^{-\frac{n+3}{2}} du.$$

Combining these last two estimates we conclude the proof of (a).

To prove (b), note that we can estimate the integral in brackets in (3.12) as follows:

$$\begin{aligned} \left| \int_0^\infty H_{m-1} \left( \frac{t+s}{2v} \right) e^{-\frac{(t+s)^2}{4v^2}} s^{m-\beta} \frac{ds}{s} \right| &\leq C_m \int_0^\infty e^{-c\frac{(t+s)^2}{4v^2}} s^{m-\beta} \frac{ds}{s} \leq C_m e^{-c\frac{t^2}{v^2}} \int_0^\infty e^{-c\frac{s^2}{v^2}} s^{m-\beta} \frac{ds}{s} \\ &= C_m e^{-c\frac{t^2}{v^2}} v^{m-\beta} \int_0^\infty e^{-cr^2} r^{m-\beta} \frac{dr}{r} = C_{m,\beta} e^{-c\frac{t^2}{v^2}} v^{m-\beta}. \end{aligned} \quad (3.13)$$

Using identity (3.12), this last inequality and Lemma 3.4(a), we get

$$|\partial_t^\beta \mathcal{P}_t(x, y)| \leq C \int_0^\infty e^{-c\frac{t^2}{v^2}} v^{-\beta} |Q_v(x, y)| \frac{dv}{v} \leq C \int_0^\infty \frac{e^{-c\frac{|x-y|^2+t^2}{v^2}}}{v^{n+\beta}} \left( 1 + \frac{v}{\rho(x)} + \frac{v}{\rho(y)} \right)^{-N} \frac{dv}{v}.$$

The last integral can be split and treated as  $I$  and  $II$  above. Hence (b) is proved.

The proof of part (c) follows parallel lines as we have just done for (b) by using identity (3.12), estimate (3.13) and Lemma 3.4(b).

For (d), let  $0 < \delta' \leq \delta$  with  $0 < \delta' < \beta$ . By Remark 3.5 and the change of variables  $w = t/v$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x, y) dy \right| &\leq Ct^\beta \int_0^\infty e^{-c\frac{t^2}{v^2}} v^{-\beta} \left| \int_{\mathbb{R}^n} Q_v(x, y) dy \right| \frac{dv}{v} \\ &\leq Ct^\beta \int_0^\infty e^{-c\frac{t^2}{v^2}} v^{-\beta} \frac{(v/\rho(x))^{\delta'}}{(1+v/\rho(x))^N} \frac{dv}{v} = C(t/\rho(x))^{\delta'} \int_0^\infty e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w}. \end{aligned}$$

On one hand,

$$\int_{t/\rho(x)}^\infty e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} \leq e^{-c\frac{t^2}{2\rho(x)^2}} \int_0^\infty e^{-c\frac{w^2}{2}} w^{\beta-\delta'} \frac{dw}{w} \leq C e^{-c\frac{t^2}{\rho(x)^2}} \leq \frac{C}{(1+t/\rho(x))^N}.$$

On the other hand, we consider two cases. If  $t/\rho(x) \leq 1$  then

$$\int_0^{t/\rho(x)} e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} \leq \int_0^1 w^{\beta-\delta'} \frac{dw}{w} \leq \frac{C}{(1+t/\rho(x))^N}.$$

If  $t/\rho(x) > 1$  then

$$\int_0^{t/\rho(x)} e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} \leq \frac{1}{(t/\rho(x))^N} \int_0^\infty e^{-cw^2} w^{\beta-\delta'+N} \frac{dw}{w} \leq \frac{C}{(1+t/\rho(x))^N}.$$

This concludes the proof of the proposition.  $\square$

To finish this section we show a reproducing formula for the operator  $t^\beta \partial_t^\beta \mathcal{P}_t$  on  $L^2(\mathbb{R}^n)$ .

**Lemma 3.7.** The operator  $t^\beta \partial_t^\beta \mathcal{P}_t$  defines an isometry from  $L^2(\mathbb{R}^n)$  into  $L^2(\mathbb{R}_+^{n+1}, \frac{dx dt}{t})$ . Moreover,

$$f(x) = \frac{4^\beta}{\Gamma(2\beta)} \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow \infty}} \int_{\varepsilon}^N (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f(x) \frac{dt}{t}, \quad \text{in } L^2(\mathbb{R}^n). \quad (3.14)$$

**Proof.** The proof is standard by using spectral techniques, see for instance [7], and we omit it here.  $\square$

#### 4. The Campanato-type space $BMO_{\mathcal{L}}^\alpha$ , $0 \leq \alpha \leq 1$ : duality and pointwise description

In this section we give the definition of space  $BMO_{\mathcal{L}}^\alpha$  introduced in [1], the relation with  $C_{\mathcal{L}}^{0,\alpha}$  and the duality result  $H_{\mathcal{L}}^p - BMO_{\mathcal{L}}^\alpha$ .

**Definition 4.1** ( $BMO^\alpha$  space for  $\mathcal{L}$ ). (See [1].) A locally integrable function  $f$  is in  $BMO_{\mathcal{L}}^\alpha$ ,  $0 \leq \alpha \leq 1$ , if there exists a constant  $C$  such that

- (i)  $\frac{1}{|B|} \int_B |f(x) - f_B| dx \leq C |B|^{\frac{\alpha}{n}}$ , for every ball  $B$  in  $\mathbb{R}^n$ , and
- (ii)  $\frac{1}{|B|} \int_B |f(x)| dx \leq C |B|^{\frac{\alpha}{n}}$ , for every  $B = B(x_0, r_0)$ , where  $x_0 \in \mathbb{R}^n$  and  $r_0 \geq \rho(x_0)$ .

As usual,  $f_B := \frac{1}{|B|} \int_B f(x) dx$ . The norm  $\|f\|_{BMO_{\mathcal{L}}^\alpha}$  is defined as the infimum of the constants  $C$  such that (i) and (ii) above hold.

**Remark 4.2.** The space  $BMO_{\mathcal{L}}^0$  is the  $BMO$  space naturally associated to  $\mathcal{L}$  given in [7]. We require  $\alpha \leq 1$  in the definition above because if  $\alpha > 1$  then the space only contains constant functions. By using the classical John–Nirenberg inequality it can be seen that if in (i) and (ii)  $L^1$ -norms are replaced by  $L^p$ -norms, for  $1 < p < \infty$ , then the space  $BMO_{\mathcal{L}}^\alpha$  does not change.

**Proposition 4.3.** Let  $f \in BMO_{\mathcal{L}}^\alpha$ ,  $0 < \alpha \leq 1$ , and  $B = B(x, r)$  with  $r < \rho(x)$ . Then there exists a constant  $C = C_\alpha$  such that  $|f_B| \leq C_\alpha \|f\|_{BMO_{\mathcal{L}}^\alpha} \rho(x)^\alpha$ .

**Proof.** Let  $j_0$  be a positive integer such that  $2^{j_0} r \leq \rho(x) < 2^{j_0+1} r$ . Since  $f \in BMO_{\mathcal{L}}^\alpha$ , we have

$$\begin{aligned} |f_B| &\leq \frac{1}{|B|} \int_B |f(z) - f_{2B}| dz + \sum_{j=1}^{j_0} |f_{2^j B} - f_{2^{j+1} B}| + |f_{2^{j_0+1} B}| \\ &\leq C \|f\|_{BMO_{\mathcal{L}}^\alpha} |B|^{\frac{\alpha}{n}} \sum_{j=1}^{j_0+1} (2^\alpha)^j = C \|f\|_{BMO_{\mathcal{L}}^\alpha} |B|^{\frac{\alpha}{n}} \frac{2^\alpha - 2^{\alpha(j_0+1)}}{1 - 2^\alpha} \\ &\leq C \|f\|_{BMO_{\mathcal{L}}^\alpha} |B|^{\frac{\alpha}{n}} 2^{\alpha(j_0+1)} = C 2^\alpha \|f\|_{BMO_{\mathcal{L}}^\alpha} (2^{j_0} r)^\alpha \leq C_\alpha \|f\|_{BMO_{\mathcal{L}}^\alpha} \rho(x)^\alpha. \quad \square \end{aligned}$$

**Remark 4.4.** From the proof of Proposition 4.3 it can be seen that if  $f$  is in  $BMO_{\mathcal{L}} = BMO_{\mathcal{L}}^0$  and  $B = B(x, r)$  with  $r < \rho(x)$  then the conclusion of Lemma 2 in [7] follows:

$$|f_B| \leq C \left(1 + \log \frac{\rho(x)}{r}\right) \|f\|_{BMO_{\mathcal{L}}}.$$

Following the works by J. Dziubański and J. Zienkiewicz [4,6,5] we introduce the Hardy space naturally associated to  $\mathcal{L}$ . An integrable function  $f$  is an element of the  $\mathcal{L}$ -Hardy space  $H_{\mathcal{L}}^p$ ,  $0 < p \leq 1$ , if the maximal function  $\mathcal{T}^* f(x) := \sup_{s>0} |\mathcal{T}_s f(x)|$ , see (3.4), belongs to  $L^p(\mathbb{R}^n)$ . The quasi-norm in  $H_{\mathcal{L}}^p$  is defined by  $\|f\|_{H_{\mathcal{L}}^p} := \|\mathcal{T}^* f\|_{L^p(\mathbb{R}^n)}$ . In [4,5] the atomic description of  $H_{\mathcal{L}}^p$  was given. Let  $\tilde{\delta} = \min\{1, \delta\}$ , with  $\delta$  as in Lemma 3.3. An atom of the  $\mathcal{L}$ -Hardy space  $H_{\mathcal{L}}^p$ ,  $\frac{n}{n+\tilde{\delta}} < p \leq 1$ , associated with a ball  $B(x_0, r)$  is a function  $a$  such that  $\text{supp } a \subseteq B(x_0, r)$  with  $r \leq \rho(x_0)$ ,  $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$  and, if  $r < \rho(x_0)/4$  then  $\int a(x) dx = 0$ . The atomic  $\mathcal{L}$ -Hardy space  $H_{\text{at}, \mathcal{L}}^p$ ,  $\frac{n}{n+\tilde{\delta}} < p \leq 1$ , is defined as the set of  $L^1$ -functions  $f$  with compact support such that  $f$  can be written as a sum  $f = \sum_i \lambda_i a_i$ , where  $\lambda_i$  are complex numbers with  $\sum_i |\lambda_i| < \infty$  and  $a_i$  are atoms in  $H_{\mathcal{L}}^p$ . The quasi-norm in the atomic Hardy space, namely the infimum of all such possible  $\sum_i |\lambda_i|$ , turns out to be equivalent to the quasi-norm  $\|f\|_{H_{\mathcal{L}}^p}$ , for that range of  $p$ . When  $n/2 < q < n$ , such equivalence can be extended to hold for Hardy spaces  $H_{\mathcal{L}}^p$  with  $\frac{n}{n+1} < p \leq \frac{n}{n+\tilde{\delta}}$ , but atoms must be redefined, see [6].

As mentioned in [1], see also [8] and [16], once an atomic decomposition of  $H_{\mathcal{L}}^p$  is at hand, the dual space can be easily described. We present the following result without proof.

**Theorem 4.5** (Duality  $H_{\mathcal{L}}^p$ – $BMO_{\mathcal{L}}^\alpha$ ). *Let  $q > n$  and  $0 \leq \alpha < 1$ . Then the dual of  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$  is the space  $BMO_{\mathcal{L}}^\alpha$ . More precisely, any continuous linear functional  $\ell$  over  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$  can be represented as*

$$\ell(a) = \int_{\mathbb{R}^n} f(x)a(x) dx,$$

for some function  $f \in BMO_{\mathcal{L}}^\alpha$  and all atoms  $a \in H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ . Moreover,  $\|\ell\| \sim \|f\|_{BMO_{\mathcal{L}}^\alpha}$ .

**Proposition 4.6** (Campanato-type description of  $C_{\mathcal{L}}^{0,\alpha}$ ). *If  $0 < \alpha \leq 1$  then the spaces  $BMO_{\mathcal{L}}^\alpha$  and  $C_{\mathcal{L}}^{0,\alpha}$  are equal and their norms are equivalent.*

The previous result was proved in [1, Proposition 4] for  $0 < \alpha < 1$  and in a weighted context. We just mention here that the proof given there is also valid for  $\alpha = 1$ . As a consequence, the functions in  $BMO_{\mathcal{L}}^\alpha$  can be modified in a set of measure zero so they become  $\alpha$ -Hölder continuous,  $0 < \alpha \leq 1$ .

## 5. Proofs of Theorems 1.3, 1.4 and 1.5

The proof of Theorem 1.3 will follow the scheme (i)  $\implies$  (ii)  $\implies$  (iii)  $\implies$  (i). The statement (iii)  $\implies$  (i) relies heavily on the duality  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ – $BMO_{\mathcal{L}}^\alpha$  developed in Section 4, so the method, rather technical, will work only for  $0 < \alpha < 1$ . Observe that the proof of (ii)  $\implies$  (iii) is immediate. To prove Theorem 1.4(I) we just note that the proofs of (i)  $\implies$  (ii)  $\implies$  (iii) in Theorem 1.3 also hold for  $\alpha = 1$ . A simple contradiction argument shows that the converse is false: if it were true then, by the comment just made,  $f \in C_{\mathcal{L}}^{0,1}$  would be equivalent to (ii) in Theorem 1.3 with  $\alpha = 1$ . But that contradicts the statement of Theorem 1.4(II) (which is proved by a counterexample). For Theorem 1.5(A) we only have to prove the necessity part since the sufficiency for  $\beta = 1$  follows the same lines as in [7]. For part (B) we give a counterexample.

### 5.1. Proof of Theorem 1.3: (i) $\implies$ (ii)

Let  $f \in C_{\mathcal{L}}^{0,\alpha}$ . Then

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)| &= \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x, z) (f(z) - f(x)) dz + f(x) \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x, z) dz \right| \\ &\leq \|f\|_{C_{\mathcal{L}}^{0,\alpha}} \int_{\mathbb{R}^n} |t^\beta \partial_t^\beta \mathcal{P}_t(x, z)| |x - z|^\alpha dz + \|f\|_{C_{\mathcal{L}}^{0,\alpha}} \rho(x)^\alpha \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x, z) dz \right| =: I + II. \end{aligned}$$

Applying Proposition 3.6(b), we obtain

$$I \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} \int_{\mathbb{R}^n} \frac{t^\beta |x - z|^\alpha}{(t + |x - z|)^{n+\beta}} dz = C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha.$$

For  $II$  we consider two cases. Assume first that  $\rho(x) \leq t$ . Then Proposition 3.6(b) gives

$$II \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha \int_{\mathbb{R}^n} \frac{t^\beta}{(t + |x - z|)^{n+\beta}} dz = C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha.$$

Suppose now that  $\rho(x) > t$ . Since  $s > n$ , we have  $\delta > 1$  in Lemma 3.3. Therefore we can choose  $\delta'$  such that  $\alpha < \delta' \leq \delta$  with  $\delta' < \beta$ . By Proposition 3.6(d),  $II \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha (t/\rho(x))^{\delta'-\alpha} \leq C \|f\|_{C_{\mathcal{L}}^{0,\alpha}} t^\alpha$ .

### 5.2. Proof of Theorem 1.3: (iii) $\implies$ (i)

Assume that  $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+\alpha+\varepsilon)} dx)$  for any  $0 < \varepsilon < \min\{\beta - \alpha, 1 - \alpha\}$ , and that the Carleson condition in (iii) holds. Let

$$[d\mu_f]_{\alpha,\beta} := \sup_B \frac{1}{|B|^{\frac{\alpha}{n}}} \left( \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \frac{dx dt}{t} \right)^{1/2}.$$

To show that  $f \in BMO_{\mathcal{L}}^\alpha$ , by Theorem 4.5, it is enough to prove that the linear functional that maps each  $g \in H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$  to  $\Phi_f(g) := \int_{\mathbb{R}^n} f(x)g(x)dx$  is continuous on  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ . In fact, we are going to prove that  $|\Phi_f(g)| \leq C[d\mu_f]_{\alpha,\beta} \|g\|_{H_{\mathcal{L}}^{\frac{n}{n+\alpha}}}$ , which implies that  $f \in BMO_{\mathcal{L}}^\alpha$  with  $\|f\|_{BMO_{\mathcal{L}}^\alpha} \leq C[d\mu_f]_{\alpha,\beta}$ .

**Step 1.** It consists in writing the functional  $\Phi$  by using extensions of  $f$  and  $g$  to the upper half-space. Define, for  $x \in \mathbb{R}^n$ ,  $t > 0$ , the extended functions  $F(x, t) := t^\beta \partial_t^\beta \mathcal{P}_t f(x)$  and  $G(x, t) := t^\beta \partial_t^\beta \mathcal{P}_t g(x)$ .

**Lemma 5.1.** Let  $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+\alpha+\varepsilon)} dx)$  for any  $\varepsilon > 0$  and  $g$  be an  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ -atom. Then

$$\frac{4^\beta}{\Gamma(2\beta)} \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx = \int_{\mathbb{R}_+^{n+1}} F(x, t) \overline{G(x, t)} \frac{dx dt}{t}.$$

The rather technical proof of the lemma above will be given at the end of this subsection. To continue we assume its validity. So we are reduced to study the integral in the right-hand side.

**Step 2.** To handle the integral in Lemma 5.1 we take a result of E. Harboure, O. Salinas and B. Viviani about tent spaces into our particular case.

**Lemma 5.2.** (See [8, p. 279].) For any pair of measurable functions  $F$  and  $G$  on  $\mathbb{R}_+^{n+1}$  we have

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} |F(x, t)| |G(x, t)| \frac{dx dt}{t} \\ & \leq C \sup_B \left( \frac{1}{|B|^{1+\frac{2\alpha}{n}}} \int_{\widehat{B}} |F(x, t)|^2 \frac{dx dt}{t} \right)^{1/2} \times \left( \int_{\mathbb{R}^n} \left( \int_{\Gamma(x)} |G(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{n}{2(n+\alpha)}} dx \right)^{\frac{n+\alpha}{n}}, \end{aligned}$$

where  $\Gamma(x)$  denotes the cone with vertex at  $x$  and aperture 1:  $\{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ .

If we take  $F(x, t) = t^\beta \partial_t^\beta \mathcal{P}_t f(x)$  in Lemma 5.2 then the supremum that appears in the inequality is exactly  $[d\mu_f]_{\alpha,\beta}$ . Hence it remains to handle the term with  $G(x, t)$ , which is done in the last step.

**Step 3.** The area function  $S_\beta$  defined by

$$S_\beta(h)(z) = \left( \iint_{\Gamma(z)} |t^\beta \partial_t^\beta \mathcal{P}_t h(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad z \in \mathbb{R}^n, \quad (5.1)$$

is a bounded operator on  $L^2(\mathbb{R}^n)$ . Indeed, by the Spectral theorem, the square function

$$g_\beta(h)(x) = \left( \int_0^\infty |t^\beta \partial_t^\beta \mathcal{P}_t h(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n, \quad (5.2)$$

satisfies  $\|g_\beta(h)\|_{L^2(\mathbb{R}^n)} = \Gamma(\beta) \|h\|_{L^2(\mathbb{R}^n)}$  and it is easy to check that  $\|S_\beta(h)\|_{L^2(\mathbb{R}^n)} = \|g_\beta(h)\|_{L^2(\mathbb{R}^n)}$ . We will finish the proof of (iii)  $\implies$  (i) in Theorem 1.3 as soon as we have proved the following

**Lemma 5.3.** There exists a constant  $C$  such that for any function  $g$  which is a linear combination of  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ -atoms we have  $\|S_\beta(g)\|_{L^{\frac{n}{n+\alpha}}} \leq C \|g\|_{H_{\mathcal{L}}^{\frac{n}{n+\alpha}}}$ .

**Proof.** Let  $g$  be an  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ -atom associated to a ball  $B = B(x_0, r)$ . We apply Hölder's inequality and the  $L^2$ -boundedness of the area function (5.1) to get

$$\int_{8B} |S_\beta(g)(x)|^{\frac{n}{n+\alpha}} dx \leq C |B|^{\frac{n+2\alpha}{2(n+\alpha)}} \|g\|_{L^2(8B)}^{\frac{n}{n+\alpha}} \leq C |B|^{\frac{n+2\alpha}{2(n+\alpha)}} |B|^{\frac{n}{2(n+\alpha)}} \|g\|_{L^\infty}^{\frac{n}{n+\alpha}} \leq C.$$

In order to complete the proof of Lemma 5.3, we must find a uniform bound for

$$\int_{(8B)^c} |S_\beta(g)(x)|^{\frac{n}{n+\alpha}} dx. \quad (5.3)$$

Let us consider first the case when  $r < \frac{\rho(x_0)}{4}$ . Then, by the moment condition on  $g$ , we have

$$\begin{aligned} (S_\beta(g)(x))^2 &= \int_0^\infty \int_{|x-y|<t} \left( \int_{\mathbb{R}^n} (t^\beta \partial_t^\beta \mathcal{P}_t(y, x') - t^\beta \partial_t^\beta \mathcal{P}_t(y, x_0)) g(x') dx' \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq \int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left( \int_B |t^\beta \partial_t^\beta \mathcal{P}_t(y, x') - t^\beta \partial_t^\beta \mathcal{P}_t(y, x_0)| \frac{dx'}{|B|^{\frac{n+\alpha}{n}}} \right)^2 \frac{dy dt}{t^{n+1}} \\ &\quad + \int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \left( \int_B |t^\beta \partial_t^\beta \mathcal{P}_t(y, x') - t^\beta \partial_t^\beta \mathcal{P}_t(y, x_0)| \frac{dx'}{|B|^{\frac{n+\alpha}{n}}} \right)^2 \frac{dy dt}{t^{n+1}} =: I_1(x) + I_2(x). \end{aligned}$$

We now use the smoothness of  $t^\beta \partial_t^\beta \mathcal{P}_t(y, x) = t^\beta \partial_t^\beta \mathcal{P}_t(x, y)$  established in Proposition 3.6(c) with  $\alpha < \delta' < \beta$  and  $N > 0$ . In the domain of integration of  $I_1(x)$  we have  $|x - x_0| \leq 2|y - x_0|$ . So

$$\begin{aligned} I_1(x) &\leq C \int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left( \int_B \left( \frac{|x' - x_0|}{t} \right)^{\delta'} \frac{t^\beta}{(|x_0 - y|^2 + t^2)^{\frac{n+\beta}{2}}} \frac{dx'}{|B|^{\frac{n+\alpha}{n}}} \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left( \frac{r}{t} \right)^{2\delta'} \frac{1}{t^{2n} \left( \frac{|x_0 - y|}{t} + 1 \right)^{2(n+\beta)}} \frac{1}{|B|^{\frac{2\alpha}{n}}} \frac{dy dt}{t^{n+1}} \\ &\leq C \int_0^{\frac{|x-x_0|}{2}} \left( \frac{r}{t} \right)^{2\delta'} \frac{1}{t^{2n} \left( \frac{|x_0 - x|}{t} \right)^{2(n+\beta)}} \frac{1}{|B|^{\frac{2\alpha}{n}}} \frac{dt}{t} \\ &\leq C \frac{r^{2(\delta' - \alpha)}}{|x - x_0|^{2(n+\beta)}} \int_0^{\frac{|x-x_0|}{2}} t^{2(\beta - \delta')} \frac{dt}{t} = C \frac{r^{2(\delta' - \alpha)}}{|x - x_0|^{2(n+\delta')}}. \end{aligned}$$

Thus, integrating over  $(8B)^c$ , we have  $\int_{(8B)^c} |I_1(x)|^{1/2} \frac{n}{n+\alpha} dx \leq C \int_{(8B)^c} \left( \frac{r^{\delta' - \alpha}}{|x - x_0|^{n+\delta'}} \right)^{\frac{n}{n+\alpha}} dx = C$ . Let us continue with  $I_2(x)$ .

If  $x \in (8B)^c$  then we have  $|x' - x_0| \leq r < \frac{|x - x_0|}{2} \leq t$ . Then, by Proposition 3.6(c) and  $x \in (8B)^c$ , we have

$$\begin{aligned} I_2(x) &\leq C \int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \left( \int_B \left( \frac{|x' - x_0|}{t} \right)^{\delta'} \frac{1}{t^n} \frac{dx'}{|B|^{\frac{n+\alpha}{n}}} \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \left( \frac{r}{t} \right)^{2\delta'} \frac{1}{t^{2n}} \frac{1}{|B|^{\frac{2\alpha}{n}}} \frac{dy dt}{t^{n+1}} = C \frac{r^{2(\delta' - \alpha)}}{|x - x_0|^{2(n+\delta')}}. \end{aligned}$$

Therefore the integral of  $|I_2(x)|^{1/2} \frac{n}{n+\alpha}$  over  $(8B)^c$  is bounded by a constant. Collecting terms we see that if  $r < \frac{\rho(x_0)}{4}$  then a uniform bound for (5.3) is obtained.

We now turn the estimate of (5.3) when  $r$  is comparable to  $\rho(x_0)$ , namely,  $\frac{\rho(x_0)}{4} < r \leq \rho(x_0)$ . For  $x \in (8B)^c$  we can split the integral in  $t > 0$  in the definition of  $S_\beta g(x)$  into three parts:

$$\begin{aligned} (S_\beta(g)(x))^2 &= \left( \int_0^{\frac{r}{2}} + \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} + \int_{\frac{|x-x_0|}{4}}^\infty \right) \int_{|x-y|<t} \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(y, x') g(x') dx' \right|^2 \frac{dy dt}{t^{n+1}} \\ &=: I'_1(x) + I'_2(x) + I'_3(x). \end{aligned}$$

In the integrand of  $I'_1(x)$ , we have  $|x' - y| \sim |x - x_0|$ . So by Proposition 3.6(b), we get

$$\begin{aligned} I'_1(x) &\leq C \int_0^{\frac{r}{2}} \int_{|x-y|<t} \left( \int_B \frac{t^\beta}{(|y-x'|+t)^{n+\beta}} \frac{1}{|B|^{\frac{n+\alpha}{n}}} dx' \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq Cr^{-2\alpha} \int_0^{\frac{r}{2}} \int_{|x-y|<t} \frac{t^{2\beta}}{(|x-x_0|+t)^{2(n+\beta)}} \frac{dy dt}{t^{n+1}} \leq C \frac{r^{2(\beta-\alpha)}}{|x-x_0|^{2(n+\beta)}}. \end{aligned}$$

For  $I'_2(x)$ , by applying Proposition 3.6(b) for any  $M > \alpha$ , together with  $|x' - y| \sim |x - x_0|$  and  $\rho(x') \sim \rho(x_0) \sim r$ , we get

$$\begin{aligned} I'_2(x) &\leq C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left( \int_B \frac{t^\beta}{(|y-x'|+t)^{n+\beta}} \left( \frac{\rho(x')}{t} \right)^M \frac{1}{|B|^{\frac{n+\alpha}{n}}} dx' \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left( \int_B \frac{1}{t^n \left( \frac{|x-x_0|}{t} + 1 \right)^{n+\beta}} \left( \frac{\rho(x_0)}{t} \right)^M \frac{1}{|B|^{\frac{n+\alpha}{n}}} dx' \right)^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left( \frac{t^{\beta-M} \rho(x_0)^M}{|x-x_0|^{n+\beta} r^\alpha} \right)^2 \frac{dy dt}{t^{n+1}} \leq C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \left( \frac{t^{\beta-M} r^{M-\alpha}}{|x-x_0|^{n+\beta}} \right)^2 \frac{dt}{t} \\ &\leq C \frac{r^{2(\beta-\alpha)}}{|x-x_0|^{2(n+\beta)}} \int_1^{\frac{|x-x_0|}{2r}} u^{2(\beta-M)} \frac{du}{u} \leq C \frac{r^{2(M-\alpha)}}{|x-x_0|^{2(n+M)}}. \end{aligned}$$

Finally, for the last term above  $I'_3(x)$ , with the same method that was used to estimate  $I'_2(x)$ , we obtain  $I'_3(x) \leq Cr^{2(M-\alpha)} |x - x_0|^{-2(n+M)}$ . Hence,  $\int_{(8B)^c} |I'_j(x)|^{1/2} \frac{n}{n+\alpha} dx \leq C$ , for  $j = 1, 2, 3$  and the uniform bound for (5.3) is established also when  $r \sim \rho(x_0)$ . The proof of Lemma 5.3 is complete.  $\square$

Now the three steps of the proof of (iii)  $\implies$  (i) in Theorem 1.3 are completed. It only remains to prove Lemma 5.1, that we took for granted before. To that end, we need the following result.

**Lemma 5.4.** Let  $q_t(x, y)$  be a function of  $x, y \in \mathbb{R}^n$ ,  $t > 0$ . Assume that for each  $N > 0$  there exists a constant  $C_N$  such that, for some  $\gamma \geq \alpha$ ,

$$|q_t(x, y)| \leq C_N \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N} t^{-n} \left( 1 + \frac{|x-y|}{t} \right)^{-(n+\gamma)}. \quad (5.4)$$

Then, for every  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ -atom  $g$  supported on  $B(x_0, r)$ , there exists  $C_{N, x_0, r} > 0$  such that

$$\sup_{t>0} \left| \int_{\mathbb{R}^n} q_t(x, y) g(y) dy \right| \leq C_{N, x_0, r} (1 + |x|)^{-(n+\gamma)}, \quad x \in \mathbb{R}^n.$$

**Proof.** Let  $I = I(x, t)$  be the integral appearing in the statement. If  $x \in B(x_0, 2r)$  then, since  $\|g\|_{L^\infty(\mathbb{R}^n)} \leq |B(x_0, r)|^{-(1+\frac{\alpha}{n})}$ , we have

$$|I| \leq C_N \frac{1}{r^{n+\alpha}} \int_{\mathbb{R}^n} t^{-n} \left( 1 + \frac{|x-y|}{t} \right)^{-(n+\gamma)} dy \leq C_N \frac{1}{r^{n+\alpha}} \int_{\mathbb{R}^n} \frac{1}{(1+|u|)^{n+\gamma}} du \leq C_{N, r}.$$

Since  $|x - x_0| \leq 2r$ , we have  $1 + |x| \leq 1 + |x - x_0| + |x_0| \leq 1 + 2r + |x_0|$ . Hence  $|I| \leq C_{N, r} \frac{(1+2r+|x_0|)^{n+\gamma}}{(1+2r+|x_0|)^{n+\gamma}} \leq C_{N, x_0, r} (1 + |x|)^{-(n+\gamma)}$ . If  $x \notin B(x_0, 2r)$  then for  $y \in B(x_0, r)$  we have  $|x - y| \sim |x - x_0|$  and, since  $r < \rho(x_0)$ , we get that  $\rho(x_0) \sim \rho(y)$ , see Lemma 3.1. Hence, choosing  $N = \gamma$  in (5.4), we get

$$|I| \leq C_\gamma \frac{t^{-\gamma} t^{-n} |x - x_0|^{-(n+\gamma)}}{\rho(x_0)^{-\gamma} t^{-(n+\gamma)}} \|g\|_{L^1(\mathbb{R}^n)} \leq C_{\gamma, x_0, r} \frac{\rho(x_0)^\gamma |x - x_0|^{-(n+\gamma)}}{r^\gamma} \leq C_{\gamma, x_0, r} |x - x_0|^{-(n+\gamma)}.$$

Since  $x \notin B(x_0, 2r)$ , we can set  $x = x_0 + 2rz$ ,  $|z| \geq 1$ . Then  $1 + |x| \leq 1 + |x_0| + 2r|z|$ , and  $\frac{1+|x_0|+2r}{2r} |x - x_0| = (1 + |x_0| + 2r)|z| \geq 1 + |x_0| + 2r|z|$ . It means that  $c_{x_0, r} |x - x_0| \geq 1 + |x|$ . Therefore  $|I| \leq C_{\gamma, x_0, r} |x - x_0|^{-(n+\gamma)} \leq C_{\gamma, x_0, r} (1 + |x|)^{-(n+\gamma)}$ .  $\square$

**Proof of Lemma 5.1.** Assume that  $g$  is an  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$ -atom associated to a ball  $B = B(x_0, r)$ . By Lemma 5.2 and Lemma 5.3, the following integral is absolutely convergent and therefore it can be described as

$$I = \int_{\mathbb{R}_+^{n+1}} F(x, t) \overline{G(x, t)} \frac{dx dt}{t} = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1/\epsilon} \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x) \overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)} \frac{dx dt}{t}.$$

Proposition 3.6(b) and  $\beta > \alpha + \varepsilon$  imply that  $q_t(x, y) := t^\beta \partial_t^\beta \mathcal{P}_t(x, y)$  satisfies (5.4) in Lemma 5.4. Therefore, since  $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+\alpha+\varepsilon)} dx)$ , Fubini's theorem can be applied to get:

$$\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x) \overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)} dx = \int_{\mathbb{R}^n} f(y) \overline{(t^\beta \partial_t^\beta \mathcal{P}_t)^2 g(y)} dy.$$

So that,

$$I = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1/\epsilon} \left[ \int_{\mathbb{R}^n} f(y) \overline{(t^\beta \partial_t^\beta \mathcal{P}_t)^2 g(y)} dy \right] \frac{dt}{t} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} f(y) \left[ \int_{\epsilon}^{1/\epsilon} \overline{t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t} g(y)} \frac{dt}{t} \right] dy. \quad (5.5)$$

We claim that

$$\sup_{\epsilon > 0} \left| \int_{\epsilon}^{1/\epsilon} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t} g(y) \frac{dt}{t} \right| \leq C(1 + |y|)^{-(n+\alpha+\varepsilon)}, \quad (5.6)$$

for any  $y \in \mathbb{R}^n$ . To prove (5.6) we first note that

$$\begin{aligned} \left| \int_{\epsilon}^{1/\epsilon} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t} g(y) \frac{dt}{t} \right| &\leq \left| \int_{\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t} g(y) \frac{dt}{t} \right| + \left| \int_{1/\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t} g(y) \frac{dt}{t} \right| \\ &= \left| \int_{\mathbb{R}^n} \int_{\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t}(x, y) \frac{dt}{t} g(x) dx \right| + \left| \int_{\mathbb{R}^n} \int_{1/\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t}(x, y) \frac{dt}{t} g(x) dx \right|. \end{aligned}$$

Hence, to prove (5.6) it is enough to check that the kernel

$$\int_{\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_{2t}(x, y) \frac{dt}{t} = 2^{[2\beta]-2\beta+1} \int_{2\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_t(x, y) \frac{dt}{t}, \quad (5.7)$$

satisfies estimate (5.4) of Lemma 5.4, for any  $\epsilon > 0$ . To verify this we consider it in three cases.

**Case I:**  $2\beta < 1$ . Making a change of variables in the definition of the fractional derivative (1.5), applying Fubini's theorem and integrating by parts, we have

$$\begin{aligned} \int_{2\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_t(x, y) \frac{dt}{t} &= C \int_{2\epsilon}^{\infty} t^{2\beta} \int_t^{\infty} \partial_u \mathcal{P}_u(x, y) (u - t)^{-2\beta} du \frac{dt}{t} \\ &= C \int_{2\epsilon}^{\infty} \partial_u \mathcal{P}_u(x, y) \int_{\frac{2\epsilon}{u}}^1 \left( \frac{w}{1-w} \right)^{2\beta} \frac{dw}{w} du = C \int_{2\epsilon}^{\infty} \mathcal{P}_u(x, y) \left( \frac{2\epsilon}{u-2\epsilon} \right)^{2\beta} \frac{du}{u} \\ &= C \int_{2\epsilon}^{\infty} \mathcal{P}_u(x, y) \left( \frac{2\epsilon}{u-2\epsilon} \right)^{2\beta} \chi_A(u) \frac{du}{u} + C \int_{2\epsilon}^{\infty} \mathcal{P}_u(x, y) \left( \frac{2\epsilon}{u-2\epsilon} \right)^{2\beta} \chi_{A^c}(u) \frac{du}{u} =: I' + II', \end{aligned}$$

where  $A = \{u - 2\epsilon \leq \epsilon + |x - y|\}$ . Observe that in the equalities above we applied the assumption  $2\beta < 1$  to have convergent integrals. Let us first estimate  $I'$ . By Proposition 3.6(a) and since  $\alpha + \epsilon < 2\beta$  we get that for any  $N > 0$ ,

$$\begin{aligned} |I'| &\leq C \frac{\epsilon^{2\beta}}{(|x - y| + \epsilon)^{n+1}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \int_{2\epsilon}^{3\epsilon + |x - y|} (u - 2\epsilon)^{-2\beta} du \\ &\leq C \epsilon^{2\beta} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} (|x - y| + \epsilon)^{-n-2\beta}, \end{aligned}$$

and the desired estimate follows. We continue now with  $II'$ . Note that in  $II'$  we have  $u - 2\epsilon > |x - y| + \epsilon$  so, again by Proposition 3.6(a), we get

$$\begin{aligned} |II'| &\leq C \left(\frac{\epsilon}{\epsilon + |x - y|}\right)^{2\beta} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \int_{2\epsilon}^{\infty} (|x - y| + u)^{-n-1} du \\ &= C \left(\frac{\epsilon}{\epsilon + |x - y|}\right)^{2\beta} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} (\epsilon + |x - y|)^{-n}, \end{aligned}$$

which implies the estimate.

**Case II:**  $2\beta = 1$ . By Proposition 3.6(b) and integrating by parts it is easy to verify condition (5.4) for  $\int_{\epsilon}^{\infty} \partial_t \mathcal{P}_{2t}(x, y) dt$ , for any  $\epsilon > 0$ .

**Case III:**  $2\beta > 1$ . Let  $k \geq 2$  be the integer such that  $k - 1 < 2\beta \leq k$ . Note that the estimate is easy when  $2\beta = k$ , just integrating by parts. When  $k - 1 < 2\beta < k$  we make a computation similar to the case  $2\beta < 1$ . In fact,

$$\begin{aligned} \int_{2\epsilon}^{\infty} t^{2\beta} \partial_t^{2\beta} \mathcal{P}_t(x, y) \frac{dt}{t} &= C \int_{2\epsilon}^{\infty} \partial_u^k \mathcal{P}_u(x, y) \int_{2\epsilon}^u t^{2\beta} (u - t)^{k-2\beta-1} \frac{dt}{t} du \\ &= C \int_{2\epsilon}^{\infty} u^{k-1} \partial_u^k \mathcal{P}_u(x, y) \int_{\frac{2\epsilon}{u}}^1 w^{2\beta} (1 - w)^{k-2\beta-1} \frac{dw}{w} du \\ &= C \int_{2\epsilon}^{\infty} u^{k-1} \partial_u^{k-1} \mathcal{P}_u(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k}}{(u - 2\epsilon)^{1+2\beta-k}} \frac{du}{u} + C \int_{2\epsilon}^{\infty} u^{k-2} \partial_u^{k-2} \mathcal{P}_u(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k}}{(u - 2\epsilon)^{1+2\beta-k}} \frac{du}{u} \\ &\quad + \cdots + C \int_{2\epsilon}^{\infty} u \partial_u \mathcal{P}_u(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k}}{(u - 2\epsilon)^{1+2\beta-k}} \frac{du}{u} + C \int_{2\epsilon}^{\infty} \mathcal{P}_u(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k}}{(u - 2\epsilon)^{1+2\beta-k}} \frac{du}{u}. \end{aligned} \quad (5.8)$$

For any  $1 \leq m \leq k - 1$  apply Proposition 3.6(b) to get that for any  $N > 0$

$$\begin{aligned} &\left| \int_{2\epsilon}^{\infty} u^m \partial_u^m \mathcal{P}_u(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k}}{(u - 2\epsilon)^{1+2\beta-k}} \frac{du}{u} \right| \\ &\leq C \frac{\epsilon^{2\beta}}{(\epsilon + |x - y|)^{n+m}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \int_{2\epsilon}^{\infty} (u - 2\epsilon)^{k-2\beta-1} \frac{du}{u^{k-m}} \\ &= C \frac{\epsilon^{2\beta}}{(\epsilon + |x - y|)^{n+m}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \int_{2\epsilon}^{3\epsilon} (u - 2\epsilon)^{k-2\beta-1} \frac{du}{u^{k-m}} \\ &\quad + C \frac{\epsilon^{2\beta}}{(\epsilon + |x - y|)^{n+m}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \int_{3\epsilon}^{\infty} (u - 2\epsilon)^{k-2\beta-1} \frac{du}{u^{k-m}} =: I'' + II''. \end{aligned}$$

For  $I''$ , since  $2\beta < k$  and  $m \geq 1 > \alpha + \epsilon$ , we obtain



$$\begin{aligned} I'' &\leq C \frac{\epsilon^m}{(\epsilon + |x - y|)^{n+m}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \\ &\leq C \frac{1}{(\epsilon + |x - y|)^n} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N} \left(\frac{\epsilon}{\epsilon + |x - y|}\right)^{\alpha+\varepsilon}, \end{aligned}$$

and the estimate follows. For  $I''$ , since  $\frac{1}{u} < \frac{1}{u-2\epsilon}$  and  $m < 2\beta$ , we also have

$$I'' \leq C \frac{\epsilon^m}{(\epsilon + |x - y|)^{n+m}} \left(1 + \frac{\epsilon}{\rho(x)} + \frac{\epsilon}{\rho(y)}\right)^{-N},$$

which gives the bound. For the last term of (5.8) we get an estimate as above by Proposition 3.6(b).

Hence, from the three cases above we see that the kernel (5.7) satisfies condition (5.4) in Lemma 5.4, for any  $\epsilon > 0$ . Therefore can pass the limit inside the integral in (5.5). Then, by Lemma 3.7, we have

$$I = \frac{4^\beta}{\Gamma(2\beta)} \int_{\mathbb{R}^n} f(y) \overline{g(y)} dy.$$

This establishes Lemma 5.1 and it finally completes the proof of (iii)  $\implies$  (i).  $\square$

### 5.3. Proof of Theorem 1.4(II)

Let us begin with the following proposition.

**Proposition 5.5.** *Let  $0 < \alpha \leq 1$  and  $f$  be a function in  $L^\infty(\mathbb{R}^n)$  such that  $|f(x)| \leq C\rho(x)^\alpha$ , for some constant  $C$  and all  $x \in \mathbb{R}^n$ . Then  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq Ct^\alpha$ , for any  $\beta > \alpha$ , if and only if  $|f(x+y) + f(x-y) - 2f(x)| \leq C|y|^\alpha$ , for all  $x, y \in \mathbb{R}^n$ .*

Let us show how this proposition can be applied to prove Theorem 1.4(II) first.

**Proof of Theorem 1.4(II).** Assume first  $n = 1$ . Consider the function, see [13, p. 148],  $f(x) = \sum_{k=1}^{\infty} 2^{-k} e^{2\pi i 2^k x}$ ,  $x \in \mathbb{R}$ . Observe that  $\rho(x) \equiv \frac{1}{\sqrt{\mu}}$ . Therefore there exists a constant  $C = 2\sqrt{\mu}$  such that  $|f(x)| \leq \sum_{k=1}^{\infty} 2^{-k} = 1 \leq \frac{C}{\sqrt{\mu}} = C\rho(x)$ , for all  $x \in \mathbb{R}$ . Now, for any  $y \in \mathbb{R}$ ,

$$f(x+y) + f(x-y) - 2f(x) = 2 \sum_{k=1}^{\infty} 2^{-k} (\cos(2\pi 2^k y) - 1) e^{2\pi i 2^k x}.$$

Since  $|\cos(2\pi 2^k y) - 1| \leq C(2^k y)^2$  and  $|\cos(2\pi 2^k y) - 1| \leq 2$ , we have

$$|f(x+y) + f(x-y) - 2f(x)| \leq C \sum_{2^k |y| \leq 1} 2^{-k} (2^k y)^2 + C \sum_{2^k |y| > 1} 2^{-k} \leq C|y|.$$

So, by Proposition 5.5, we obtain  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq Ct$ . Let us see that  $f$  cannot be a function in  $C_{\mathcal{L}_\mu}^{0,1}$ . To arrive to a contradiction suppose that  $|f(x+y) - f(x)| \leq C_f |y|$ , for any  $x, y \in \mathbb{R}$ . Then by Bessel's inequality for  $L^2$  periodic functions we would have

$$(C_f |y|)^2 \geq \int_0^1 |f(x+y) - f(x)|^2 dx = \sum_{k=1}^{\infty} 2^{-2k} |e^{2\pi i 2^k y} - 1|^2 \geq |y|^2 \sum_{2^k |y| \leq 1} |e^{2\pi i 2^k y} - 1|^2.$$

Note that in the range  $2^k |y| \leq 1$  we have  $|e^{2\pi i 2^k y} - 1|^2 \geq c(2^k y)^2$ . Hence we arrive to the contradiction  $C_f^2 \geq c|y|^2 \sum_{2^k |y| \leq 1} 2^{2k}$ .

For the case  $n \geq 2$ , note that we can write  $\mathcal{L}_\mu = \mathcal{L}_\mu^1 - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}$ , where  $\mathcal{L}_\mu^1 = -\frac{\partial^2}{\partial x_1^2} + \mu$ . The operator  $\mathcal{L}_\mu^1$  acts only in the one-dimensional variable  $x_1$ . Let us define  $g(x_1, \dots, x_n) = f(x_1)$ , with  $f$  as above. Then, with an easy computation using the subordination formula (3.9), we have  $\|t^\beta \partial_t^\beta \mathcal{P}_t g\|_{L^\infty(\mathbb{R}^n)} = \|t^\beta \partial_t^\beta e^{-t\sqrt{\mathcal{L}_\mu^1}} f\|_{L^\infty(\mathbb{R})} \leq Ct$ , and, for any  $x, x' \in \mathbb{R}^n$ , the inequality  $|g(x) - g(x')| = |f(x_1) - f(x'_1)| \leq C|x_1 - x'_1| \leq C|x - x'|$  fails for any  $C > 0$ .  $\square$

To prove Proposition 5.5 we need the following two lemmas.

**Lemma 5.6.** Let  $f$  be a locally integrable function on  $\mathbb{R}^n$ ,  $n \geq 3$ , and  $\alpha > 0$ . If there exists  $\beta > \alpha$  such that  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C_\beta t^\alpha$ , for all  $t > 0$ , then for any  $\sigma > \alpha$  we also have  $\|t^\sigma \partial_t^\sigma \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C_\sigma t^\alpha$ , for all  $t > 0$ . Moreover, the constants  $C_\beta$  and  $C_\sigma$  are comparable.

**Proof.** Assume first that  $\sigma > \beta > \alpha$ . Then, by hypothesis and Proposition 3.6(b), we have

$$\begin{aligned} |t^\sigma \partial_t^\sigma \mathcal{P}_t f(x)| &= |t^\sigma \partial_t^{\sigma-\beta} \mathcal{P}_{t/2}(\partial_t^\beta \mathcal{P}_{t/2} f)(x)| = t^\sigma \left| \int_{\mathbb{R}^n} \partial_t^{\sigma-\beta} \mathcal{P}_{t/2}(x, y) \partial_t^\beta \mathcal{P}_{t/2} f(y) dy \right| \\ &\leq C t^{\sigma+\alpha-\beta} \int_{\mathbb{R}^n} \frac{1}{(|y|+t)^{n+\sigma-\beta}} dy = C t^\alpha. \end{aligned}$$

Suppose now that  $\alpha < \sigma < \beta$ . Let  $k$  be the least positive integer for which  $\sigma < \beta \leq \sigma + k$ . Applying the case just proved above, we get

$$\begin{aligned} |t^\sigma \partial_t^\sigma \mathcal{P}_t f(x)| &\leq t^\sigma \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty |\partial_{s_k}^{k+\sigma} \mathcal{P}_{s_k} f(x)| ds_k \cdots ds_2 ds_1 \\ &\leq C t^\sigma \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty s_k^{\alpha-(k+\sigma)} ds_k \cdots ds_2 ds_1 = C t^\alpha. \quad \square \end{aligned}$$

**Lemma 5.7.** Let  $0 < \alpha \leq 1$ . If a function  $f$  satisfies  $|f(x)| \leq C \rho(x)^\alpha$  for all  $x \in \mathbb{R}^n$  then for any  $\beta > \alpha$ ,  $\|t^\beta \partial_t^\beta (\mathcal{P}_t - P_t) f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ , for all  $t > 0$ , where  $P_t$  is the classical Poisson semigroup (1.2) with kernel (3.11).

**Proof.** Let  $\beta > \alpha$  and  $m = [\beta] + 1$ . In a parallel way as in (3.12), we can derive a formula for the kernel  $D_\beta(x, y, t)$  of the operator  $t^\beta \partial_t^\beta (\mathcal{P}_t - P_t)$  in terms of the heat kernels for  $\mathcal{L}$  and  $-\Delta$  given in (3.4) and (3.6):

$$\begin{aligned} D_\beta(x, y, t) &= t^\beta \partial_t^\beta \int_0^\infty \frac{t e^{-\frac{t^2}{4u}}}{2\sqrt{\pi}} (k_u(x, y) - h_u(x - y)) \frac{du}{u^{3/2}} \\ &= C t^\beta \int_0^\infty \int_0^\infty H_{m+1}\left(\frac{t+s}{2\sqrt{u}}\right) e^{-\frac{(t+s)^2}{4u}} \left(\frac{1}{\sqrt{u}}\right)^{m+1} s^{m-\beta} \frac{ds}{s} (k_u(x, y) - h_u(x - y)) \frac{du}{u^{1/2}}. \end{aligned}$$

Then, by Lemma 3.3, we have

$$\begin{aligned} |D_\beta(x, y, t)| &\leq C t^\beta \int_0^\infty \int_0^\infty e^{-c \frac{(t+s)^2}{4u}} \left(\frac{1}{\sqrt{u}}\right)^{m+1} s^{m-\beta} \frac{ds}{s} |k_u(x, y) - h_u(x - y)| \frac{du}{u^{1/2}} \\ &\leq C \int_0^\infty e^{-c \frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta \left(\frac{\sqrt{u}}{\rho(y)}\right)^\alpha w_u(x - y) \frac{du}{u}, \end{aligned}$$

where the function  $w \in \mathcal{S}$  is nonnegative. Hence, for all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} |t^\beta \partial_t^\beta (\mathcal{P}_t - P_t) f(x)| &\leq C \int_{\mathbb{R}^n} \int_0^\infty e^{-c \frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta \left(\frac{\sqrt{u}}{\rho(y)}\right)^\alpha w_u(x - y) \frac{du}{u} \rho(y)^\alpha dy \\ &\leq C \int_0^\infty e^{-c \frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta (\sqrt{u})^\alpha \frac{du}{u} = C t^\alpha \int_0^\infty e^{-v} v^{\frac{\beta-\alpha}{2}} \frac{dv}{v} = C t^\alpha. \quad \square \end{aligned}$$

**Proof of Proposition 5.5.** Assume that  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ , for any  $\beta > \alpha$ . Then, by Lemma 5.7, we obtain  $\|t^\beta \partial_t^\beta P_t f\|_{L^\infty(\mathbb{R}^n)} \leq \|t^\beta \partial_t^\beta (P_t - \mathcal{P}_t) f\|_{L^\infty(\mathbb{R}^n)} + \|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ . Therefore, as  $f$  is bounded,  $f$  is in the classical  $\alpha$ -Lipschitz space  $\Lambda^\alpha$ , see [13]. Hence  $|f(x+y) + f(x-y) - 2f(x)| \leq C|y|^\alpha$ , for all  $x, y \in \mathbb{R}^n$ .

For the converse, since  $f \in L^\infty(\mathbb{R}^n)$ , then, by [13],  $\|t^2 \partial_t^2 P_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ . So Lemma 5.7 gives  $\|t^2 \partial_t^2 \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq \|t^2 \partial_t^2 (P_t - \mathcal{P}_t) f\|_{L^\infty(\mathbb{R}^n)} + \|t^2 \partial_t^2 P_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$ . Thus, by Lemma 5.6, we get  $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leq C t^\alpha$  for any  $\beta > \alpha$ .  $\square$

#### 5.4. Proof of Theorem 1.5(A)

As explained at the beginning of this section, we only need to prove the necessity part. Let  $f \in BMO_{\mathcal{L}}$ . Let us fix a ball  $B = B(x_0, r)$  and write  $f = f_1 + f_2 + f_3$ , with  $f_1 = (f - f_B)\chi_{2B}$ ,  $f_2 = (f - f_B)\chi_{(2B)^c}$  and  $f_3 = f_B$ .

For  $f_1$ , by the boundedness of the area function (5.1) on  $L^2(\mathbb{R}^n)$  and Remark 4.2 with  $p = 2$ ,

$$\begin{aligned} \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \frac{dx dt}{t} &= \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \int_{\mathbb{R}^n} \chi_{|x-z|<t}(z) \frac{dx dt}{t^{n+1}} \\ &\leq \frac{1}{|B|} \int_{|x_0-z|<2r} \int_0^\infty \int_{\mathbb{R}^n} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \chi_{|x-z|<t}(z) \frac{dx dt}{t^{n+1}} dz \\ &= \frac{1}{|B|} \int_{|x_0-z|<2r} \iint_{\Gamma(z)} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \frac{dx dt}{t^{n+1}} dz \leq \frac{C}{|B|} \int_{2B} |f(z) - f_B|^2 dz \leq C \|f\|_{BMO_{\mathcal{L}}}^2. \end{aligned}$$

For  $f_2$  and  $x \in B$ , apply Proposition 3.6(b) and the classical annuli argument to get

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t f_2(x)| &\leq C \sum_{k=2}^\infty \int_{2^k B \setminus 2^{k-1} B} |f(z) - f_{2^k B}| \frac{t^\beta}{(t + |x - z|)^{n+\beta}} dz \\ &\quad + \sum_{k=2}^\infty \sum_{j=1}^k |f_{2^j B} - f_{2^{j-1} B}| \int_{2^k B \setminus 2^{k-1} B} \frac{t^\beta}{(t + |x - z|)^{n+\beta}} dz \\ &\leq C \left(\frac{t}{r}\right)^\beta \left( \sum_{k=2}^\infty \frac{1}{2^{k\beta}} \frac{1}{(2^k r)^n} \int_{2^k B} |f(z) - f_{2^k B}| dz + \|f\|_{BMO_{\mathcal{L}}} \sum_{k=2}^\infty \frac{k}{2^{k\beta}} \right) \\ &\leq C \left(\frac{t}{r}\right)^\beta \|f\|_{BMO_{\mathcal{L}}} \sum_{k=2}^\infty \frac{1+k}{2^{k\beta}} = C \left(\frac{t}{r}\right)^\beta \|f\|_{BMO_{\mathcal{L}}}. \end{aligned}$$

Therefore  $\frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_2(x)|^2 \frac{dx dt}{t} \leq C \|f\|_{BMO_{\mathcal{L}}}^2 \int_0^r \left(\frac{t}{r}\right)^{2\beta} \frac{dt}{t} = C \|f\|_{BMO_{\mathcal{L}}}^2$ .

Let us finally consider  $f_3$ . Assume that  $r \geq \rho(x_0)$ . By Proposition 3.6(d), for some  $0 < \delta' \leq \delta$  with  $\delta' < \beta$ , we have

$$|t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)| \leq C |f_B| \frac{(t/\rho(x))^{\delta'}}{(1 + t/\rho(x))^N} \leq C \|f\|_{BMO_{\mathcal{L}}} \frac{(t/\rho(x))^{\delta'}}{(1 + t/\rho(x))^N}.$$

Hence

$$\begin{aligned} \frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \frac{dx dt}{t} &\leq C \|f\|_{BMO_{\mathcal{L}}}^2 \frac{1}{|B|} \int_{\widehat{B}} \frac{(t/\rho(x))^{2\delta'}}{(1 + t/\rho(x))^{2N}} \frac{dx dt}{t} \\ &\leq C \|f\|_{BMO_{\mathcal{L}}}^2 \frac{1}{|B|} \int_B \left( \int_0^{\rho(x)} + \int_{\rho(x)}^\infty \right) \frac{(t/\rho(x))^{2\delta'}}{(1 + t/\rho(x))^{2N}} \frac{dt}{t} dx. \end{aligned} \quad (5.9)$$

On one hand,

$$\int_0^{\rho(x)} \frac{(t/\rho(x))^{2\delta'}}{(1 + t/\rho(x))^{2N}} \frac{dt}{t} \leq \int_0^{\rho(x)} (t/\rho(x))^{2\delta'} \frac{dt}{t} = C.$$

On the other hand,

$$\int_{\rho(x)}^\infty \frac{(t/\rho(x))^{2\delta'}}{(1 + t/\rho(x))^{2N}} \frac{dt}{t} \leq \int_{\rho(x)}^\infty (t/\rho(x))^{2\delta'-2N} \frac{dt}{t} = C.$$

Therefore from (5.9) we obtain that if  $r \geq \rho(x_0)$  then  $\frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \frac{dx dt}{t} \leq C \|f\|_{BMO_{\mathcal{L}}}^2$ . Suppose that  $r < \rho(x_0)$ . By Remark 4.4, Proposition 3.6(d) with some  $\delta' > 1/2$  and Lemma 3.1, we get

$$\begin{aligned}
\frac{1}{|B|} \int_{\frac{B}{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \frac{dx dt}{t} &\leq C \|f\|_{BMO_{\mathcal{L}}}^2 \left(1 + \log \frac{\rho(x_0)}{r}\right)^2 \frac{1}{|B|} \int_{\frac{B}{B}} \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}} \frac{dx dt}{t} \\
&\leq C \|f\|_{BMO_{\mathcal{L}}}^2 \left(1 + \log \frac{\rho(x_0)}{r}\right)^2 \frac{1}{|B|} \int_B \int_0^r (t/\rho(x_0))^{2\delta'} \frac{dt}{t} dx \\
&= C \|f\|_{BMO_{\mathcal{L}}}^2 \left(1 + \log \frac{\rho(x_0)}{r}\right)^2 \left(\frac{r}{\rho(x_0)}\right)^{2\delta'} \leq C \|f\|_{BMO_{\mathcal{L}}}^2,
\end{aligned}$$

for all  $r < \rho(x_0)$ . This finishes the proof.

### 5.5. Proof of Theorem 1.5(B)

As in the argument of the proof of Theorem 1.4(II), we only need to consider the case  $n = 1$ . We will take  $\beta = 1$ . Let  $f(x) = \max\{\log \frac{1}{|x|}, 0\}$ ,  $x \in \mathbb{R}$ . It is well known that  $f$  belongs to the classical  $BMO(\mathbb{R})$ . Observe that the function  $f$  is nonnegative and it is supported in  $[-1, 1]$ . For every  $x$  we have  $\rho(x) = \frac{1}{\sqrt{\mu}}$ . Hence, for  $r \geq \rho(x)$  and  $B(x_0, r) = [x_0 - r, x_0 + r]$ ,  $\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |f(x)| dx \leq \frac{1}{2r} \int_{B(0, 1)} |f(x)| dx \leq C \sqrt{\mu}$ . So  $f \in BMO_{\mathcal{L}\mu}$ . Now,

$$\begin{aligned}
t \partial_t \mathcal{P}_t f(0) &= C \int_0^\infty t \left(1 - \frac{t^2}{2s}\right) \frac{e^{-t^2/(4s)}}{s^{3/2}} \int_{|y|<1} \frac{e^{-y^2/(4s)}}{s^{1/2}} (-\log |y|) dy e^{-s\mu} ds \\
&= C \int_0^\infty w^2 (1 - w^2) e^{-w^2/2} \int_{|zt|<1} \frac{e^{-(zw)^2/2}}{s^{1/2}} (-\log |zt|) dz e^{-\frac{t^2}{2w^2}\mu} \frac{dw}{w} \\
&= C \int_0^\infty w (1 - w^2) e^{-w^2/2} \int_{|zt|<1} e^{-(zw)^2/2} (-\log |z|) dz e^{-\frac{t^2}{2w^2}\mu} dw \\
&\quad + C \int_0^\infty w (1 - w^2) e^{-w^2/2} \int_{|zt|<1} e^{-(zw)^2/2} (-\log |t|) dz e^{-\frac{t^2}{2w^2}\mu} dw =: I + II.
\end{aligned}$$

Observe that

$$\begin{aligned}
|I| &\leq C \int_0^\infty w e^{-w^2/c} \int_{\mathbb{R}} e^{-(zw)^2/2} |\log |z|| dz dw \\
&\leq C \int_0^\infty w e^{-w^2/c} \left( \int_{|z|<1} (-\log |z|) dz + \int_{|z|>1} e^{-(zw)^2/2} |z|^\delta dz \right) dw \\
&\leq C \int_0^\infty w e^{-w^2/c} \left( 1 + \frac{1}{w^\delta} \right) dw \leq C,
\end{aligned}$$

where  $\delta < 1$ . For the second integral,

$$|III| \leq C |\log |t|| \int_0^\infty w e^{-w^2/c} \int_{\mathbb{R}} e^{-(zw)^2/2} dz dw = C |\log |t|| \int_0^\infty e^{-w^2/c} dw = C |\log |t||.$$

Therefore the two integrals that define  $t \partial_t \mathcal{P}_t f(0)$  are (absolutely) convergent. The limit when  $t \rightarrow 0$  of the second term  $II$  above is infinity. Thus  $t \partial_t \mathcal{P}_t f(0) \rightarrow \infty$  as  $t \rightarrow 0$ .

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