



Quasiconformal mappings and sharp estimates for the distance to L^∞ in some function spaces

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ABSTRACT

We provide several estimates which involve the distance to L^∞ in some function spaces, the composition operator induced by a quasiconformal mapping and the logarithm of the Jacobian of a quasiconformal mapping. Our results are sharp in the two dimensional case.

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1. Introduction and main results

Let Ω be a domain of \mathbb{R}^n with $n \geq 2$. A homeomorphism $f : \Omega \rightarrow \mathbb{R}^n$ is a K -quasiconformal mapping for a constant $K \geq 1$ if $f \in W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$ and

$$|Df(x)|^n \leq K J_f(x) \quad \text{for a.e. } x \in \Omega. \quad (1)$$

Here $Df(x)$ stands for the differential matrix of f and $J_f(x) = \det Df(x)$ denotes the Jacobian determinant of f . The norm $|Df(x)|$ of $Df(x)$ in (1) is defined as $|Df(x)| = \sup \{|Df(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\}$.

Let $BMO(\mathbb{R}^n)$ be the space of functions of bounded mean oscillation in \mathbb{R}^n . We recall that a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to $BMO(\mathbb{R}^n)$ if u is locally integrable in \mathbb{R}^n and satisfies

$$\|u\|_{BMO(\mathbb{R}^n)} = \sup_Q \int_Q |u(x) - u_Q| dx < \infty. \quad (2)$$

The supremum in (2) is taken over all cubes $Q \subset \mathbb{R}^n$. Here and in what follows the notation

$$u_E = \int_E u(x) dx = \frac{1}{|E|} \int_E u(x) dx,$$

is used whenever $E \subset \mathbb{R}^n$ is a Lebesgue measurable set of positive bounded measure $|E|$.

Let us introduce the composition operator $T_f[u] = u \circ f^{-1}$ where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a K -quasiconformal mapping and $u \in BMO(\mathbb{R}^n)$. It is well-known that T_f maps $BMO(\mathbb{R}^n)$ into itself continuously, as stated by the following result of Reimann [27].

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Theorem 1.1 ([27]). Let $n \geq 2$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping for some constant $K \geq 1$. Then there exists $C = C(n, K) \geq 1$ such that, for every $u \in \text{BMO}(\mathbb{R}^n)$,

$$\frac{1}{C(n, K)} \|u\|_{\text{BMO}(\mathbb{R}^n)} \leq \|u \circ f^{-1}\|_{\text{BMO}(\mathbb{R}^n)} \leq C(n, K) \|u\|_{\text{BMO}(\mathbb{R}^n)}. \quad (3)$$

It is worth pointing out that spaces of functions of bounded mean oscillation are not the only ones which are stable under quasiconformal changes of variables. Indeed, quasiconformal mappings (and their suitable generalizations) turn to be the class of homeomorphisms for which the composition operator acts continuously between Sobolev spaces (see [13, 16, 29–31] and the references therein), logarithmic Orlicz–Sobolev spaces (see [15]), fractional Sobolev spaces (see [17, 22]), spaces of functions which are absolutely continuous (see [14]). More than that, the study of composition operators between Sobolev spaces seems to have a connection with the problem of the regularity of the inverse of a Sobolev homeomorphism considered for instance in [7, 10, 18, 19, 24].

Let us introduce the distance of a function $u \in \text{BMO}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ as

$$\text{dist}_{\text{BMO}(\mathbb{R}^n)}(u, L^\infty(\mathbb{R}^n)) = \inf_{\varphi \in L^\infty(\mathbb{R}^n)} \|u - \varphi\|_{\text{BMO}(\mathbb{R}^n)}. \quad (4)$$

The first result of this paper provides quantitative estimates as in (3) where the BMO norms are replaced by the distances in BMO to L^∞ . More precisely, given a function $u \in \text{BMO}(\mathbb{R}^n)$, we consider the quantity

$$\varepsilon(u) = \inf \left\{ \lambda > 0 : \sup_Q \int_Q \exp \frac{|u - u_Q|}{\lambda} dx < \infty \right\}, \quad (5)$$

introduced in [11] by Garnett and Jones. The supremum in (5) is taken over all cubes $Q \subset \mathbb{R}^n$. The main result of [11] states that $\varepsilon(\cdot)$ is equivalent to the distance to $L^\infty(\mathbb{R}^n)$ in the space $\text{BMO}(\mathbb{R}^n)$ defined in (4). Before we state our results, we need to recall a well-known property of quasiconformal mappings. More precisely, as a corollary of the Gehring's Lemma [12], for fixed $K \geq 1$ and $n \geq 2$, there exist positive constant $C_0 = C_0(n, K)$ and $\alpha = \alpha(n, K)$ with $0 < \alpha \leq 1 \leq C_0$ such that, for every K -quasiconformal mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\frac{|f(E)|}{|f(Q)|} \leq C_0 \left(\frac{|E|}{|Q|} \right)^\alpha \quad \text{if } Q \subset \mathbb{R}^n \text{ is a cube and } E \subset Q \text{ is a measurable set.} \quad (6)$$

Finally, let

$$C(n, K) = \frac{1}{\alpha(n, K)}. \quad (7)$$

We are in a position to state our first result.

Theorem 1.2. Let $n \geq 2$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping. Then, for every $u \in \text{BMO}(\mathbb{R}^n)$,

$$\varepsilon(u \circ f^{-1}) \leq C(n, K) \varepsilon(u), \quad (8)$$

where $C(n, K)$ is the constant defined in (7).

We mention here that to each function $u \in \text{BMO}(\mathbb{R}^n)$ there corresponds a weight in the A_∞ class of Muckenhoupt (see Section 2 for the definition) given by $e^{\delta u}$, for some $\delta > 0$ depending on n and $\|u\|_{\text{BMO}}$. For this reason, Theorem 1.2 is deeply related to a paper by Johnson and Neugebauer [21] where the composition problem for the classes of Muckenhoupt is treated (for the definitions of such classes see again Section 2).

Let us point out that some sharp results for planar quasiconformal mappings (see Theorem 1.1 in [1] and Corollary 10 in [3]) implies that

$$C(2, K) = K.$$

Therefore, in the two-dimensional case, the inequality (8) reads as

$$\varepsilon(u \circ f^{-1}) \leq K \varepsilon(u) \quad \text{for every } u \in \text{BMO}(\mathbb{R}^2). \quad (9)$$

We address that the above estimate is sharp, in the sense that there exists a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a function $u \in \text{BMO}(\mathbb{R}^2)$ such that the inequality (9) occurs with equal sign. This is shown by means of Example 1.

With regard to the Jacobian determinant J_f of a quasiconformal mapping in [27] it is proved that if $n \geq 2$ and if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a K -quasiconformal mapping then $\log J_f \in \text{BMO}(\mathbb{R}^n)$. Furthermore, in [28] it is proved that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a K -quasiconformal mapping then there exists a constant $B = B(K)$ depending only on K such that

$$\|\log J_f\|_{\text{BMO}(\mathbb{R}^2)} \leq B(K). \quad (10)$$

Moreover, the constant $B(K)$ is of the form

$$B(K) = \sigma \int_1^K \frac{C(\kappa)}{\kappa} d\kappa,$$

where σ is some universal constant and $C = C(\cdot)$ is the constant $C(2, K)$ which appears in (3) when $n = 2$.

Our next result provides quantitative estimates close to (10).

Theorem 1.3. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a K -quasiconformal mapping. Then*

$$\varepsilon(\log J_f) \leq K - 1. \quad (11)$$

We address that the above result is sharp, in the sense that there exists a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that the inequality (11) occurs with equal sign. This is shown by means of Example 2.

Now, we want to extend our results to a space which is close to $\text{BMO}(\mathbb{R}^n)$. Let Ω be a bounded domain of \mathbb{R}^n . The exponential Orlicz space $\text{EXP}(\Omega)$ is the set of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx < \infty.$$

We recall (see e.g. [26]) that $\text{EXP}(\Omega)$ is a Banach space equipped with the *Luxemburg norm* defined as

$$\|u\|_{\text{EXP}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx \leq 2 \right\}. \quad (12)$$

We also remark that $L^\infty(\Omega)$ is not a dense subspace of $\text{EXP}(\Omega)$ (see e.g. [26]). Similarly to (4) the distance to $L^\infty(\Omega)$ in the space $\text{EXP}(\Omega)$ is defined as

$$\text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) = \inf_{\varphi \in L^\infty(\Omega)} \|u - \varphi\|_{\text{EXP}(\Omega)}. \quad (13)$$

Appealing to the results in [5,9], the distance to $L^\infty(\Omega)$ in $\text{EXP}(\Omega)$ evaluated with respect to the Luxemburg norm (12) is given by

$$\text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx < \infty \right\},$$

for every $u \in \text{EXP}(\Omega)$.

The link between the spaces $\text{BMO}(\mathbb{R}^n)$ and $\text{EXP}(\Omega)$ is given by a result of Iwaniec and Sbordone in [20] which states that a measurable function $u : \Omega \rightarrow \mathbb{R}$ belongs to $\text{EXP}(\Omega)$ if and only if there exists $v \in \text{BMO}(\mathbb{R}^n)$ such that

$$|u(x)| \leq v(x) \quad \text{for a.e. } x \in \Omega.$$

This implies that $u \in \text{EXP}(\Omega)$ if and only if $u \circ f^{-1} \in \text{EXP}(f(\Omega))$ provided $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a quasiconformal mapping (see Lemma 3.2 in [8]).

Our next theorems are similar to Theorems 1.2 and 1.3 respectively and feature the space of exponentially integrable functions.

Theorem 1.4. *Let $n \geq 2$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping and let Ω be bounded domain of \mathbb{R}^n . Then, for every $u \in \text{EXP}(\Omega)$,*

$$\text{dist}_{\text{EXP}(f(\Omega))}(u \circ f^{-1}, L^\infty(f(\Omega))) \leq C(n, K) \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)), \quad (14)$$

where $C(n, K)$ is the constant defined in (7).

In the planar case the inequality (14) reads as

$$\text{dist}_{\text{EXP}(f(\Omega))}(u \circ f^{-1}, L^\infty(f(\Omega))) \leq K \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) \quad \text{if } \Omega \subset \mathbb{R}^2 \text{ and for every } u \in \text{EXP}(\Omega). \quad (15)$$

This estimate is proved in [8] and it is sharp in the sense that, given any bounded domain $\Omega \subset \mathbb{R}^2$, there exist a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a function $u \in \text{EXP}(\Omega)$ such that the inequality (15) occurs with equal sign. This is shown by means of Example 3.3 in [8] and of Example 3, where a larger class of quasiconformal mappings is considered.

Theorem 1.5. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a K -quasiconformal mapping and let Ω be a bounded open subset of \mathbb{R}^2 . Then*

$$\text{dist}_{\text{EXP}(\Omega)}(\log J_f, L^\infty(\Omega)) \leq K - 1. \quad (16)$$

We address that the above result is sharp, in the sense that there exists a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that the inequality (16) occurs with equal sign. This is shown by means of Example 4.

2. Definitions and preliminary results

We will need to recall the basic properties of the *Muckenhoupt class* A_p (see [25]). A function $w : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a *weight* if w is positive a.e. and locally integrable in \mathbb{R}^n . A weight w belongs to the Muckenhoupt class A_p for $1 < p < \infty$ if

$$A_p(w) = \sup_Q \left(\int_Q w \, dx \right) \left(\int_Q w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty. \quad (17)$$

The supremum in (17) is taken over all cubes $Q \subset \mathbb{R}^n$. We say that $A_p(w)$ is the A_p -constant of w . The Muckenhoupt class A_∞ is defined as

$$A_\infty = \bigcup_{p>1} A_p.$$

It should be mentioned that an example of weight in A_p ($1 < p < \infty$) is given by the function

$$w(x) = |x|^\alpha \quad \forall x \in \mathbb{R}^n, \quad (18)$$

if α is in the range

$$-n < \alpha < n(p-1). \quad (19)$$

Let us mention here (see for instance [6]) that $w \in A_\infty$ if and only if for every cube $Q \subset \mathbb{R}^n$ and every measurable set $E \subset Q$ it holds

$$\frac{\int_E w(x) \, dx}{\int_Q w(x) \, dx} \leq C_0 \left(\frac{|E|}{|Q|} \right)^\alpha, \quad (20)$$

for some $0 < \alpha \leq 1 \leq C_0$.

It is straightforward to prove the following result (already stated in [11]), which links the space of functions of bounded mean oscillation and A_2 weights.

Proposition 2.1. *Let $u \in \text{BMO}(\mathbb{R}^n)$. There exists $\lambda > 0$ such that $e^{\frac{u}{\lambda}} \in A_2$. Moreover*

$$\varepsilon(u) = \inf \left\{ \lambda > 0 : e^{\frac{u}{\lambda}} \in A_2 \right\}. \quad (21)$$

We will also need some well-known properties of quasiconformal mappings. Our main references here will be [2,4,23,29]. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping. As already mentioned, the Jacobian of f is a weight in A_∞ . This is equivalent to (6) since f satisfies the identity

$$|f(G)| = \int_G J_f(x) \, dx,$$

for every Lebesgue measurable set $G \subset \mathbb{R}^n$. More generally, it is possible to prove that suitable positive powers of the Jacobian of a planar quasiconformal mapping are A_p weights, as shown by the following theorem, proved in Theorem 12.4.2 in [2].

Theorem 2.2. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a K -quasiconformal mapping. Then, for every $s \in (0, \frac{K}{K-1})$,*

$$J_f^s \in A_p \quad \text{for every } p > 1 + (K-1)s. \quad (22)$$

As a consequence of the sharp result of Astala [1], it is possible to establish the optimal integrability for positive and negative powers of the Jacobian of a planar quasiconformal mapping. This is shown in the following theorem.

Theorem 2.3. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a K -quasiconformal mapping. If $E \subset \mathbb{R}^2$ is bounded domain then*

$$J_f^p \in L^1(E) \quad \text{for } p \in \left(0, \frac{K}{K-1} \right), \quad (23)$$

$$\frac{1}{J_f^b} \in L^1(E) \quad \text{for } b \in \left(0, \frac{1}{K-1} \right). \quad (24)$$

Finally, we need to recall a property of the image of a cube under a quasiconformal mapping, proved in Lemma 4 in [12] (see also Lemma 4 in [27]).

Lemma 2.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping and let $P' \subset \mathbb{R}^n$ be a cube. There exists a cube $Q \subset \mathbb{R}^n$ and a constant $C = C(n, K) \geq 1$ depending only on n and K such that $P' \subset f(Q)$ and $|f(Q)| \leq C(n, K)|P'|$.*

3. Proofs

We dedicate this section to the proofs of [Theorems 1.2–1.5](#). Various constants which occur at each stage of the proofs are denoted by C_1, C_2, \dots .

Proof of Theorem 1.2. Let $u \in \text{BMO}(\mathbb{R}^n)$. Let us fix a constant λ such that

$$\lambda > \varepsilon(u), \quad (25)$$

and let us define

$$s(u, \lambda) = \sup_Q \int_Q \exp \frac{|u - u_Q|}{\lambda} dx. \quad (26)$$

The supremum in (26) is taken over all cubes $Q \subset \mathbb{R}^n$. By (25) we see that $s(u, \lambda) < \infty$. Let us fix some cube $P' \subset \mathbb{R}^n$. From [Lemma 2.4](#) in Section 2 there exists a cube $Q \subset \mathbb{R}^n$ such that

$$P' \subset f(Q), \quad (27)$$

and

$$|f(Q)| \leq C_1 |P'|, \quad (28)$$

for some constant $C_1 = C_1(n, K)$ depending only on n and K . Consider the sets E_k defined as

$$E_k = \{x \in Q : k \leq |u(x) - u_Q| < k + 1\} \quad \text{for } k = 0, 1, 2, \dots \quad (29)$$

Therefore

$$\begin{aligned} s(u, \lambda) &\geq \int_Q \exp \frac{|u(x) - u_Q|}{\lambda} dx \\ &= \sum_{k=0}^{\infty} \frac{1}{|Q|} \int_{E_k} \exp \frac{|u(x) - u_Q|}{\lambda} dx \\ &\geq \sum_{k=0}^{\infty} \frac{|E_k|}{|Q|} \exp \frac{k}{\lambda}. \end{aligned} \quad (30)$$

Let us fix some constant μ such that

$$\mu > C(n, K)\lambda, \quad (31)$$

where $C(n, K)$ is the constant defined in (7). First observe that

$$\int_{P'} \exp \frac{|u \circ f^{-1}(z) - (u \circ f^{-1})_{P'}|}{\mu} dz \leq \exp \left(\frac{|(u \circ f^{-1})_{P'} - u_Q|}{\mu} \right) \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz. \quad (32)$$

By means of the Jensen inequality we observe that

$$\begin{aligned} \exp \left(\frac{|(u \circ f^{-1})_{P'} - u_Q|}{\mu} \right) &= \exp \left(\left| \int_{P'} \frac{u \circ f^{-1}(z) - u_Q}{\mu} dz \right| \right) \\ &\leq \exp \left(\int_{P'} \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz \right) \\ &\leq \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz. \end{aligned}$$

The estimate above and (32) yields

$$\int_{P'} \exp \frac{|u \circ f^{-1}(z) - (u \circ f^{-1})_{P'}|}{\mu} dz \leq \left(\int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz \right)^2. \quad (33)$$

Let us define

$$G_k = \{z \in P' : k \leq |u \circ f^{-1}(z) - u_Q| < k + 1\} \quad \text{for } k = 0, 1, 2, \dots \quad (34)$$

This clearly implies

$$\begin{aligned} \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz &= \sum_{k=0}^{\infty} \frac{1}{|P'|} \int_{G_k} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz \\ &\leq \sum_{k=0}^{\infty} \frac{|G_k|}{|P'|} \exp \frac{k+1}{\mu}. \end{aligned} \quad (35)$$

Recalling the definition of the sets E_k given by (29) and using the relations (27), (28) and (35) we get

$$\begin{aligned} \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz &\leq C_2 \sum_{k=0}^{\infty} \frac{|\{z \in f(Q) : k \leq |u \circ f^{-1}(z) - u_Q| < k+1\}|}{|f(Q)|} \exp \frac{k+1}{\mu} \\ &= C_2 \sum_{k=0}^{\infty} \frac{|f(E_k)|}{|f(Q)|} \exp \frac{k+1}{\mu}. \end{aligned} \quad (36)$$

In what follows, we simply denote the constant $\alpha(n, K)$ in (6) by α . It follows from (6) and from the definition of $C(n, K)$ given by (7) that

$$\begin{aligned} \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz &\leq C_3 \sum_{k=0}^{\infty} \left(\frac{|E_k|}{|Q|} \right)^{\alpha} \exp \frac{k+1}{\mu} \\ &= C_3 \sum_{k=0}^{\infty} \left[\left(\frac{|E_k|}{|Q|} \right)^{\alpha} \exp \frac{k\alpha}{\lambda} \right] \exp \left(\frac{k+1}{\mu} - \frac{k\alpha}{\lambda} \right). \end{aligned} \quad (37)$$

Taking into account that the conjugate exponent of $p = 1/\alpha$ is given by $p' = 1/(1 - \alpha)$, we use first Hölder inequality and subsequently (30) to obtain

$$\begin{aligned} \int_{P'} \exp \frac{|u \circ f^{-1}(z) - u_Q|}{\mu} dz &\leq C_4 \left\{ \sum_{k=0}^{\infty} \frac{|E_k|}{|Q|} \exp \frac{k}{\lambda} \right\}^{\alpha} \left\{ \sum_{k=0}^{\infty} \exp \left[\frac{1}{1-\alpha} \left(\frac{k+1}{\mu} - \frac{k\alpha}{\lambda} \right) \right] \right\}^{1-\alpha} \\ &\leq C_5 [s(u, \lambda)]^{\alpha} e^{1/\mu} \left\{ \sum_{k=0}^{\infty} \theta^k \right\}^{1-\alpha}, \end{aligned} \quad (38)$$

where θ is defined by

$$\theta = \exp \left[\frac{1}{1-\alpha} \left(\frac{1}{\mu} - \frac{\alpha}{\lambda} \right) \right].$$

From (31) it follows that $\theta \in (0, 1)$ which readily implies $\sum_{k=0}^{\infty} \theta^k = 1/(1 - \theta)$. Hence, from (33) and (38) we may conclude that there exists a constant C_6 independent of P' such that

$$\int_{P'} \exp \frac{|u \circ f^{-1}(z) - (u \circ f^{-1})_{P'}|}{\mu} dz \leq C_6, \quad (39)$$

provided (31) holds. Let I be the set defined as

$$I = \left\{ \mu > 0 : \sup_{P'} \int_{P'} \exp \frac{|u \circ f^{-1}(z) - (u \circ f^{-1})_{P'}|}{\mu} dz < \infty \right\}. \quad (40)$$

The supremum in (40) is taken over all cubes $P' \subset \mathbb{R}^n$. The inclusion

$$(C(n, K)\lambda, \infty) \subset I,$$

is proved. We observe that $\inf I = \varepsilon(u \circ f^{-1})$ and we conclude that

$$\varepsilon(u \circ f^{-1}) \leq C(n, K)\lambda. \quad (41)$$

Since λ is any constant satisfying (25), we may pass to the limit for $\lambda \searrow \varepsilon(u)$ in (41) and finally get the desired inequality (8). \square

Proof of Theorem 1.3. We want to estimate $\varepsilon(\log J_f)$ by means of (21). Let us fix some constant λ such that

$$\lambda > K - 1, \quad (42)$$

and define $s = 1/\lambda$. Hence $0 < s < 1/(K - 1)$. Observe that

$$e^{\frac{\log J_f}{\lambda}} = J_f^s.$$

Now we appeal to (22). By (42) we see that $1 + (K - 1)s < 2$. In particular, we conclude that $J_f^s \in A_2$. Let I be the set defined as

$$I = \{\lambda > 0 : J_f^s \in A_2\}.$$

The inclusion

$$(K - 1, \infty) \subset I,$$

is proved. The estimate (16) follows from the fact that $\varepsilon(\log J_f) = \inf I$. The proof is complete. \square

Proof of Theorem 1.4. Let $u \in \text{EXP}(\Omega)$. Let us fix a constant λ such that

$$\lambda > \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)), \quad (43)$$

Consider the sets E_k defined as

$$E_k = \{x \in \Omega : k \leq |u(x)| < k + 1\} \quad \text{for } k = 0, 1, 2, \dots \quad (44)$$

Therefore

$$\infty > \int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx = \sum_{k=0}^{\infty} \int_{E_k} \exp \frac{|u(x)|}{\lambda} dx \geq \sum_{k=0}^{\infty} |E_k| \exp \frac{k}{\lambda}. \quad (45)$$

Let us fix a constant μ such that

$$\mu > C(n, K)\lambda, \quad (46)$$

where $C(n, K)$ is the constant defined in (7). This clearly implies

$$\begin{aligned} \int_{f(\Omega)} \exp \frac{|u \circ f^{-1}(z)|}{\mu} dz &= \sum_{k=0}^{\infty} \int_{f(E_k)} \exp \frac{|u \circ f^{-1}(z)|}{\mu} dz \\ &\leq \sum_{k=0}^{\infty} |f(E_k)| \exp \frac{k+1}{\mu}. \end{aligned} \quad (47)$$

In what follows, we simply denote the constant $\alpha(n, K)$ in (6) by α . We fix a cube Q such that $\Omega \subset Q$. It follows from (6) and from the definition of $C(n, K)$ given by (7) that

$$\begin{aligned} \int_{f(\Omega)} \exp \frac{|u \circ f^{-1}(z)|}{\mu} dz &\leq C_1 |f(Q)| \sum_{k=0}^{\infty} \left(\frac{|E_k|}{|Q|} \right)^{\alpha} \exp \frac{k+1}{\mu} \\ &= C_1 |f(Q)| \sum_{k=0}^{\infty} \left[\left(\frac{|E_k|}{|Q|} \right)^{\alpha} \exp \frac{k\alpha}{\lambda} \right] \exp \left(\frac{k+1}{\mu} - \frac{k\alpha}{\lambda} \right). \end{aligned} \quad (48)$$

Taking into account that the conjugate exponent of $p = 1/\alpha$ is given by $p' = 1/(1 - \alpha)$, we use Hölder inequality to obtain

$$\int_{f(\Omega)} \exp \frac{|u \circ f^{-1}(z)|}{\mu} dz \leq C_1 |f(Q)| \left\{ \sum_{k=0}^{\infty} \frac{|E_k|}{|Q|} \exp \frac{k}{\lambda} \right\}^{\alpha} e^{1/\mu} \left\{ \sum_{k=0}^{\infty} \theta^k \right\}^{1-\alpha}, \quad (49)$$

where θ is defined by

$$\theta = \exp \left[\frac{1}{1-\alpha} \left(\frac{1}{\mu} - \frac{\alpha}{\lambda} \right) \right].$$

From (46) it follows that $\theta \in (0, 1)$ which readily implies $\sum_{k=0}^{\infty} \theta^k = 1/(1 - \theta)$. Hence, from (45) and (49) we may conclude that

$$\exp \frac{|u \circ f^{-1}|}{\mu} \in L^1(f(\Omega)),$$

provided (46) holds. Let I be the set defined as

$$I = \left\{ \mu > 0 : \exp \frac{|u \circ f^{-1}|}{\mu} \in L^1(f(\Omega)) \right\}. \quad (50)$$

The inclusion

$$(C(n, K)\lambda, \infty) \subset I,$$

is proved. We observe that $\inf I = \text{dist}_{\text{EXP}(f(\Omega))}(u \circ f^{-1}, L^\infty(f(\Omega)))$ and we conclude that

$$\text{dist}_{\text{EXP}(f(\Omega))}(u \circ f^{-1}) \leq C(n, K)\lambda. \quad (51)$$

Since λ is any constant satisfying (43), we may pass to the limit for $\lambda \searrow \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega))$ in (51) and finally get the desired inequality (14). \square

Proof of Theorem 1.5. The proof relies on the integrability property stated in Theorem 2.3. Let us fix some constant λ such that

$$\lambda > K - 1. \quad (52)$$

We observe that

$$\int_{\Omega} \exp \frac{|\log J_f|}{\lambda} dx = \int_{\{x \in \Omega : J_f < 1\}} J_f^{-\frac{1}{\lambda}} dx + \int_{\{x \in \Omega : J_f \geq 1\}} J_f^{\frac{1}{\lambda}} dx. \quad (53)$$

From (23) and (24) it follows that

$$\exp \frac{|\log J_f|}{\lambda} \in L^1(\Omega).$$

Let I be the set defined as

$$I = \left\{ \lambda > 0 : \exp \frac{|\log J_f|}{\lambda} \in L^1(\Omega) \right\}.$$

The inclusion

$$(K - 1, \infty) \subset I, \quad (54)$$

is proved. The estimate (16) follows from the fact that $\text{dist}_{\text{EXP}(\Omega)}(\log J_f, L^\infty(\Omega)) = \inf I$. The proof is complete. \square

4. Examples

We dedicate this section to the construction of examples proving that statements of Theorems 1.2 and 1.4 in dimension $n = 2$ and Theorems 1.3 and 1.5 are sharp. Our examples rely on the fact that the functionals $\varepsilon(\cdot)$ and $\text{dist}_{\text{EXP}(\Omega)}(\cdot, L^\infty(\Omega))$ satisfy the properties

$$\varepsilon(\alpha u) = |\alpha| \varepsilon(u) \quad \forall u \in \text{BMO}(\mathbb{R}^n) \quad \forall \alpha \in \mathbb{R}, \quad (55)$$

and

$$\text{dist}_{\text{EXP}(\Omega)}(\alpha u, L^\infty(\Omega)) = |\alpha| \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) \quad \forall u \in \text{EXP}(\Omega) \quad \forall \alpha \in \mathbb{R}, \quad (56)$$

and also on the fact that the Jacobian of the radial stretching

$$f(z) = \rho(|z|) \frac{z}{|z|},$$

is given by

$$J_f(z) = \frac{\rho(|z|)\dot{\rho}(|z|)}{|z|}.$$

Here $\rho(\cdot)$ is a smooth increasing function such that $\rho(0) = 0$ and $\dot{\rho}(\cdot)$ is its derivative.

Example 1. We show that for every $K \geq 1$ there exist a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a function $u \in \text{BMO}(\mathbb{R}^2)$ such that (9) holds as an equality. To this aim, it is sufficient to consider the function

$$u(x) = \log |x| \quad \forall x \in \mathbb{R}^2, \quad (57)$$

and to take the radial stretching

$$f(y) = |y|^{\frac{1}{K}} \frac{y}{|y|} \quad \forall y \in \mathbb{R}^2.$$

The inverse of f is given by

$$f^{-1}(z) = |z|^K \frac{z}{|z|} \quad \forall z \in \mathbb{R}^2.$$

Therefore $u \circ f^{-1}$ and u are related by

$$u \circ f^{-1}(z) = Ku(z) \quad \forall z \in \mathbb{R}^2.$$

We appeal to (55) and we get

$$\varepsilon(u \circ f^{-1}) = K\varepsilon(u),$$

as desired.

Example 2. We show that for every $K \geq 1$ there exists a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that (11) holds as an equality. We consider the K -quasiconformal mapping given by

$$f(z) = |z|^K \frac{z}{|z|} \quad \forall z \in \mathbb{R}^2.$$

We compute the Jacobian of f and we get

$$J_f(z) = K|z|^{2(K-1)} \quad \forall z \in \mathbb{R}^2.$$

We want to evaluate $\varepsilon(\log J_f)$ by means of (21). If we replace cubes by balls in (17) we obtain a quantity which is equivalent to the A_p -constant of w . This implies that $w \in A_2$ if and only if

$$\sup_B \left(\int_B w \, dx \right) \left(\int_B \frac{1}{w} \, dx \right) < \infty. \quad (58)$$

The supremum in (58) is taken over all balls $B \subset \mathbb{R}^2$. It is straightforward to see that $w \in A_2$ if and only if $1/w \in A_2$. It is therefore sufficient to establish for which values of $\lambda > 0$ the function $\exp \left\{ -\frac{\log J_f}{\lambda} \right\}$ belongs to A_2 . To this aim, we observe that

$$\exp \left\{ -\frac{\log J_f(z)}{\lambda} \right\} = K^{-\frac{1}{\lambda}} |z|^{-\frac{2(K-1)}{\lambda}}.$$

It is clear from (18) and (19) that $e^{-\frac{\log J_f}{\lambda}} \in A_2$ for $\lambda > K - 1$ while $e^{-\frac{\log J_f}{\lambda}} \notin A_2$ for $\lambda \leq K - 1$ (actually $e^{-\frac{\log J_f}{\lambda}}$ is not even locally integrable on each ball with center in the origin). It follows that $\varepsilon(\log J_f) = K - 1$ as claimed.

Example 3. We show that for every bounded domain Ω of \mathbb{R}^2 and for every $K \geq 1$ there exist a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a function $u \in \text{EXP}(\Omega)$ such that (15) holds as an equality. Up to a translation, we may assume that $0 \in \Omega$. It is sufficient to prove the claimed result with this additional hypothesis, since the distance to L^∞ in the space of exponentially integrable function satisfies the following property

$$\text{dist}_{\text{EXP}(\Omega_0 + \Omega)}(u(x_0 + \cdot), L^\infty(x_0 + \Omega)) = \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) \quad \forall x_0 \in \mathbb{R}^2,$$

where $x_0 + \Omega = \{x \in \mathbb{R}^2 : x - x_0 \in \Omega\}$. Let us take some radius $R > 0$ such that $B_R \Subset \Omega$. Let $u : \Omega \rightarrow \mathbb{R}$ be the function

$$u(x) = -2 \log \frac{|x|}{R} \quad \forall x \in B_R, \quad u(x) = 0 \quad \forall x \in \Omega \setminus B_R.$$

It is easily seen that $u \in \text{EXP}(\Omega)$; especially,

$$\text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) = 1.$$

We consider the K -quasiconformal mapping

$$f(y) = R^{1-\frac{1}{K}} |y|^{\frac{1}{K}} \frac{y}{|y|} \quad \forall y \in B_R, \quad f(y) = y \quad \forall y \in \mathbb{R}^2 \setminus B_R. \quad (59)$$

Since f agrees with the identity outside B_R it follows that $f(\Omega \setminus B_R) = \Omega \setminus B_R$; on the other hand f maps B_R onto itself and this implies that $f(\Omega) = \Omega$. Notice that $f(0) = 0$. The inverse of f is given by

$$f^{-1}(z) = \frac{|z|^K}{R^{K-1}} \frac{z}{|z|} \quad \forall z \in B_R, \quad f(z) = z \quad \forall z \in \Omega \setminus B_R.$$

Therefore $u \circ f^{-1}$ and u are related by

$$u \circ f^{-1}(z) = Ku(z) \quad \forall z \in \mathbb{R}^2.$$

We appeal to (56) and we get

$$\text{dist}_{\text{EXP}(f(\Omega))}(u \circ f^{-1}, L^\infty(f(\Omega))) = K \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega))$$

as desired.

Example 4. We show that for every bounded domain Ω of \mathbb{R}^2 and for every $K \geq 1$ there exists a K -quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that (16) holds as an equality. As for Example 3 we may assume that $0 \in \Omega$ and then fix some $R > 0$ such that $B_R \subseteq \Omega$. We consider the K -quasiconformal mapping given by the inverse of the one defined in (59), namely

$$f(z) = \frac{|z|^K}{R^{K-1}} \frac{z}{|z|} \quad \forall z \in B_R, \quad f(z) = z \quad \forall z \in \Omega \setminus B_R.$$

We compute the Jacobian of f and we get

$$J_f(z) = \frac{K}{R^{2(K-1)}} |z|^{2(K-1)} \quad \forall z \in B_R, \quad J_f(z) = 1 \quad \forall z \in \Omega \setminus B_R,$$

and therefore

$$\log J_f(z) = \log K + 2(K-1) \log \frac{|z|}{R} \quad \forall z \in B_R, \quad \log J_f(z) = 0 \quad \forall z \in \Omega \setminus B_R.$$

Since $\log J_f$ is supported in B_R and since

$$\log J_f(z) = \log K + v(z) \quad \forall z \in B_R,$$

where

$$v(z) = 2(K-1) \log \frac{|z|}{R} \quad \forall z \in B_R,$$

it follows that

$$\text{dist}_{\text{EXP}(\Omega)}(\log J_f, L^\infty(\Omega)) = \text{dist}_{\text{EXP}(B_R)}(v, L^\infty(B_R)). \quad (60)$$

Hence

$$\begin{aligned} \int_{B_R} \exp \frac{|v|}{\lambda} dz &= \int_{B_R} \left(\frac{|z|}{R} \right)^{-\frac{2(K-1)}{\lambda}} dz \\ &= 2\pi \int_0^R \left(\frac{r}{R} \right)^{-\frac{2(K-1)}{\lambda}} r dr \\ &= 2\pi R \int_0^1 t^{1-\frac{2(K-1)}{\lambda}} dt. \end{aligned} \quad (61)$$

A straightforward computation gives us that $\exp \frac{|v|}{\lambda} \in L^1(B_R)$ for $\lambda > K-1$ while $\exp \frac{|v|}{\lambda} \notin L^1(B_R)$ for $\lambda \leq K-1$. Finally, from (60) it follows that

$$\text{dist}_{\text{EXP}(\Omega)}(\log J_f, L^\infty(\Omega)) = K-1$$

as claimed.

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References

- [1] K. Astala, Area distortion of quasiconformal mappings, *Acta Math.* 173 (1) (1994) 37–60.
- [2] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, in: Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009.
- [3] K. Astala, T. Iwaniec, E. Saksman, Beltrami operators in the plane, *Duke Math. J.* 107 (1) (2001) 27–56.
- [4] B. Bojarski, T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in \mathbb{R}^n , *Ann. Acad. Sci. Fenn. Ser. A I Math.* 8 (2) (1983) 257–324.
- [5] M. Carozza, C. Sbordone, The distance to L^∞ in some function spaces and applications, *Differential Integral Equations* 10 (4) (1997) 599–607.
- [6] R.R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* 51 (1974) 241–250.
- [7] M. Csörnyei, S. Hencl, J. Maly, Homeomorphism in the Sobolev space $W^{1,n-1}$, *J. Reine Angew. Math.* 644 (2010) 221–235.

- [8] F. Farroni, R. Giova, Quasiconformal mappings and exponentially integrable functions, *Studia Math.* 203 (2) (2011) 195–203.
- [9] N. Fusco, P.L. Lions, C. Sbordone, Sobolev imbedding theorems in borderline cases, *Proc. Amer. Math. Soc.* 124 (2) (1996) 561–565.
- [10] N. Fusco, G. Moscarriello, C. Sbordone, The limit of $W^{1,1}$ homeomorphism with finite distortion, *Calc. Var. Partial Differential Equations* 33 (3) (2008) 377–390.
- [11] J.B. Garnett, P.W. Jones, The distance in BMO to L^∞ , *Ann. of Math.* (2) 108 (2) (1978) 373–393.
- [12] F.W. Gehring, The L^p -integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* 130 (1973) 265–277.
- [13] V. Gol'dshtein, A. Ukhlov, Weighted Sobolev spaces and embedding theorems, *Trans. Amer. Math. Soc.* 361 (2009) 3829–3850.
- [14] S. Hencl, Absolutely continuous functions of several variables and quasiconformal mappings, *Z. Anal. Anwend.* 22 (4) (2003) 767–778.
- [15] S. Hencl, L. Kleprlik, Composition of q -quasiconformal mappings and functions in Orlicz–Sobolev space, 2011. Preprint.
- [16] S. Hencl, P. Koskela, Mappings of finite distortion: composition operator, *Ann. Acad. Sci. Fenn. Math.* 33 (1) (2008) 65–80.
- [17] S. Hencl, P. Koskela, Composition of quasiconformal mappings and functions in Triebel–Lizorkin spaces, *Math. Nachr.* (in press).
- [18] S. Hencl, P. Koskela, Regularity of the inverse of a planar Sobolev homeomorphism, *Arch. Ration. Mech. Anal.* 180 (1) (2006) 7595.
- [19] S. Hencl, P. Koskela, J. Maly, Regularity of the inverse of a Sobolev homeomorphism in space, *Proc. Roy. Soc. Edinburgh Sect. A* 136 (6) (2006) 1267–1285.
- [20] T. Iwaniec, C. Sbordone, Quasiharmonic fields, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 18 (5) (2001) 519–572.
- [21] R. Johnson, C.J. Neugebauer, Homeomorphisms preserving A_p , *Rev. Mat. Iberoam.* 3 (2) (1987) 249–273.
- [22] P. Koskela, D. Yang, Y. Zhou, Pointwise characterization of Besov and Triebel–Lizorkin spaces and quasiconformal mappings, *Adv. Math.* 226 (4) (2011) 3579–3621.
- [23] O. Lehto, K.I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973.
- [24] G. Moscarriello, A. Passarelli di Napoli, The regularity of the inverse of Sobolev homeomorphism with finite distortion, *J. Geom. Anal.* (in press).
- [25] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* 165 (1972) 207–226.
- [26] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, in: *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 146, Marcel Dekker, Inc., New York, 1991.
- [27] H.M. Reimann, Functions of bounded mean oscillation and quasiconformal mappings, *Comment. Math. Helv.* 49 (1974) 260–276.
- [28] H.M. Reimann, On the parametric representation of quasiconformal mappings, in: *Convegno sulle Trasformazioni Quasiconformi e Questioni Connesse*, INDAM, Rome, 1974, in: *Symposia Mathematica*, vol. XVIII, Academic Press, London, 1976, pp. 421–428.
- [29] S. Rickman, Quasiregular Mappings, in: *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 26 (3), Springer-Verlag, Berlin, 1993.
- [30] S.K. Vodopyanov, A.D. Ukhlov, Weighted Sobolev spaces and quasiconformal mappings, *Dokl. Akad. Nauk* 403 (5) (2005) 583–588.
- [31] W. Ziemer, *Weakly Differentiable Functions*, in: *Graduate Texts in Mathematics*, vol. 120, Springer-Verlag, 1989.