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Pointwise estimate for the Hardy–Littlewood maximal operator on the orbits of contractive mappings

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ABSTRACT

Let M_n denote the Hardy–Littlewood maximal operator on the n-th iteration of a given iterated function system (IFS). We give sufficient conditions on the IFS in order to obtain a pointwise estimate for M_n in terms of the composition of M_0 and a discrete Hardy–Littlewood type maximal operator. As a corollary we prove the uniform preservation of Muckenhoupt condition along the Hutchinson orbits induced by such an IFS.

0. Introduction

We shall start by introducing our result for the most elementary self-similar settings; the interval $[0, 1]$. The interval $[0, 1] = X$ can be regarded as the Banach fixed point for the mapping T on the compact sets K of the real line defined as

$$T(K) = \psi_1(K) \cup \psi_2(K),$$

where $\psi_1(x) = \frac{x}{2}$, $\psi_2(x) = \frac{x}{2} + \frac{1}{2}$. The standard one dimensional Lebesgue length λ on $[0, 1]$, can also be seen as the invariant measure induced by the IFS $\Psi = \{\psi_1, \psi_2\}$. In fact, λ is the fixed point of the mapping S on the Borel probabilities μ on $[0, 1]$ defined by

$$S(\mu)(E) = \frac{1}{2}\mu\left(\psi_1^{-1}(E)\right) + \frac{1}{2}\mu\left(\psi_2^{-1}(E)\right),$$

for E a Borel subset of X.

The system $\Psi = \{\psi_1, \psi_2\}$ is by no means the only IFS producing $[0, 1]$ as the self-similar set and λ as the invariant measure. Let us write T_Ψ and S_Ψ to denote the mappings T and S introduced above to emphasize its dependence on Ψ. The system $\Phi = \{\phi_1, \phi_2\}$ with $\phi_1(x) = \psi_1(x) = \frac{x}{2}$ and $\phi_2(x) = 1 - \frac{x}{2}$ (see Fig. 1), induces the mappings T_Φ and S_Φ changing ψ_i by ϕ_i. The fixed points for T_Φ and S_Φ are, again, $[0, 1]$ and λ. It is easy to realize that the system Φ has some advantages over the system Ψ from the, let us say, analytical point of view. In fact, if μ is absolutely continuous with respect to λ with density w, i.e. $d\mu(x) = w(x)dx$, it is easy to check that $S_\Psi(\mu)$ is also absolutely continuous and that its Radon–Nikodym derivative is given by

$$w_\Psi = \begin{cases} w \circ \psi_1^{-1} & \text{on } X_1, \\ w \circ \psi_2^{-1} & \text{on } X_2, \end{cases}$$

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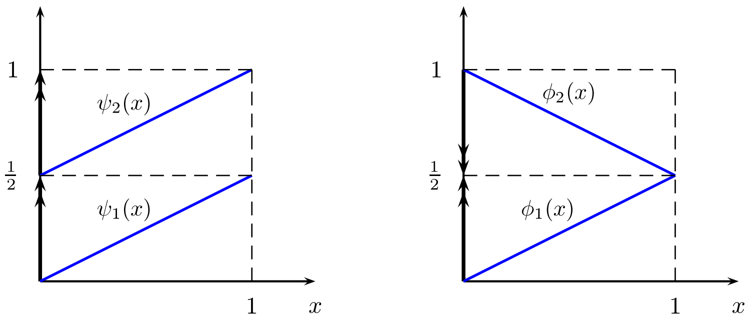


Fig. 1. $\Psi = \{\psi_1, \psi_2\}$ and $\Phi = \{\phi_1, \phi_2\}$.

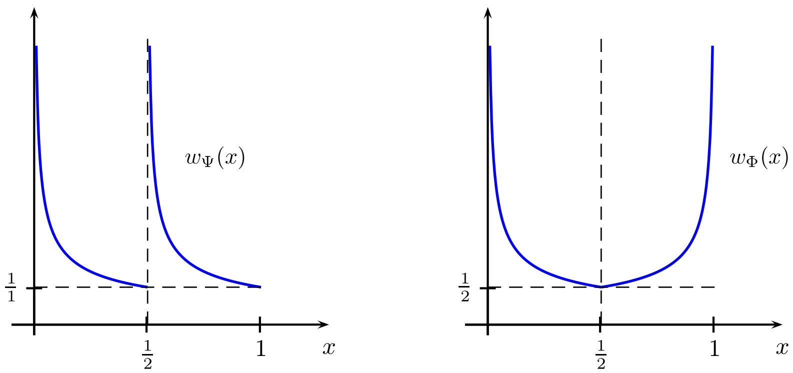


Fig. 2. Densities for $S_\Psi(\mu)$ and $S_\Phi(\mu)$.

where $X_i = \psi_i(X) = \psi_i([0,1])$. Of course $S_\Phi(\mu)$ has the density

$$w_\Phi = \begin{cases} w \circ \phi_1^{-1} & \text{on } X_1, \\ w \circ \phi_2^{-1} & \text{on } X_2. \end{cases}$$

It is easy to see that Φ is continuity preserving but Ψ is not, in the sense that $w_\Phi(x)$ if continuous if w is. The function $w_\Psi(x)$, instead, is generically discontinuous for w continuous.

Not only continuity is preserved by Φ but also some precise quantitative integral properties such as the Muckenhoupt conditions. Take μ to be an absolutely continuous measure on $[0,1]$ with a density belonging to a Muckenhoupt class. To fix ideas, take $d\mu(x) = \frac{1}{2}w(x)dx$, with $w(x) = x^{-1/2}$. Hence μ is a Borel probability measure on $[0,1]$. Moreover, μ is doubling. In other words, regarding $X = [0,1]$ as a metric space with the restriction of the usual distance, we easily see that $\mu(B(x,2r)) \leq 4\mu(B(x,r))$ for every $x \in X$ and every $r > 0$. Here $B(y,s)$ is the open ball in $[0,1]$ centered at $y \in X$ with radius $s > 0$. Precisely, $B(y,s) = (y-s, y+s) \cap [0,1]$. Actually the doubling property can be deduced from the fact that $w(x)$ is an A_2 Muckenhoupt weight. We shall introduce later these classes of densities. Notice that while w_Ψ is no longer doubling, w_Φ is. In fact

$$2\sqrt{2}\, w_\Psi(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1/2, \\ \left(x - \dfrac{1}{2}\right)^{-1/2} & \text{if } 1/2 < x < 1, \end{cases}$$

(see Fig. 2), and

$$2\sqrt{2}\, w_\Phi(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1/2, \\ (1-x)^{-1/2} & \text{if } 1/2 < x < 1. \end{cases}$$

For our purposes, two facts deserve to be emphasized. First, these behaviors persist along the iterations S_Ψ^n of S_Ψ and S_Φ^n of S_Φ (see Fig. 3). Second, the densities associated to the measures $S_\Phi^n(\mu)$ are all A_2-Muckenhoupt weights. Moreover, the A_2 constants are bounded uniformly with respect to n.

After the original work by Benjamin Muckenhoupt contained in [1] (see also [2,3]) it is well known that the Muckenhoupt condition on a density w reflects the behavior of the Hardy–Littlewood maximal operator on the spaces $L^p(\mu)$ with $d\mu(x) = w(x)dx$. Hence, it looks natural to ask whether the above observed behavior of $S_\Phi^n(\mu)$ can be predicted from the analysis of Hardy–Littlewood maximal functions.

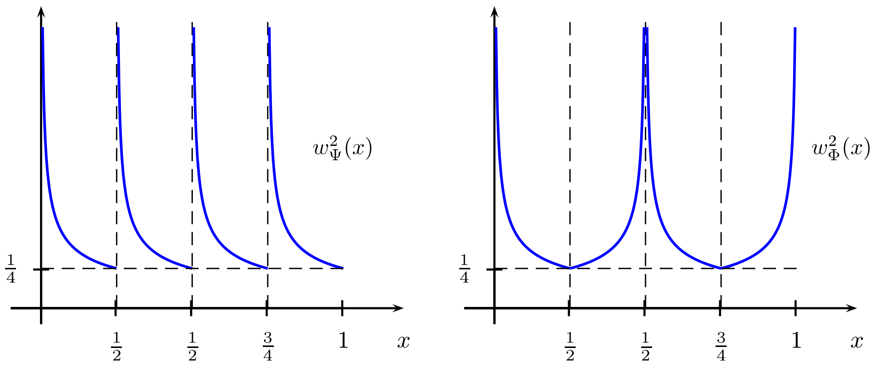


Fig. 3. Densities for $S_{\Psi}^{2}(\mu)$ and $S_{\Phi}^{2}(\mu)$.

To state our result in the setting defined by the IFS Φ on $X = [0, 1]$, we start by some basic notation. For any Borel measurable function f on X and any $x \in X$, set

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y)|\, dy,$$

to denote the standard centered Hardy–Littlewood maximal function on X. Here, as before, λ denotes the one dimensional Lebesgue measure on X and $B(x, r) = (x - r, x + r) \cap X$, $x \in X$.

For a given (large) positive integer N, we may regard the set $I_N := \{1, 2, \ldots, N\}$ with the counting measure and the usual distance inherited from $\mathbb{R}^1$, as a metric measure space. In such a setting the Hardy–Littlewood maximal operator is well defined. In fact, for g a real function (finite sequence) defined on I_N and let $i \in I_N$, the Hardy–Littlewood maximal function is given by

$$\mathfrak{M}_N g(i) = \sup_{r>0} \frac{1}{\operatorname{card}(\mathfrak{I}(i,r))} \sum_{j\in\mathfrak{I}(i,r)} |g(j)|,$$

where $\mathfrak{I}(i, r) = (i - r, i + r) \cap I_N$.

It is easy to see directly by the standard covering arguments or to deduce from the general setting of spaces of homogeneous type, that the operators $\mathfrak{M}_N$ are uniformly of weak type $(1, 1)$ and hence uniformly bounded on each $L^p(I_N, \operatorname{card})$ for $1 < p \le \infty$.

Notice that for each $n \in \mathbb{N}$ and for each $j = 1, 2, 3, \ldots, 2^n$ there exists one and only one sequence $\{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_i \in \{1, 2\}$ such that $\phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_n}([0, 1]) = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]$. In fact, it is enough to take $\alpha_i = \beta_i + 1$, where β_i, $i = 1, \ldots, n$, are the n first terms in the binary expansion of any number in $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$. This fact allows us to write

$$[0, 1] = \bigcup_{j=1}^{2^n} \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] = \bigcup_{j=1}^{2^n} X_j^n = \bigcup_{j=1}^{2^n} \phi_j^n(X), \quad \text{with } \phi_j^n = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_n}.$$

To simplify our statement, let us introduce the following notation. For a given Borel measurable f on $[0, 1]$ and a fixed $z \in [0, 1]$, we write $M(f \circ \phi^n)(z)$ to denote the sequence $g_z(j) = M(f \circ \phi_j^n)(z)$, for $j \in I_{2^n} = \{1, 2, 3, \ldots, 2^n\}$.

Theorem 1. *There exists a constant C such that the inequality*

$$(Mf)(\phi_i^n(z)) \le C\mathfrak{M}_{2^n}[M(f \circ \phi^n)(z)](i) \tag{0.1}$$

holds for every $i = 1, 2, 3, \ldots, 2^n$, every $n \in \mathbb{N}$, every measurable function f defined on $[0, 1]$ and every $z \in [0, 1]$.

Inequality (0.1) reads, somehow more explicitly

$$(Mf)(\phi_i^n(z)) \le C \sup_{r>0} \frac{1}{\operatorname{card}(\mathfrak{I}(i,r))} \sum_{j\in\mathfrak{I}(i,r)} M(f \circ \phi_j^n)(z).$$

Let us show here how to use (0.1) to prove that the Muckenhoupt classes are preserved along the Hutchinson orbits. Following [1] (see also [3]) we say that a non-negative integrable function w defined on $[0, 1]$ is an $A_p = A_p([0, 1])$ Muckenhoupt weight, with $1 < p < \infty$, if there exists a constant C such that the inequality

$$\left(\int_{B(x,r)} w(y)\, dy\right) \left(\int_{B(x,r)} w^{-\frac{1}{p-1}}(y)\, dy\right)^{p-1} \le C\, (\lambda(B(x, r)))^p$$

holds for every $x \in X$ and $r > 0$. Here $B(x, r)$ and λ have the same meaning as in the definition of the operator M.

Corollary 2. *If $w \in A_p([0,1])$ and $d\nu = w(x)dx$, then there exists a constant C such that*

$$\int_{[0,1]} |Mf|^p \, d\nu_n \le C \int_{[0,1]} |f|^p \, d\nu_n, \tag{0.2}$$

for every $n \in \mathbb{N}$ and every measurable function f, where $\nu_n = S_\Phi^n(\nu)$. Hence ν_n is absolutely continuous with respect to dx and its Radon–Nikodym derivative belongs uniformly to $A_p([0,1])$.

Proof. Notice first that with the above notation we have that $d\nu_n(x) = w_n(x)dx$ with $w_n = w \circ (\phi_j^n)^{-1}$ on $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]$, for every $j = 1, \ldots, 2^n$. Hence for a given measurable function h we have

$$\int_{[0,1]} h \, d\nu_n = \sum_{i=1}^{2^n} \int_{X_i^n} h(z) w((\phi_i^n)^{-1}(z)) \, dz = \frac{1}{2^n} \sum_{i=1}^{2^n} \int_X h(\phi_i^n(x)) w(x) \, dx. \tag{0.3}$$

To prove (0.2) we apply (0.3), (0.1), the uniform L^p boundedness of $\mathfrak{M}_{2^n}$ with the counting measure, the $L^p(w dx)$ boundedness of M and (0.3) again, as follows.

$$\begin{aligned}
\int_{[0,1]} |Mf|^p \, d\nu_n &= \frac{1}{2^n} \sum_{i=1}^{2^n} \int_{[0,1]} |Mf(\phi_i^n(x))|^p w(x) \, dx \\
&\le \frac{C}{2^n} \sum_{i=1}^{2^n} \int_{[0,1]} \left| \mathfrak{M}_{2^n}[M(f \circ \phi^n)(x)](i) \right|^p w(x) \, dx \\
&= \frac{C}{2^n} \int_{[0,1]} \left(\sum_{i=1}^{2^n} \left| \mathfrak{M}_{2^n}[M(f \circ \phi^n)(x)](i) \right|^p \right) w(x) \, dx \\
&\le \frac{C}{2^n} \int_{[0,1]} \left(\sum_{i=1}^{2^n} \left| M(f \circ \phi_i^n)(x) \right|^p \right) w(x) \, dx \\
&= \frac{C}{2^n} \sum_{i=1}^{2^n} \int_{[0,1]} \left| M(f \circ \phi_i^n)(x) \right|^p w(x) \, dx \\
&\le \frac{C}{2^n} \sum_{i=1}^{2^n} \int_{[0,1]} \left| (f \circ \phi_i^n)(x) \right|^p w(x) \, dx \\
&= C \int_{[0,1]} |f|^p \, d\nu_n.
\end{aligned}$$

The constant C may change from line to line. The absolute continuity of ν_n and the uniform Muckenhoupt condition for its Radon–Nikodym derivative follows from Muckenhoupt's theorem and the fact that the constant C in the above inequality does not depend on n and f. □

We shall obtain Theorem 1 as a consequence of the more general result contained in Theorem 3 which we state and prove, after some notation, in Section 1. In Section 2 we generalize Corollary 2, and in Section 3 we exhibit examples of the general results applied to some classical situations.

1. The main result

We shall describe the general setting from a somehow axiomatic point of view. The approach allows us to state and prove the main result in a concise and quite general form containing many classical situations.

(A) *The underlying space* (X, d, μ). Let (X, d) be a compact metric space with diameter 1. Let μ be a Borel probability on X such that the functions of $r \in (0, 1]$ defined by $\mu_x(r) = \mu(B(x, r))$, $x \in X$, are uniformly equivalent to a positive power of r. Precisely, there exist constants K_1, K_2 and $\gamma > 0$ such that the inequalities

$$K_1 r^\gamma \le \mu_x(r) \le K_2 r^\gamma$$

hold for every $x \in X$ and $r \in (0, 1]$. Sometimes this property is called Ahlfors condition or is described by saying that (X, d, μ) is a *normal space* of dimension γ. In fact γ is the Hausdorff dimension of each ball in X. It is easy to see that if (X, d, μ) is a normal space, then (X, d, μ) is a *space of homogeneous type*. This means that there exists a constant $A \ge 1$ (called doubling constant) such that $0 < \mu_x(2r) \le A\mu_x(r) < \infty$ for every $x \in X$ and every $r > 0$.

(B) *The family Φ of similitudes.* A finite set $\Phi = \{\phi_i : X \to X, i = 1, 2, \dots, H\}$ of contractive similitudes with the same contraction rate is given. Precisely, each ϕ_i satisfies

$$d(\phi_i(x), \phi_i(y)) = \beta d(x, y)$$

for every $x, y \in X$ and some constant $0 < \beta < 1$. For $n \in \mathbb{N}$, set $\mathfrak{I}^n = \{1, 2, \dots, H\}^n$. Given $\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in \mathfrak{I}^n$, we denote with $\boldsymbol{\phi}_{\boldsymbol{i}}^n$ the composition $\phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}$. Then for any subset E of X we have $\boldsymbol{\phi}_{\boldsymbol{i}}^n(E) = \left(\phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}\right)(E)$. Set $X_{\boldsymbol{i}}^n = \boldsymbol{\phi}_{\boldsymbol{i}}^n(X)$ and $X^n = \bigcup_{\boldsymbol{i} \in \mathfrak{I}^n} X_{\boldsymbol{i}}^n$. We shall assume that Φ satisfies:

(B1) *Open Set Condition (OSC).* There exists a non-empty open set $U \subset X$ such that

$$\bigcup_{i=1}^{H} \phi_i(U) \subseteq U,$$

and $\phi_i(U) \cap \phi_j(U) = \emptyset$ if $i \neq j$. We shall say that U is a *set for the OSC* for Φ.

(B2) *Adjacency.* There exists a positive constant c such that the inclusion

$$B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \cap X_{\boldsymbol{j}}^n \subseteq B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z), cr) \cap X_{\boldsymbol{j}}^n$$

holds for every $n \in \mathbb{N}$, every $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$, every $r > 0$ and every $z \in X$.

To avoid dilations for the statement of the general result, we only remark at this point that the setting $X = [0, 1]$ with the usual distance and length, and $\Phi = \left\{\phi_1(x) = \frac{x}{2}, \phi_2(x) = 1 - \frac{x}{2}\right\}$ presented in the introduction satisfies all these properties. Notice also that the system $\Psi = \left\{\psi_1(x) = \frac{x}{2}, \psi_2(x) = \frac{1}{2} + \frac{x}{2}\right\}$ satisfies all the above properties except (B2), which does not hold if $n = 1$ with $i = 1, j = 2, z = 1$ and r small. That is why we call it the "adjacency" property of the system.

We proceed to define precisely the three maximal operators involved. Let h be an integrable real function defined on X. The Hardy–Littlewood centered maximal function associated to h is given by

$$Mh(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |h(y)|\, d\mu(y).$$

To define a discrete version of the Hardy–Littlewood maximal operator, let us fix $x_0 \in U$ and for $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$ define $\tilde{d}(\boldsymbol{i}, \boldsymbol{j}) = d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0), \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0))$. For $n \in \mathbb{N}$, $\boldsymbol{i} \in \mathfrak{I}^n$ and $r > 0$, set $\mathcal{B}(\boldsymbol{i}, r)$ to denote the $\tilde{d}$-ball of radius r in $(\mathfrak{I}^n, \tilde{d})$. More precisely, $\mathcal{B}(\boldsymbol{i}, r) = \{\boldsymbol{j} \in \mathfrak{I}^n : d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0), \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)) < r\}$. As our second operator, we shall consider a Hardy–Littlewood type maximal function defined using the family $\mathcal{B}(\boldsymbol{i}, r)$. Precisely, given a real function g defined on $\mathfrak{I}^n$,

$$\mathfrak{M}_n g(\boldsymbol{i}) = \sup_{r>0} \frac{1}{\operatorname{card}(\mathcal{B}(\boldsymbol{i}, r))} \sum_{\boldsymbol{j} \in \mathcal{B}(\boldsymbol{i}, r)} |g(\boldsymbol{j})|.$$

We have to point out that $\tilde{d}$ and hence the $\mathfrak{M}_n$'s depend on $x_0 \in U$, but we shall fix it from now on.

To introduce the third Hardy–Littlewood maximal operator considered in this note, we shall make use of the natural "uniformly distributed" probability measure induced by μ on X^n given by

$$\mu^n(E) = \frac{1}{H^n} \sum_{\boldsymbol{j} \in \mathfrak{I}^n} \mu\left((\boldsymbol{\phi}_{\boldsymbol{j}}^n)^{-1}(E)\right)$$

for E a Borel set in X^n. In other words, $\mu^n = H^{-n} \sum_{\boldsymbol{j} \in \mathfrak{I}^n} \mu_{\boldsymbol{j}}^n$, with $\mu_{\boldsymbol{j}}^n(E) = \mu\left((\boldsymbol{\phi}_{\boldsymbol{j}}^n)^{-1}(E)\right)$. The third maximal operator involved in our main result is the Hardy–Littlewood operator on the space (X^n, d, μ^n). Precisely, for a Borel measurable function f on X^n we define, for $v \in X^n$,

$$M_n f(v) = \sup_{r>0} \frac{1}{\mu^n(B(v, r))} \int_{B(v,r)} |f(y)|\, d\mu^n(y).$$

Here $B(v, r)$ is the d-ball in X^n. Notice that $M_0 = M$ under the standard assumption $X^0 = X$ and $\mu^0 = \mu$.

Theorem 3. *There exists a geometric constant C such that the inequality*

$$M_n f\left(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z)\right) \le C \mathfrak{M}_n\left(M(f \circ \boldsymbol{\phi}^n)(z)\right)(\boldsymbol{i})$$

holds for every $f \in L^1(X^n, \mu^n)$, $z \in X$, $\boldsymbol{i} \in \mathfrak{I}^n$ and $n \in \mathbb{N}$, where $M(f \circ \boldsymbol{\phi}^n)(z)$ denotes the function g on $\mathfrak{I}^n$ defined by $g(\boldsymbol{j}) = M(f \circ \boldsymbol{\phi}_{\boldsymbol{j}}^n)(z)$.

Before proving Theorem 3 we shall collect in the next lemma some elementary properties of a system $((X, d, \mu), \Phi)$ satisfying (A) and (B) above. Item (1) in Lemma 4 is contained in [4, Theorem 2.1(III)], and Item (2b) is contained in [5, Lemma 2.4]. The proofs of (2a), (3)–(5) are given after the proof of Theorem 3.

Lemma 4. (1) *The sequence $\{(X^n, d, \mu^n) : n \in \mathbb{N}\}$ is a uniform family of spaces of homogeneous type. In other words, there exists a constant $\tilde{A}$ such that*

$$0 < \mu^n(B(x, 2r)) \le \tilde{A}\mu^n(B(x, r))$$

for every $r > 0$, $x \in X^n$ and $n \in \mathbb{N}$.

(2) *Let* $x_0 \in U$ *be fixed, and for each* $n \in \mathbb{N}$ *we consider the set* $\Delta_n = \left\{\boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0) : \boldsymbol{j} \in \mathfrak{I}^n\right\}$. *Then*
   (a) *for every* $n \in \mathbb{N}$ *we have that* Δ_n *is a* $\delta\beta^n$*-disperse set, with* $\delta = \operatorname{dist}(x_0, \partial U)$. *This means that* $d(\boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0), \boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0)) \geq \delta\beta^n$ *for every* $\boldsymbol{i} \neq \boldsymbol{j}$ *in* $\mathfrak{I}^n$;
   (b) $\{(\Delta_n, d, \operatorname{card}) : n \in \mathbb{N}\}$ *is a sequence of spaces of homogeneous type with a uniform doubling constant A.*
(3) *Given* $a > 0$, *there exists a constant* $N = N(a)$ *such that* $\operatorname{card}(\mathcal{B}(\boldsymbol{i}, a\beta^n)) \leq N$ *for every* $\boldsymbol{i} \in \mathfrak{I}^n$ *and every* $n \in \mathbb{N}$.
(4) *For each* $n \in \mathbb{N}$ *we have that*

$$\mu^n(B(y, r)) \geq \frac{K_1}{H^n}\frac{r^\gamma}{\beta^{\gamma n}},$$

for every $0 < r \leq \beta^n/2$ *and every* $y \in X^n$.
(5) *If h is an integrable real function on* (X, μ) *then for each* $n \in \mathbb{N}$ *and* $\boldsymbol{j} \in \mathfrak{I}^n$ *the function* $h \circ \boldsymbol{\phi}_{\boldsymbol{j}}^n$ *is integrable on* $(X_{\boldsymbol{j}}^n, \mu_{\boldsymbol{j}}^n)$ *and*

$$\int_X h \circ \boldsymbol{\phi}_{\boldsymbol{j}}^n \, d\mu = \int_{X_{\boldsymbol{j}}^n} h \, d\mu_{\boldsymbol{j}}^n.$$

Proof of Theorem 3. Fix $n \in \mathbb{N}, z \in X$ and $\boldsymbol{i} \in \mathfrak{I}^n$. Notice that since $\boldsymbol{\phi}_{\boldsymbol{i}}^n(z) \in X^n$, $M_n f(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z))$ is well defined for any measurable function f on X^n. We shall estimate a general mean of the form

$$\frac{1}{\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r))}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)} |f(y)| \, d\mu^n(y),$$

for $0 < r \leq 1$. Recall the fact that $B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r)$ is to be understood as the d-ball on X^n, or in an equivalent way one may think that is the d-ball on X since μ^n is supported on $X^n \subseteq X$. Let us divide our analysis in two cases depending on the relative sizes of r and β^n.

Assume first that $r \leq 3\beta^n$. Let us start by estimating $\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r))$. Notice that

$$\frac{c_1}{H^n}\frac{r^\gamma}{\beta^{\gamma n}} \leq \mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r)),$$

for some constant c_1. In fact, to estimate $\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r))$ we use property (4) in Lemma 4 when $r \leq \frac{\beta^n}{2}$. If $\frac{\beta^n}{2} < r \leq 3\beta^n$, the estimates are trivial since

$$\frac{K_1}{H^n}\frac{r^\gamma}{6^\gamma\beta^{\gamma n}} \leq \mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r/6)) \leq \mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r)).$$

Then the desired inequality holds with $c_1 = \min\left\{K_1, \frac{K_1}{6^\gamma}\right\}$.

To estimate $\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)} |f(y)| \, d\mu^n(y)$ we shall use the adjacency property for Φ. If $\mathfrak{I}_{(\boldsymbol{i},z,r)}^n$ denotes the set of those $\boldsymbol{j}$ in $\mathfrak{I}^n$ for which $X_{\boldsymbol{j}}^n$ intersects $B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r)$, we have that

$$\begin{aligned}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)} |f(y)| \, d\mu^n(y) &= \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}_{(\boldsymbol{i},z,r)}^n}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)} |f(y)| \, d\mu_{\boldsymbol{j}}^n(y)\\ &= \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}_{(\boldsymbol{i},z,r)}^n}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)\cap X_{\boldsymbol{j}}^n} |f(y)| \, d\mu_{\boldsymbol{j}}^n(y).\end{aligned}$$

Using the adjacency property (B2) of Φ for the domain of integration in the above integral, we get that

$$\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)} |f(y)| \, d\mu^n(y) \leq \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}_{(\boldsymbol{i},z,r)}^n}\int_{B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z),cr)\cap X_{\boldsymbol{j}}^n} |f(y)| \, d\mu_{\boldsymbol{j}}^n(y).$$

Let us estimate any of the integrals in the last sum by "changing variables" in the sense of property (5) in Lemma 4. For $\boldsymbol{j} \in \mathfrak{I}_{(\boldsymbol{i},z,r)}^n$ we have that

$$\begin{aligned}\int_{B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z),cr)\cap X_{\boldsymbol{j}}^n} |f(y)| \, d\mu_{\boldsymbol{j}}^n(y) &= \int_{X_{\boldsymbol{j}}^n} \mathcal{X}_{B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z),cr)}(y)|f(y)| \, d\mu_{\boldsymbol{j}}^n(y)\\ &= \int_X \mathcal{X}_{B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z),cr)}\left(\boldsymbol{\phi}_{\boldsymbol{j}}^n(u)\right)\left|\left(f \circ \boldsymbol{\phi}_{\boldsymbol{j}}^n\right)(u)\right| \, d\mu(u)\\ &= \int_{B(z,cr\beta^{-n})}\left|\left(f \circ \boldsymbol{\phi}_{\boldsymbol{j}}^n\right)\right| \, d\mu.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)}|f(y)|\,d\mu^n(y) &\le \frac{1}{c_1}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}}\frac{\beta^{\gamma n}}{r^\gamma}\int_{B\left(z,\frac{cr}{\beta^n}\right)}\left|\left(f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n\right)\right|\,d\mu\\ &\le \frac{c^\gamma K_2}{c_1}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}} M(f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n)(z).\end{aligned}$$

Notice now that $\mathfrak{I}^n_{(\boldsymbol{i},z,r)}\subseteq\mathcal{B}(\boldsymbol{i},5\beta^n)$. In fact, if $\boldsymbol{j}\in\mathfrak{I}^n$ is such that $B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)\cap X_{\boldsymbol{j}}^n\neq\emptyset$, then there exists $y\in X_{\boldsymbol{j}}^n$ such that $d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),y)<r$. Hence

$$\begin{aligned}d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0),\boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)) &\le d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0),\boldsymbol{\phi}_{\boldsymbol{i}}^n(z))+d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),y)+d(y,\boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0))\\ &< \beta^n+r+\beta^n\\ &\le 5\beta^n.\end{aligned}$$

From property (3) in Lemma 4 we also have that $\mathrm{card}(\mathcal{B}(\boldsymbol{i},5\beta^n))\le N$ for some constant N. So that

$$\begin{aligned}\frac{1}{\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)}|f(y)|\,d\mu^n(y) &\le \frac{Nc^\gamma K_2}{c_1\mathrm{card}(\mathcal{B}(\boldsymbol{i},5\beta^n))}\sum_{\boldsymbol{j}\in\mathcal{B}(\boldsymbol{i},5\beta^n)} M(f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n)(z)\\ &\le c_1^{-1}c^\gamma K_2 N\mathfrak{M}_n\left(M(f\circ\boldsymbol{\phi}^n)(z)\right)(\boldsymbol{i}).\end{aligned}$$

Assume next that $r>3\beta^n$. Again we have to provide an adequate estimate for the mean value

$$\frac{1}{\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)}|f(y)|\,d\mu^n(y).$$

Let us first get a lower bound for $\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))$. From the definition of μ^n we see that

$$\begin{aligned}\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)) &= \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}^n}\mu\left((\boldsymbol{\phi}_{\boldsymbol{j}}^n)^{-1}(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))\right)\\ &\ge \frac{1}{H^n}\mathrm{card}\left(\{\boldsymbol{j}\in\mathfrak{I}^n:X_{\boldsymbol{j}}^n\subseteq B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)\}\right).\end{aligned}$$

Let us observe that the dispersion property given in (2a) in Lemma 4 allows to regard the uniform homogeneity contained in (2b) of this lemma, as equivalent to the uniform homogeneity of the sequence $(\mathfrak{I}^n,\tilde{d},\mathrm{card})$. Now, since in this case $\mathcal{B}(\boldsymbol{i},r/3)\subseteq\{\boldsymbol{j}\in\mathfrak{I}^n:X_{\boldsymbol{j}}^n\subseteq B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)\}$, we get that

$$\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))\ge\frac{1}{H^n}\mathrm{card}(\mathcal{B}(\boldsymbol{i},r/3))\ge\frac{1}{A^3H^n}\mathrm{card}(\mathcal{B}(\boldsymbol{i},2r)).$$

On the other hand

$$\begin{aligned}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)}|f(y)|\,d\mu^n(y) &= \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}}\int_{B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z),r)\cap X_{\boldsymbol{j}}^n}|f(y)|\,d\mu_{\boldsymbol{j}}^n(y)\\ &\le \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}}\int_{X_{\boldsymbol{j}}^n}|f(y)|\,d\mu_{\boldsymbol{j}}^n(y)\\ &= \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}}\int_X|f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n|\,d\mu\\ &\le \frac{1}{H^n}\sum_{\boldsymbol{j}\in\mathfrak{I}^n_{(\boldsymbol{i},z,r)}} M(f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n)(z).\end{aligned}$$

So that, since $\mathfrak{I}^n_{(\boldsymbol{i},z,r)}\subseteq\mathcal{B}(\boldsymbol{i},2r)$, we have

$$\begin{aligned}\frac{1}{\mu^n(B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r))}\int_{B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z),r)}|f(y)|\,d\mu^n(y) &\le \frac{A^3}{\mathrm{card}(\mathcal{B}(\boldsymbol{i},2r))}\sum_{\boldsymbol{j}\in\mathcal{B}(\boldsymbol{i},2r)} M(f\circ\boldsymbol{\phi}_{\boldsymbol{j}}^n)(z)\\ &\le A^3\mathfrak{M}_n\left(M(f\circ\boldsymbol{\phi}^n)(z)\right)(\boldsymbol{i}).\quad\square\end{aligned}$$

Proof of Lemma 4. As we already said the proof of (1) is contained in [4], and the proof of (2b) in [5].

Let us prove that the OSC implies (2a). In fact, take $\boldsymbol{j}, \boldsymbol{i} \in \{1, \ldots, H\}^n$ with $\boldsymbol{j} \neq \boldsymbol{i}$, and set $x_{\mathbf{j}}^n = \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)$ and $x_{\mathbf{i}}^n = \boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0)$. Since U is an open set, we have that $B(x_0, \delta) \subseteq U$, with $\delta = d(x_0, \partial U)$. Then

$$B(x_{\mathbf{j}}^n, \delta\beta^n) = \boldsymbol{\phi}_{\boldsymbol{j}}^n\,(B(x_0, \delta)) \subseteq \boldsymbol{\phi}_{\boldsymbol{j}}^n(U),$$
$$B(x_{\mathbf{i}}^n, \delta\beta^n) = \boldsymbol{\phi}_{\boldsymbol{i}}^n\,(B(x_0, \delta)) \subseteq \boldsymbol{\phi}_{\boldsymbol{i}}^n(U),$$

and since $\boldsymbol{\phi}_{\boldsymbol{j}}^n(U)$ and $\boldsymbol{\phi}_{\boldsymbol{i}}^n(U)$ are disjoint, we have $B(x_{\mathbf{j}}^n, \delta\beta^n) \cap B(x_{\mathbf{i}}^n, \delta\beta^n) = \emptyset$. This implies that $d(x_{\mathbf{j}}^n, x_{\mathbf{i}}^n) \geq \delta\beta^n$.

The estimate in (3) is an immediate consequence of the results in [6]. Since the spaces $(\Delta_n, d, \text{card})$ are uniformly of homogeneous type and the set Δ_n is $\delta\beta^n$-disperse, every d-ball of radius bounded above by a constant times β^n has at most N elements of Δ_n, where N is independent of n and of the center of the given ball. In other words, there exists a constant $N = N(a)$ such that

$$\text{card}(\mathcal{B}(\boldsymbol{i}, a\beta^n)) \leq N$$

uniformly in n and $\boldsymbol{i} \in \mathfrak{I}^n$.

To prove (4), fix $n \in \mathbb{N}$ and take $y \in X^n$. Let $\boldsymbol{i} \in \mathfrak{I}^n$ be such that $y \in X_{\boldsymbol{i}}^n$. Since $(\boldsymbol{\phi}_{\mathbf{i}}^n)^{-1}(B(y, r)) = B\left((\boldsymbol{\phi}_{\mathbf{i}}^n)^{-1}(y), \frac{r}{\beta^n}\right)$, we have that

$$\begin{aligned}\mu^n(B(y, r)) &= \frac{1}{H^n} \sum_{\boldsymbol{j} \in \mathfrak{I}^n} \mu\left((\boldsymbol{\phi}_{\boldsymbol{j}}^n)^{-1}(B(y, r))\right) \\ &\geq \frac{1}{H^n} \mu\left(B\left((\boldsymbol{\phi}_{\mathbf{i}}^n)^{-1}(y), \frac{r}{\beta^n}\right)\right) \\ &\geq \frac{K_1}{H^n} \frac{r^\gamma}{\beta^{\gamma n}}.\end{aligned}$$

The identity in (5) is a consequence of the fact that when h is the indicator function of a measurable set E, we have

$$\int_X \mathcal{X}_E\left(\boldsymbol{\phi}_{\boldsymbol{j}}^n\right) d\mu(x) = \mu\left((\boldsymbol{\phi}_{\boldsymbol{j}}^n)^{-1}(E)\right) = \mu_{\boldsymbol{j}}^n(E) = \int_{X_{\boldsymbol{j}}^n} \mathcal{X}_E\, d\mu_{\boldsymbol{j}}^n. \quad \square$$

2. On the stability of Muckenhoupt classes

In the next result our setting is as in Section 1, in other words (X, d, μ) satisfies (A) and $\boldsymbol{\Phi} = \{\boldsymbol{\phi}_{\boldsymbol{i}}^n : \boldsymbol{i} \in \mathfrak{I}^n, n \in \mathbb{N}\}$ satisfies (B). Given a Borel measure ν on X, we define for each $n \in \mathbb{N}$

$$S_{\boldsymbol{\Phi}}^n(\nu)(E) = \frac{1}{H^n} \sum_{\mathrm{i} \in \mathfrak{I}^n} \nu\left(\left(\boldsymbol{\phi}_{\mathbf{i}}^n\right)^{-1}(E)\right).$$

Theorem 5. *If $w \in A_p(X, d, \mu)$ and $d\nu = w\,d\mu$, then there exists a constant C such that*

$$\int_{X^n} |M_n f|^p\, d\nu^n \leq C \int_{X^n} |f|^p\, d\nu^n, \tag{2.1}$$

for every $n \in \mathbb{N}$ and every measurable function f in X^n, where $\nu^n = S_{\boldsymbol{\Phi}}^n(\nu)$. Hence ν^n is absolutely continuous with respect to μ^n and its Radon–Nikodym derivative belongs uniformly to $A_p(X^n, d, \mu^n)$.

Proof. Notice first that

$$\nu^n(E) = \frac{1}{H^n} \sum_{\mathrm{i} \in \mathfrak{I}^n} \int_X (\mathcal{X}_E \circ \boldsymbol{\phi}_{\mathbf{i}}^n)(z) w(z)\, d\mu(z).$$

Hence

$$\int_{X^n} g\, d\nu^n = \frac{1}{H^n} \sum_{\mathrm{i} \in \mathfrak{I}^n} \int_X g(\boldsymbol{\phi}_{\mathbf{i}}^n(z)) w(z)\, d\mu(z).$$

Then, using the above remark, Theorem 3, the uniform L^p boundedness of $\mathfrak{M}_{2^n}$ with the counting measure and the $L^p(wd\mu)$ boundedness of M we obtain

$$\int_{X^n} |M_n f|^p\, d\nu^n = \frac{1}{H^n} \sum_{\mathrm{i} \in \mathfrak{I}^n} \int_X \left|M_n f(\boldsymbol{\phi}_{\mathbf{i}}^n(z))\right|^p w(z)\, d\mu(z)$$

$$\le \frac{C}{H^n} \sum_{\boldsymbol{i} \in \mathfrak{I}^n} \int_X \left| \mathfrak{M}_n \left(M(f \circ \boldsymbol{\phi}^n)(z) \right) (\boldsymbol{i}) \right|^p w(z)\, d\mu(z)$$
$$\le \frac{C}{H^n} \int_X \sum_{\boldsymbol{i} \in \mathfrak{I}^n} |M(f \circ \boldsymbol{\phi}_{\boldsymbol{i}}^n)|^p w(z)\, d\mu(z)$$
$$\le \frac{C}{H^n} \int_X \sum_{\boldsymbol{i} \in \mathfrak{I}^n} |f \circ \boldsymbol{\phi}_{\boldsymbol{i}}^n|^p w(z)\, d\mu(z)$$
$$= C \int_{X^n} |f|^p\, d\nu^n.$$

Since from (1) in Lemma 4 we have that the spaces (X^n, d, μ^n) are uniformly spaces of homogeneous type, we can conclude that ν^n is absolutely continuous with respect to μ^n and its Radon–Nikodym derivative belongs uniformly to $A_p(X^n, d, \mu^n)$. $\square$

3. Some examples

In this section we show how some classical fractals can be obtained through somehow non-standard IFSs satisfying the adjacency property (B2).

The classical Sierpinski IFSs can be slightly modified in order to preserve the adjacency. For the Sierpinski gasket, the usual IFS is $\Psi = \{\psi_1, \psi_2, \psi_3\}$, with

$$\psi_1(x, y) = \frac{1}{2}(x, y), \qquad \psi_2(x, y) = \frac{1}{2}(x + 1, y), \qquad \psi_3(x, y) = \frac{1}{2}\left(x + \frac{1}{2}, y + \frac{\sqrt{3}}{2}\right),$$

defined on the triangle X with vertices at $a = (0, 0)$, $b = \left(1/2, \sqrt{3}/2\right)$ and $c = (1, 0)$.

If ρ_θ denotes the rotation of θ radians about the origin of $\mathbb{R}^2$ in the positive sense, we have that the IFS given by $\Phi = \{\phi_1, \phi_2, \phi_3\}$, where

$$\phi_1(x, y) = \frac{1}{2}(x, y),$$
$$\phi_2(x, y) = \frac{1}{2}(\rho_{4\pi/3}(x, y)) + \boldsymbol{v},$$
$$\phi_3(x, y) = \frac{1}{2}(\rho_{2\pi/3}(x, y)) + \boldsymbol{v},$$

with $\boldsymbol{v} = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$, satisfies the adjacency property, the OSC and gives rise to the standard Sierpinski triangle (see Fig. 4).

Property (B2) for Φ follows from the following lemma, which can be applied also to some other fractals like the Sierpinski carpet after a redefinition of the IFS preserving adjacency.

Lemma 6. *Let $\Phi = \{\phi_1, \ldots, \phi_H\}$ be a finite family of contractive similitudes on X with the same contraction rate β. Let us assume that Φ satisfies the following properties:*

(1) *if $x \in X_i \cap X_j$ then $d(x, \phi_i(z)) = d(x, \phi_j(z))$ for every $z \in X$ and every $i, j \in \{1, \ldots, H\}$;*
(2) *for every $z \in X$ and every $r \le \beta^n$ such that $B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \cap X_{\boldsymbol{j}}^n \neq \emptyset$, we have that $X_{\boldsymbol{i}}^n \cap X_{\boldsymbol{j}}^n \cap B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \neq \emptyset$.*

Then for every $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$ and every $n \in \mathbb{N}$, we have that

(i) *if $x \in X_{\boldsymbol{i}}^n \cap X_{\boldsymbol{j}}^n$ then there exists $x_0 \in X$ such that $x = \boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0) = \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)$;*
(ii) *if $x \in X_{\boldsymbol{i}}^n \cap X_{\boldsymbol{j}}^n$ then $d(x, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) = d(x, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z))$ for every $z \in X$;*
(iii) *$B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \cap X_{\boldsymbol{j}}^n \subseteq B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z), 3r) \cap X_{\boldsymbol{j}}^n$ for every $z \in X$.*

Proof. Let us prove (i) by induction on n. For $n = 1$, let us assume that $x = \phi_i(x_0) = \phi_j(x_1)$ for some $x_0, x_1 \in X$. Applying hypothesis (1) with $z = x_1$ we have that $d(x, \phi_i(x_1)) = d(x, \phi_j(x_1)) = 0$. Then $x = \phi_i(x_1)$, and we have $\phi_i(x_1) = x = \phi_i(x_0)$. Since ϕ_i is one to one we conclude that $x_0 = x_1$. Let us now show that if (i) holds for n then also holds for $n + 1$. In fact, take $x \in X_{\boldsymbol{k}}^{n+1} \cap X_{\boldsymbol{\ell}}^{n+1}$. Then there exist $\boldsymbol{i}, \boldsymbol{j} \in \mathfrak{I}^n$, $k, \ell \in \{1, \ldots, H\}$ and $x_1, x_2 \in X$ such that $x = \boldsymbol{\phi}_{\boldsymbol{i}}^n(\phi_k(x_1)) = \boldsymbol{\phi}_{\boldsymbol{j}}^n(\phi_\ell(x_2))$. Since we are assuming (i) for n, there exists $x_0 \in X$ such that $x = \boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0) = \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)$. Since $\boldsymbol{\phi}_{\boldsymbol{i}}^n$ and $\boldsymbol{\phi}_{\boldsymbol{j}}^n$ are one to one, we have that $x_0 = \phi_k(x_1) = \phi_\ell(x_2)$. Then $x_0 \in X_k \cap X_\ell$, so that there exists $\widetilde{x} \in X$ such that $x_0 = \phi_k(\widetilde{x}) = \phi_\ell(\widetilde{x})$. Hence $x = \boldsymbol{\phi}_{\boldsymbol{i}}^n(\phi_k(\widetilde{x})) = \boldsymbol{\phi}_{\boldsymbol{j}}^n(\phi_\ell(\widetilde{x}))$, which proves (i).

To prove (ii) we shall use (i) and the similarity condition of the IFS. Let us fix $z \in X$ and $x \in X_{\boldsymbol{i}}^n \cap X_{\boldsymbol{j}}^n$. Let $x_0 \in X$ such that $x = \boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0) = \boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0)$. Then

$$d(x, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) = d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(x_0), \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) = \beta^n d(x_0, z),$$

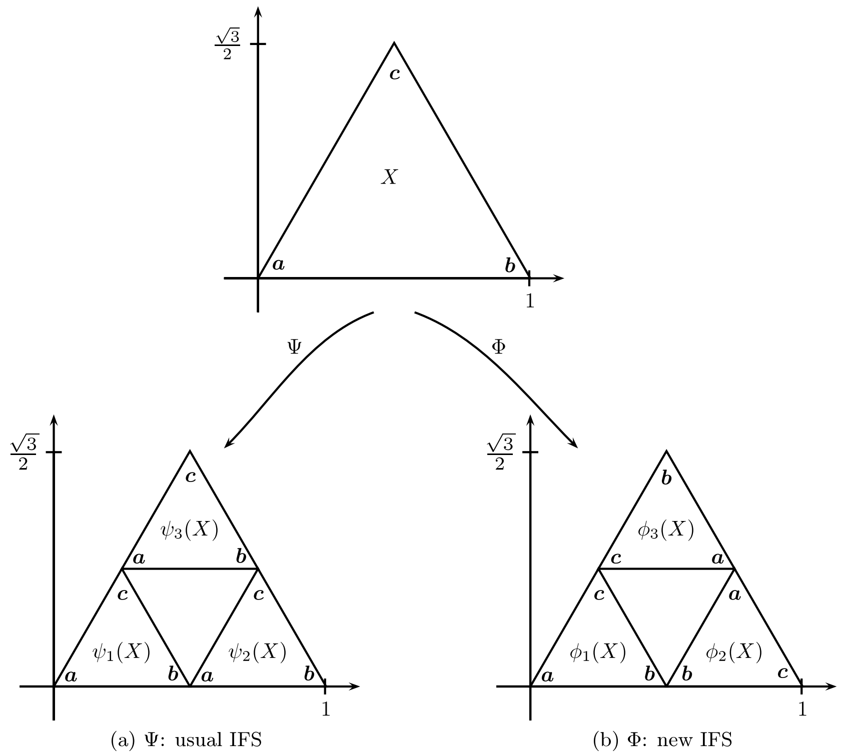


Fig. 4. $X^1 = \bigcup_{i=1}^{3} \phi_i(X) = \bigcup_{i=1}^{3} \psi_i(X)$.

and

$$d(x, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z)) = d(\boldsymbol{\phi}_{\boldsymbol{j}}^n(x_0), \boldsymbol{\phi}_{\boldsymbol{j}}^n(z)) = \beta^n d(x_0, z),$$

so that $d(x, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) = d(x, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z))$, and we prove (ii).

To prove (iii), let us assume that $B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \cap X_{\boldsymbol{j}}^n \neq \emptyset$. If $r > \beta^{-n}$ the inclusion holds since $\operatorname{diam}(X_{\boldsymbol{j}}^n) = \beta^{-n}$ implies $B(\boldsymbol{\phi}_{\boldsymbol{j}}^n(z), 3r) \cap X_{\boldsymbol{j}}^n = X_{\boldsymbol{j}}^n$, so that we can assume $r \leq \beta^{-n}$. Fix $y \in B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r) \cap X_{\boldsymbol{j}}^n$. From (2) there exists $x \in X_{\boldsymbol{i}}^n \cap X_{\boldsymbol{j}}^n \cap B(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), r)$, and from (ii) we have that $d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), x) = d(x, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z))$. Then

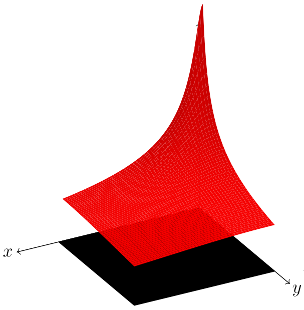
$$\begin{aligned} d(y, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z)) &\leq d(y, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) + d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), x) + d(x, \boldsymbol{\phi}_{\boldsymbol{j}}^n(z)) \\ &= d(y, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) + d(\boldsymbol{\phi}_{\boldsymbol{i}}^n(z), x) + d(x, \boldsymbol{\phi}_{\boldsymbol{i}}^n(z)) \\ &< r + r + r \\ &= 3r. \quad \square \end{aligned}$$

With this lemma, in order to prove that Φ satisfies the required properties to apply Theorem 3 to the Sierpinski gasket, we only need to check (1) and (2). Property (1) follows immediately. To verify (2) we only have to observe that for $r \leq 2^{-n}$, if a ball intersects two components of X^n and it is centered in one of them, then these two components share a vertex belonging to that ball.

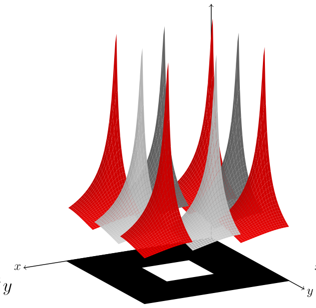
Let us finally observe and depict an illustration of Theorem 5 for the Sierpinski carpet. Let Φ be the classical IFS for the Sierpinski carpet, and let $\Phi = \{\phi_i : 1 \leq i \leq 8\}$ be given by

$$\begin{aligned} &\phi_1(x, y) = \frac{1}{3}(x, y), \qquad \phi_2(x, y) = T_{\frac{2}{3},0}(S_2(\phi_1(x, y))), \\ &\phi_3(x, y) = T_{\frac{2}{3},0}(\phi_1(x, y)), \qquad \phi_4(x, y) = T_{0,\frac{2}{3}}(S_1(\phi_1(x, y))), \\ &\phi_5(x, y) = T_{\frac{2}{3},\frac{2}{3}}(S_1(\phi_1(x, y))), \qquad \phi_6(x, y) = T_{0,\frac{2}{3}}(\phi_1(x, y)), \\ &\phi_7(x, y) = T_{\frac{2}{3},\frac{2}{3}}(S_2(\phi_1(x, y))), \qquad \phi_8(x, y) = T_{\frac{2}{3},\frac{2}{3}}(\phi_1(x, y)), \end{aligned}$$

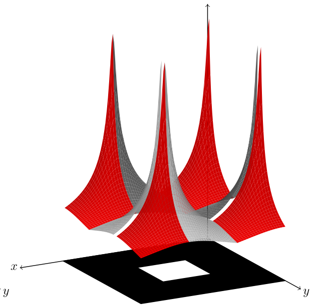
defined on the unit square X of $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, where $T_{a,b}(x, y) = (x + a, y + b)$, $S_1(x, y) = (x, -y)$ and $S_2(x, y) = (-x, y)$. The basic weight function considered is $w(x, y) = (x^2 + y^2)^{-1/4}$ and the basic measure is $d\mu = dxdy$. The following figure illustrate the Radon–Nikodym derivatives w_Ψ^1 and w_Φ^1 of ν_Ψ^1 and ν_Φ^1.



(A) $w(x,y)$ on X



(B) $w^1_\Psi(x,y)$ on X^1



(C) $w^1_\Phi(x,y)$ on X^1

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