

The finite Hartley new convolutions and solvability of the integral equations with Toeplitz plus Hankel kernels

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ABSTRACT

The main aim of this work is to consider integral equations of convolution type with the Toeplitz plus Hankel kernels firstly posed by Tsitsiklis and Levy (1981) [11]. By constructing eight new generalized convolutions for the finite Hartley transforms we obtain a necessary and sufficient condition for the solvability and unique explicit L^2 -solution of those equations. Thanks to this convolution approach the solvability condition obtained here is remarkably different from those in Tsitsiklis and Levy (1981) [11] and in other papers.

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1. Introduction

Consider the integral equation of the form

$$\lambda\varphi(x) + \int_a^b [p(x-u) + q(x+u)]\varphi(u)du = f(x), \quad (1.1)$$

where $\lambda \in \mathbb{C}$, $-\infty \leq a < b \leq +\infty$, and $K(x, u) := p(x-u) + q(x+u)$ is the kernel of the equation. Eq. (1.1) with a Toeplitz $p(x-u)$ or Hankel $q(x+u)$ kernel attracts attention of many authors as they have practical applications in such diverse fields as scattering theory, fluid dynamics, linear filtering theory, and inverse scattering problems in quantum-mechanics, problems in radiative wave transmission, and further applications in Medicine and Biology (see [1–11]). The historical development concerning Eq. (1.1) as well as its applications can be found in the above-mentioned works. In particular, Eq. (1.1) has been carefully studied when $K(x, u)$ is a Toeplitz $p(x-u)$, Hankel kernel $q(x+u)$, or $K(x, u) = p(x-u) + q(x+u)$, i.e. the Toeplitz and Hankel kernels are generated by the same function p .

In 1981, Tsitsiklis and Levy [11] considered Eq. (1.1) with general Toeplitz plus Hankel kernels $p(x-u) + q(x+u)$. Toeplitz plus Hankel kernels also appear in the study of a circular punch penetrating a finitely thick elastic layer resting on a rigid foundation, in that of atmospheric scattering, and in rarefied gas dynamics. Moreover, Eq. (1.1) is a generalization of Levinson equations considered by Chanda and Sabatier [4] for the Toeplitz case and of that studied by Agranovich and Marchenko [1] for the Hankel kernel.

Integral operators defined by the Toeplitz plus Hankel kernels are closely related to Wiener–Hopf plus Hankel type operators that are the vigorously studied objects. With respect to both numerical and theoretical investigations there have been many efforts, implicit or explicit, to study Eq. (1.1) in different spaces of functions. For example, by assuming that the

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functions p, q are twice continuously differentiable over all \mathbb{R} , the work [11] investigated some operator characteristics and gave numerical solutions to Eq. (1.1). Under an assumption that the kernel $K(x, u)$ is continuously differentiable in each of variables x and u , Levinson reduced the problem to an appealing set of first order partial differential equations that admit a recursive fast numerical solution and have been of particular use in the field of fast signal processing (see [3,12,11] and references therein). Feldman et al. [13] provided a sufficient and necessary condition via partial indices for the invertibility of the Toeplitz integral operator (without Hankel term) in the space of Lebesgue integrable functions on finite intervals. In recent times, the papers [14–18] dealing with the cases of infinite domains of integration have been published.

Our study of Eq. (1.1) is motivated by continuous efforts of those studies as well as a sufficiently long list of above-mentioned materials.

The aim of this paper is to study the solvability of the equation by a finite convolution approach. Our idea is constructing convolutions for the Hartley transforms and reducing the initial integral equation to systems of two linear algebraic equations. By determining solutions of those systems of equations, we obtain the L^2 -solution which is an explicit Hartley series. Thus, instead of the smoothness assumption for the kernel functions p and q , widely used in other works, we need only their Lebesgue integrability and 2π -periodicity.

The paper is divided into three sections and organized as follows.

In Section 2.1 of Section 2, the known notions of the finite Fourier sine, cosine transforms and that of the finite Hartley transforms are recalled, and some preparing lemmas related to the finite Hartley transforms are proved. In Section 2.2, we construct eight new generalized convolutions for the finite Hartley transforms, and prove the main theorem with a mind that the obtained generalized convolutions could be useful for other applications such as digital filtering, etc. Besides, it is proved that, the Riemann integrability, 2π -periodicity and boundedness of only function f are sufficient for ensuring the continuity of the convolution functions defined by (2.10)–(2.13), and (2.19)–(2.22).

Section 3 deals with the application involved in studying Eq. (1.1) which is our main interest. In Theorem 3.1, by assuming that the functions p and q are 2π -periodic and square-integrable on $[0, 2\pi]$ we obtain a necessary and sufficient condition for the solvability of Eq. (1.1) and its explicit solution is given by the Hartley series. It should be emphasized that the condition in Theorem 3.1 is remarkably different from those in other papers (cf. [1,4,13,11]). Furthermore, we show the fact that classical partial-differential equations on finite domains (for example, that are the elliptic, hyperbolic, or parabolic types) can be solved effectively by using the Hartley transform and their convolutions.

2. Hartley transforms and their convolutions

2.1. Hartley transforms

We now begin with the precise definition of the Fourier series of a function. Write $\mathbb{N} := \{0, 1, 2, 3, \dots\}$.

Definition 2.1 (See [19,20]). (a) Let f be a Lebesgue integrable function on a finite interval $[0, 2\pi]$. The finite Fourier cosine, sine transforms of f are defined respectively as

$$\mathcal{F}_c\{f(x)\}(n) = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx := \tilde{f}_c(n), \quad n \in \mathbb{N}, \quad (2.2)$$

$$\mathcal{F}_s\{f(x)\}(n) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx := \tilde{f}_s(n), \quad n \in \mathbb{N}. \quad (2.3)$$

(b) The infinite sum

$$(F_f)(x) := \frac{\tilde{f}_c(0)}{2} + \sum_{n=1}^{\infty} [\tilde{f}_c(n) \cos(nx) + \tilde{f}_s(n) \sin(nx)] \quad (2.4)$$

is called the Fourier series of f on $[0, 2\pi]$ where $\tilde{f}_c(n), \tilde{f}_s(n)$ are the Fourier coefficients of f .

At this point, we do not say anything about the convergence of series (2.4). Namely, there exists a function $f \in L^1[0, 2\pi]$ such that its series (2.4) diverges at every $x \in [0, 2\pi]$. However, if $1 < p < +\infty$ and if $f \in L^p[0, 2\pi]$, then its series (2.4) converges in mean (L^p -norm) to a function in $L^p[0, 2\pi]$. In the framework of this paper, only two spaces $L^1[0, 2\pi], L^2[0, 2\pi]$ are concerned.

Series (2.4) may be rewritten as follows

$$(F_f)(x) = \frac{\tilde{f}_c(0) + \tilde{f}_s(0)}{2} + \sum_{n=1}^{\infty} \left\{ \frac{\tilde{f}_c(n) + \tilde{f}_s(n)}{2} [\cos(nx) + \sin(nx)] + \frac{\tilde{f}_c(n) - \tilde{f}_s(n)}{2} [\cos(nx) - \sin(nx)] \right\}. \quad (2.5)$$

The representation in the form (2.5) suggests us to define the finite Hartley transforms as follows.

Definition 2.2 (Finite Hartley Transforms, See Also [21,22]). (a) Let f be a Lebesgue integrable function on $[0, 2\pi]$. The finite Hartley transforms of f are defined by

$$\begin{aligned}\mathcal{H}_1\{f(x)\}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \operatorname{cas}(nx) dx := \tilde{f}_1(n), \quad n \in \mathbb{N}, \\ \mathcal{H}_2\{f(x)\}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \operatorname{cas}(-nx) dx := \tilde{f}_2(n), \quad n \in \mathbb{N},\end{aligned}\quad (2.6)$$

where the integral kernel, known as the cosine-and-sine or *cas function*, is defined as $\operatorname{cas} x := \cos x + \sin x$.

(b) The infinite sum

$$(\mathcal{H}_f)(x) := \tilde{f}_1(0) + \sum_{n=1}^{\infty} [\tilde{f}_1(n) \operatorname{cas}(nx) + \tilde{f}_2(n) \operatorname{cas}(-nx)] \quad (2.7)$$

is called Hartley series of f on $[0, 2\pi]$. According to the above-mentioned convergence of (2.4) we can see that if $f \in L^2[0, 2\pi]$, then the right-hand side of (2.7) defines a function in space $L^2[0, 2\pi]$.

We call $\tilde{f}_1(n), \tilde{f}_2(n)$ the n -th Hartley coefficients of f corresponding to $\mathcal{H}_1, \mathcal{H}_2$, respectively. It is easily seen that for every $n \in \mathbb{N}$,

$$\tilde{f}_1(-n) = \tilde{f}_2(n), \quad \text{and} \quad \mathcal{H}_1\{f(-x)\}(n) = \mathcal{H}_2\{f(x)\}(n), \quad (2.8)$$

$$\tilde{f}_1(n) = \frac{1}{2}(\tilde{f}_c(n) + \tilde{f}_s(n)), \quad \tilde{f}_2(n) = \frac{1}{2}(\tilde{f}_c(n) - \tilde{f}_s(n)). \quad (2.9)$$

Theorem 2.1 below is an immediate consequence of the known results in Fourier analysis (see [20]).

Theorem 2.1. *The set of functions*

$$\left\{ 1/\sqrt{2\pi}; \operatorname{cas}(nx)/\sqrt{2\pi}; \operatorname{cas}(-nx)/\sqrt{2\pi} : n \geq 1 \right\}$$

is an orthonormal basis of the Hilbert space $L^2[0, 2\pi]$, and the following identities yield:

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \operatorname{cas} m(x+u) \operatorname{cas}(nu) du &= \delta_{mn} \cos nx, \\ \frac{1}{2\pi} \int_0^{2\pi} \operatorname{cas} m(x-u) \operatorname{cas}(nu) du &= \delta_{mn} \sin nx,\end{aligned}$$

where δ_{mn} is the Kronecker delta.

The Hartley transforms \mathcal{H}_1 and \mathcal{H}_2 also have the uniqueness theorem and the Riemann–Lebesgue lemma. Namely, if $f \in L^1[0, 2\pi]$ with $\tilde{f}_1(n) = \tilde{f}_2(n) = 0$ for all $n \in \mathbb{N}$, then $f = 0$; and if $f \in L^1[0, 2\pi]$, then

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} \operatorname{cas}(nx) f(x) dx = 0.$$

2.2. Convolutions

In general, Fourier coefficients of the product of two functions fg are not the product of the Fourier coefficients of f and g . The notion of convolution of two functions is a nice idea focusing on the so-called factorization identity which plays a fundamental role in Fourier analysis; namely, factorization identity of convolution says that the Fourier coefficients of $f * g$ are the product of the Fourier coefficients of f and g . Finite Fourier convolution appears naturally not only in the context of Fourier series but also serves more generally in the analysis of functions in other settings (see [19,20,23–25]).

Let $\|f\|_1$ denote the norm of a function f in $L^1[0, 2\pi]$.

Theorem 2.2 (Convolution Theorem). Suppose that the function f defined on \mathbb{R} is 2π -periodic. If f, g are Lebesgue integrable on $[0, 2\pi]$, then each of integral transforms (2.10)–(2.13) below defines a generalized Hartley convolution followed by its norm inequality and factorization identity:

$$\begin{aligned}(f *_{\mathcal{H}_1} g)(x) &= \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) + f(x-u) + f(-x+u) - f(-x-u)] g(u) du, \\ \|f *_{\mathcal{H}_1} g\|_1 &\leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_1\{(f *_{\mathcal{H}_1} g)(x)\}(n) = \tilde{f}_1(n) \tilde{g}_1(n).\end{aligned}\quad (2.10)$$

$$(f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) - f(x-u) + f(-x+u) + f(-x-u)]g(u)du, \quad (2.11)$$

$$\|f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_1\{(f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2}{*} g)(x)\}(n) = \tilde{f}_2(n)\tilde{g}_2(n).$$

$$(f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [-f(x+u) + f(x-u) + f(-x+u) + f(-x-u)]g(u)du, \quad (2.12)$$

$$\|f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_1\{(f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1}{*} g)(x)\}(n) = \tilde{f}_2(n)\tilde{g}_1(n).$$

$$(f \underset{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) + f(x-u) - f(-x+u) + f(-x-u)]g(u)du, \quad (2.13)$$

$$\|f \underset{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_1\{(f \underset{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2}{*} g)(x)\}(n) = \tilde{f}_1(n)\tilde{g}_2(n).$$

Moreover, if f, g are Riemann integrable and bounded on $[0, 2\pi]$, then the functions defined by the integrals on the right-hand side of (2.10)–(2.13) are continuous.

To prove Theorem 2.2 we need the following lemmas.

Lemma 2.1. Let f be a 2π -periodic function. Assume that f is Lebesgue integrable on $[0, 2\pi]$. The following identities hold for any $u \in [0, 2\pi]$ and for every $n \in \mathbb{N}$:

$$\mathcal{H}_1\{f(x+u) + f(x-u) + f(-x+u) - f(-x-u)\}(n) = 2\tilde{f}_1(n)\text{cas}(nu), \quad (2.14)$$

$$\mathcal{H}_2\{f(x-u) + f(-x+u) + f(-x-u) - f(x+u)\}(n) = 2\tilde{f}_2(n)\text{cas}(nu). \quad (2.15)$$

Proof. Let us first prove (2.14). Due to the assumptions, f is integrable on every finite interval $[a, a + 2\pi]$. We have

$$\begin{aligned} 2\tilde{f}_1(n)\text{cas}(nu) &= 2\text{cas}(nu) \frac{1}{2\pi} \int_0^{2\pi} f(x)\text{cas}(nx)dx \\ &= \frac{1}{\pi} \int_0^{2\pi} f(x)[\cos(nu)\cos(nx) + \cos(nu)\sin(nx) + \sin(nu)\cos(nx) + \sin(nu)\sin(nx)]dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x)[2\cos n(x-u) + 2\sin n(x+u)]dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x)[\text{cas } n(x-u) + \text{cas } n(-x+u) + \text{cas } n(x+u) - \text{cas } n(-x-u)]dx, \end{aligned}$$

which is, by putting $x-u = t$, $-x+u = t$, $x+u = t$ and $-x-u = t$,

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-u}^{2\pi-u} f(t+u)\text{cas}(nt)dt + \frac{1}{2\pi} \int_{-2\pi+u}^{+u} f(-t+u)\text{cas}(nt)dt + \frac{1}{2\pi} \int_{+u}^{2\pi+u} f(t-u)\text{cas}(nt)dt \\ &\quad - \frac{1}{2\pi} \int_{-2\pi-u}^{-u} f(-t-u)\text{cas}(nt)dt. \end{aligned}$$

As the inner integral functions are 2π -periodic with respect to corresponding to variable t , the limits of integration can be replaced with limits from 0 to 2π . Therefore,

$$\begin{aligned} 2\tilde{f}_1(n)\text{cas}(nu) &= \frac{1}{2\pi} \int_0^{2\pi} f(t+u)\text{cas}(nt)dt + \frac{1}{2\pi} \int_0^{2\pi} f(-t+u)\text{cas}(nt)dt \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} f(t-u)\text{cas}(nt)dt - \frac{1}{2\pi} \int_0^{2\pi} f(-t-u)\text{cas}(nt)dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} [f(x+u) + f(-x+u) + f(x-u) - f(-x-u)]\text{cas}(nx)dx \\ &= \mathcal{H}_1\{f(x+u) + f(x-u) + f(-x+u) - f(-x-u)\}(n). \end{aligned}$$

Identity (2.14) is proved. Identity (2.15) may be proved analogously. The lemma is proved. \square

Proof of Theorem 2.2. Firstly, we prove the norm inequality of convolution (2.10). Since f is 2π -periodic and integrable on $[0, 2\pi]$ we have

$$\begin{aligned} \int_0^{2\pi} |f(x-u)|dx &= \int_{-u}^{2\pi-u} |f(t)|dt = \int_{-u}^0 |f(t)|dt + \int_0^{2\pi-u} |f(t)|dt \\ &= \int_{2\pi-u}^{2\pi} |f(s-2\pi)|ds + \int_0^{2\pi-u} |f(t)|dt = \int_0^{2\pi} |f(t)|dt = \|f\|_1, \end{aligned} \quad (2.16)$$

for any $u \in \mathbb{R}$. Similarly,

$$\int_0^{2\pi} |f(x+u)|dx = \int_0^{2\pi} |f(-x+u)|dx = \int_0^{2\pi} |f(-x-u)|dx = \|f\|_1. \quad (2.17)$$

We then have

$$\begin{aligned} \|f *_{\mathcal{H}_1} g\|_1 &= \frac{1}{4\pi} \int_0^{2\pi} dx \left| \int_0^{2\pi} [f(x+u) + f(x-u) + f(-x+u) - f(-x-u)] g(u) du \right| \\ &\leq \frac{1}{4\pi} \int_0^{2\pi} |g(u)| du \times \int_0^{2\pi} [|f(x+u)| + |f(x-u)| + |f(-x+u)| + |f(-x-u)|] dx = \frac{1}{\pi} \|g\|_1 \|f\|_1. \end{aligned}$$

We shall prove the factorization identity. By Lemma 2.1,

$$\begin{aligned} \tilde{f}_1(n) \tilde{g}_1(n) &= \frac{1}{2\pi} \int_0^{2\pi} g(u) \tilde{f}_1(n) \text{cas}(nu) du \\ &= \frac{1}{4\pi} \int_0^{2\pi} \mathcal{H}_1\{f(x+u) + f(x-u) + f(-x+u) - f(-x-u)\}(n) g(u) du \\ &= \frac{1}{4\pi} \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} [f(x+u) + f(x-u) + f(-x+u) - f(-x-u)] \text{cas}(nx) dx \right) \times g(u) du \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{4\pi} \int_0^{2\pi} [f(x+u) + f(x-u) + f(-x+u) - f(-x-u)] \times g(u) du \right) \text{cas}(nx) dx \\ &= \mathcal{H}_1\{(f *_{\mathcal{H}_1} g)(x)\}(n), \end{aligned}$$

which is desired.

The norm inequalities and factorization identities of convolutions (2.11)–(2.13) may be proved analogously.

Concerning the second statement of the theorem, it suffices to prove that

$$(f * g)(x) := \int_0^{2\pi} f(x-u) g(u) du \quad (2.18)$$

defines a continuous function, whose continuity follows immediately from the following lemma.

Lemma 2.2 (See [20]). Suppose f is Riemann integrable and bounded by B . Then there exists a sequence $\{f_k\}_{k=1}^\infty$ of continuous functions so that

$$\sup_{x \in [0, 2\pi]} |f_k(x)| \leq B, \quad \text{and} \quad \int_0^{2\pi} |f(x) - f_k(x)| dx \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Clearly, $f *_{\mathcal{H}_1} g$ is a 2π -periodic function. Therefore, the proof for convolution (2.10) is completed.

Convolutions (2.11)–(2.13) may be proved in the same way as in the proof of (2.10). Theorem 2.2 is proved. \square

Corollary 2.1 (Convolution Theorem). If the functions f, g fulfill the conditions as in Theorem 2.2, then each one of integral transforms (2.10)–(2.13) below defines a convolution:

$$\begin{aligned} (f *_{\mathcal{H}_2} g)(x) &= \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) + f(x-u) + f(-x+u) - f(-x-u)] g(u) du, \\ \|f *_{\mathcal{H}_2} g\|_1 &\leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_2\{(f *_{\mathcal{H}_2} g)(x)\}(n) = \tilde{f}_2(n) \tilde{g}_2(n). \end{aligned} \quad (2.19)$$

$$(f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) - f(x-u) + f(-x+u) + f(-x-u)]g(u)du, \quad (2.20)$$

$$\|f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_2\{(f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1}{*} g)(x)\}(n) = \tilde{f}_1(n)\tilde{g}_1(n).$$

$$(f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [-f(x+u) + f(x-u) + f(-x+u) + f(-x-u)]g(u)du, \quad (2.21)$$

$$\|f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_2\{(f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2}{*} g)(x)\}(n) = \tilde{f}_1(n)\tilde{g}_2(n).$$

$$(f \underset{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_1}{*} g)(x) = \frac{1}{4\pi} \int_0^{2\pi} [f(x+u) + f(x-u) - f(-x+u) + f(-x-u)]g(u)du, \quad (2.22)$$

$$\|f \underset{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_1}{*} g\|_1 \leq \|f\|_1 \|g\|_1; \quad \mathcal{H}_2\{(f \underset{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_1}{*} g)(x)\}(n) = \tilde{f}_2(n)\tilde{g}_1(n).$$

Moreover, if f, g are Riemann integrable and bounded on $[0, 2\pi]$, then the functions defined by these convolutions are continuous on $[0, 2\pi]$.

The convolutions (2.19)–(2.22) can be proved similarly to those in Theorem 2.2. In fact, the factorization identities can be proved directly in other way as: since (2.10), $\mathcal{H}_1\{(f \underset{\mathcal{H}_1}{*} g)(x)\} = \mathcal{H}_1\{f(x)\}\mathcal{H}_1\{g(x)\}$. Replacing x with $-x$ in this identity and using (2.8), we obtain (2.19). The other ones of convolutions (2.20)–(2.22) might be proved similarly.

Note that $L^2[0, 2\pi] \subset L^1[0, 2\pi]$. We have the following theorem.

Theorem 2.3. Suppose that f, g fulfill the assumptions in Theorem 2.2. If f, g are squares-integrable on $[0, 2\pi]$ then the following norm inequalities hold:

$$\begin{aligned} \|f \underset{\mathcal{H}_1}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; & \|f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; \\ \|f \underset{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; & \|f \underset{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; \\ \|f \underset{\mathcal{H}_2}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; & \|f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; \\ \|f \underset{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; & \|f \underset{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_2}{*} g\|_2 &\leq M_0 \|f\|_2 \|g\|_2; \end{aligned}$$

where $M_0 := 2\sqrt{2}$.

Proof. It is sufficient to prove the first inequality as the others may be completed similarly. By the Cauchy–Schwarz inequality,

$$(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2). \quad (2.23)$$

In the same way as in the proof of (2.16), we can prove four identities presented in (2.24) below:

$$\int_0^{2\pi} |f(\alpha x + \beta u)|^2 du = \|f\|_2^2, \quad (2.24)$$

where the coefficients α and β may be chosen arbitrarily from the set $\{-1, 1\}$. Note that function f is 2π -periodic. Using the Schwarz–Bunyakovsky integral inequality and (2.23) and (2.24), we have the following inequalities for every $x \in [0, 2\pi]$

$$\begin{aligned} |(f \underset{\mathcal{H}_1}{*} g)(x)|^2 &= \frac{1}{4\pi} \left| \int_0^{2\pi} [f(x+u) + f(x-u) + f(u-x) - f(-x-u)]g(u)du \right|^2 \\ &\leq \frac{1}{4\pi} \int_0^{2\pi} |g(u)|^2 du \int_0^{2\pi} |f(x+u) + f(x-u) + f(-x+u) - f(-x-u)|^2 du \\ &\leq \frac{1}{\pi} \int_0^{2\pi} |g(u)|^2 du \int_0^{2\pi} [|f(x+u)|^2 + |f(x-u)|^2 + |f(-x+u)|^2 \\ &\quad + |f(-x-u)|^2] du \leq \frac{4}{\pi} \|f\|_2^2 \|g\|_2^2. \end{aligned}$$

This implies $\|f \underset{\mathcal{H}_1}{*} g\|_2^2 \leq 8\|f\|_2^2 \|g\|_2^2$. The theorem is proved. \square

The following examples show the continuity of new convolutions (2.10)–(2.13) and (2.19)–(2.22) even if the functions f, g are discontinuous. Moreover, we will see that each of those convolutions is totally different from others and from the classical Fourier convolutions.

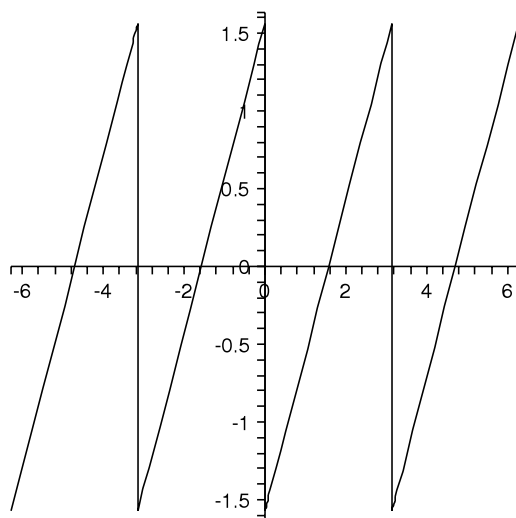


Fig. 1. $f(x) = x - \frac{\pi}{2} - \pi \left[\frac{x}{\pi} \right]$.

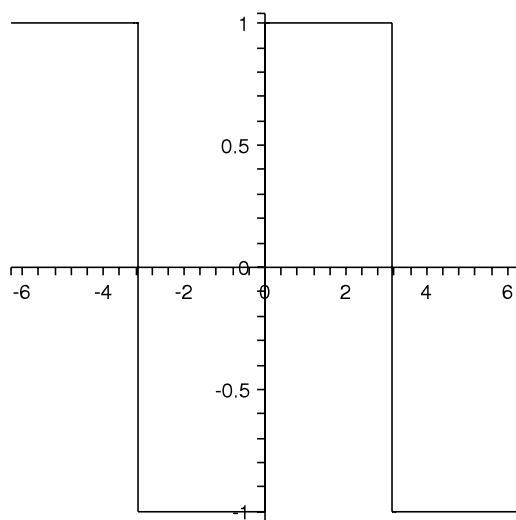


Fig. 2. $f(x) = (-1)^{\left[\frac{x}{\pi} \right]}$.

Example 2.1. Let $f(x) = x - \frac{\pi}{2} - \pi \left[\frac{x}{\pi} \right]$ (see Fig. 1), and $g(x) = 2x$. We have Figs. 3 and 4.

Example 2.2. Let $f(x) = (-1)^{\left[\frac{x}{\pi} \right]}$ (see Fig. 2), $g(x) = 8x^3 - 12x$. We have Figs. 5 and 6.

3. Application

We now consider the integral equation

$$\lambda \varphi(x) + \frac{1}{\pi} \int_0^{2\pi} [p(x-u) + q(x+u)] \varphi(u) du = f(x), \quad (3.25)$$

where $\lambda \in \mathbb{C}$ is predetermined, p, q, f are given functions, and φ is to be determined. The functions p, q are known as the Toeplitz and Hankel kernels, respectively. Being different from other approaches, our idea is to reduce Eq. (3.25) to systems of two linear equations by using a group of eight new finite Hartley convolutions.

Our results of Theorem 3.1 are based on the assumption that functions q, p are piecewise continuous on $[0, 2\pi]$. As every continuous or piecewise continuous function on $[0, 2\pi]$ has its 2π -periodic extension we can assume that the functions p, q are piecewise continuous and 2π -periodic on \mathbb{R} .

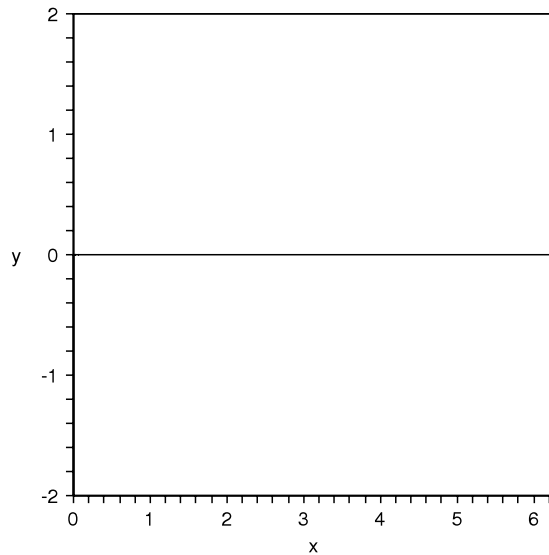


Fig. 3. $(f *_{\mathcal{F}_c} g)(x)$.

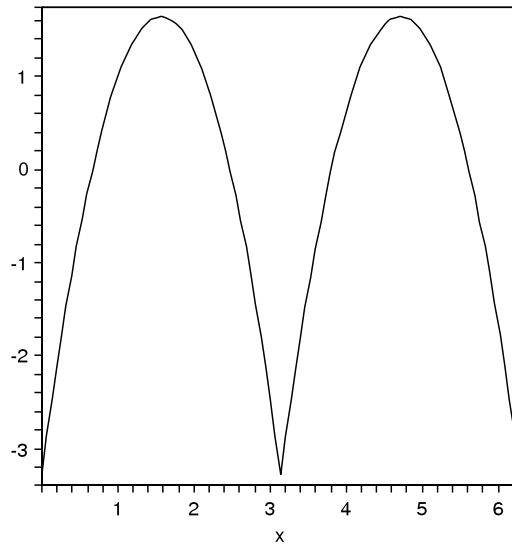


Fig. 4. $(f *_{\mathcal{F}_S, \mathcal{F}_c, \mathcal{F}_S} g)(x)$.

In what follows, let $\tilde{p}_1(n), \tilde{p}_2(n), \tilde{q}_1(n), \tilde{q}_2(n)$ denote the Hartley coefficients of the functions $p(x), q(x)$, respectively. Write:

$$\begin{aligned} \mathbf{A}(n) &:= \lambda + \tilde{p}_1(n) + \tilde{p}_2(n) + \tilde{q}_1(n) - \tilde{q}_2(n); & \mathbf{B}(n) &:= \tilde{p}_1(n) - \tilde{p}_2(n) + \tilde{q}_1(n) + \tilde{q}_2(n); \\ \mathbf{D}(n) &:= \mathbf{A}(n)\mathbf{A}(-n) - \mathbf{B}(n)\mathbf{B}(-n); & \mathbf{D}_1(n) &:= \mathbf{A}(-n)\tilde{f}_1(n) - \mathbf{B}(n)\tilde{f}_2(n); \\ \mathbf{D}_2(n) &:= \mathbf{A}(n)\tilde{f}_2(n) - \mathbf{B}(-n)\tilde{f}_1(n). \end{aligned} \quad (3.26)$$

Theorem 3.1. Suppose that the functions p, q are piecewise continuous on $[0, 2\pi]$ and f is a given square-integrable function.

- (i) If $\lambda \neq 0$, then there exists an integer K^* such that $\mathbf{D}(n) \neq 0$ for every $n \geq K^*$.
- (ii) If $\mathbf{D}(n) \neq 0$ for every $n \in \mathbb{N}$, then Eq. (3.25) has a unique L^2 -solution for every $f \in L^2[0, 2\pi]$ which is given by

$$\varphi(x) = \frac{\mathbf{D}_1(0)}{\mathbf{D}(0)} + \sum_{n=1}^{\infty} \left[\frac{\mathbf{D}_1(n)}{\mathbf{D}(n)} \text{cas}(nx) + \frac{\mathbf{D}_2(n)}{\mathbf{D}(n)} \text{cas}(-nx) \right]. \quad (3.27)$$

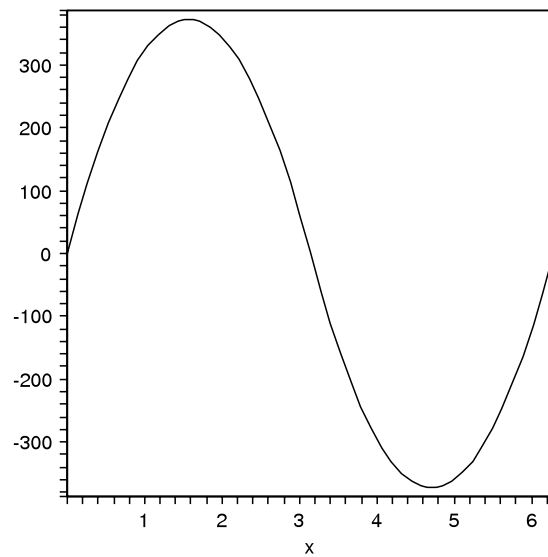


Fig. 5. $(f *_{\mathcal{F}_C} g)(x)$.

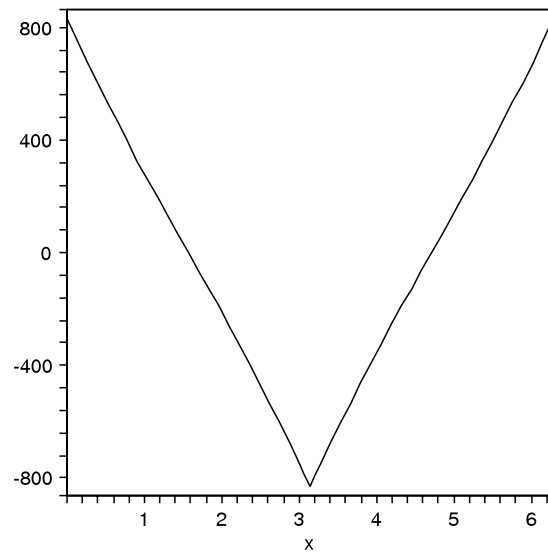


Fig. 6. $(f *_{\mathcal{F}_S, \mathcal{F}_C, \mathcal{F}_S} g)(x)$.

Proof. (i). By the Riemann–Lebesgue lemma as: $\lim_{n \rightarrow \infty} \tilde{p}_j(n) = \lim_{n \rightarrow \infty} \tilde{q}_j(n) = 0$ ($j = 1, 2$), we have $\lim_{n \rightarrow \infty} \mathbf{D}(n) = \lambda^2 \neq 0$. Therefore, there exists an integer $K^* \in \mathbb{N}$ such that $\mathbf{D}(n) \neq 0 \forall n \geq K^*$. Item (i) is proved.

(ii) From convolutions (2.10)–(2.13) and (2.19)–(2.22) it follows that

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} f(x-u)g(u)du &= (f *_{\mathcal{H}_1} g)(x) - (f *_{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2} g)(x) + (f *_{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1} g)(x) + (f *_{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2} g)(x) \\ &= (f *_{\mathcal{H}_2} g)(x) - (f *_{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1} g)(x) + (f *_{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2} g)(x) + (f *_{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_1} g)(x), \end{aligned} \quad (3.28)$$

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} f(x+u)g(u)du &= (f *_{\mathcal{H}_1} g)(x) + (f *_{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_2} g)(x) - (f *_{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1} g)(x) + (f *_{\mathcal{H}_1, \mathcal{H}_1, \mathcal{H}_2} g)(x) \\ &= (f *_{\mathcal{H}_2} g)(x) + (f *_{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_1} g)(x) - (f *_{\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2} g)(x) + (f *_{\mathcal{H}_2, \mathcal{H}_2, \mathcal{H}_1} g)(x). \end{aligned} \quad (3.29)$$

Applying \mathcal{H}_1 , \mathcal{H}_2 to both sides of (3.28) and (3.29) and using the factorization identities of those convolutions that appeared in the right-hand sides, we obtain

$$\mathcal{H}_1 \left(\frac{1}{\pi} \int_0^{2\pi} f(x-u)g(u)du \right) (n) = \tilde{f}_1(n)\tilde{g}_1(n) - \tilde{f}_2(n)\tilde{g}_2(n) + \tilde{f}_2(n)\tilde{g}_1(n) + \tilde{f}_1(n)\tilde{g}_2(n), \quad (3.30)$$

$$\mathcal{H}_2 \left(\frac{1}{\pi} \int_0^{2\pi} f(x-u)g(u)du \right) (n) = \tilde{f}_2(n)\tilde{g}_2(n) - \tilde{f}_1(n)\tilde{g}_1(n) + \tilde{f}_1(n)\tilde{g}_2(n) + \tilde{f}_2(n)\tilde{g}_1(n), \quad (3.31)$$

$$\mathcal{H}_1 \left(\frac{1}{\pi} \int_0^{2\pi} f(x+u)g(u)du \right) (n) = \tilde{f}_1(n)\tilde{g}_1(n) + \tilde{f}_2(n)\tilde{g}_2(n) - \tilde{f}_2(n)\tilde{g}_1(n) + \tilde{f}_1(n)\tilde{g}_2(n), \quad (3.32)$$

$$\mathcal{H}_2 \left(\frac{1}{\pi} \int_0^{2\pi} f(x+u)g(u)du \right) (n) = \tilde{f}_2(n)\tilde{g}_2(n) + \tilde{f}_1(n)\tilde{g}_1(n) - \tilde{f}_1(n)\tilde{g}_2(n) + \tilde{f}_2(n)\tilde{g}_1(n), \quad (3.33)$$

for any 2π -periodic integrable functions f and integrable functions g .

Come back to Eq. (3.25). Let us first prove the uniqueness of the solution of Eq. (3.25) by the Fredholm alternative theorem. Suppose that the homogeneous equation corresponding to Eq. (3.25) (i.e. $f = 0$) has a solution $\varphi_* \in L^2[0, 2\pi]$ (note that it has at least trivial solution $\varphi = 0$), i.e.,

$$\lambda\varphi_*(x) + \frac{1}{\pi} \int_0^{2\pi} [p(x-u) + q(x+u)]\varphi_*(u)du = 0.$$

Applying \mathcal{H}_1 , \mathcal{H}_2 to both sides of the above identity and using the identities (3.30)–(3.33), we obtain a system of two linear equations

$$\begin{cases} \mathbf{A}(n)\tilde{\varphi}_{*1}(n) + \mathbf{B}(n)\tilde{\varphi}_{*2}(n) = 0 \\ \mathbf{B}(-n)\tilde{\varphi}_{*1}(n) + \mathbf{A}(-n)\tilde{\varphi}_{*2}(n) = 0, \end{cases} \quad (3.34)$$

for defining the unknown Hartley coefficients $\tilde{\varphi}_{*1}(n)$, $\tilde{\varphi}_{*2}(n)$ of φ_* . For $n \in \mathbb{N}$, the determinants of (3.34) are defined as in (3.26). Since $\mathbf{D}(n) \neq 0$ for every $n \in \mathbb{N}$, $\tilde{\varphi}_{*1}(n) = \tilde{\varphi}_{*2}(n) = 0 \forall n \geq 0$. Due to the uniqueness theorem of the Hartley transforms, we obtain $\varphi_* = 0$. Thus, the homogeneous equation to (3.25) has only a trivial solution, hence by the Fredholm alternative theorem, Eq. (3.25) has a unique solution.

We shall establish the solution formula (3.27). Suppose that φ is a square-integrable function satisfying (3.25). In the same way as obtaining the system (3.34), we have the system of two linear equations for every $n \geq 0$

$$\begin{cases} \mathbf{A}(n)\tilde{\varphi}_1(n) + \mathbf{B}(n)\tilde{\varphi}_2(n) = \tilde{f}_1(n) \\ \mathbf{B}(-n)\tilde{\varphi}_1(n) + \mathbf{A}(-n)\tilde{\varphi}_2(n) = \tilde{f}_2(n). \end{cases} \quad (3.35)$$

Since $\mathbf{D}(n) \neq 0$ for every $n \in \mathbb{N}$, system (3.35) has a unique solution given by

$$\tilde{\varphi}_1(n) = \frac{\mathbf{D}_1(n)}{\mathbf{D}(n)}, \quad \tilde{\varphi}_2(n) = \frac{\mathbf{D}_2(n)}{\mathbf{D}(n)}, \quad n = 0, 1, \dots \quad (3.36)$$

By Theorem 2.1,

$$\|\varphi\|_2 = \left| \frac{\mathbf{D}_1(0)}{\mathbf{D}(0)} \right|^2 + \sum_{n=1}^{\infty} \left[\left| \frac{\mathbf{D}_1(n)}{\mathbf{D}(n)} \right|^2 + \left| \frac{\mathbf{D}_2(n)}{\mathbf{D}(n)} \right|^2 \right] < +\infty.$$

Therefore, the function φ_0 given by

$$\varphi_0(x) = \frac{\mathbf{D}_1(0)}{\mathbf{D}(0)} + \sum_{n=1}^{\infty} \left[\frac{\mathbf{D}_1(n)}{\mathbf{D}(n)} \cos(nx) + \frac{\mathbf{D}_2(n)}{\mathbf{D}(n)} \cos(-nx) \right]$$

belongs to $L^2[0, 2\pi]$, and $\varphi_0(x) = \varphi(x)$ for almost every $x \in [0, 2\pi]$. But, φ satisfies (3.25), so does φ_0 . Item (ii) is proved. \square

Example 3.1. Find the solution of the following linear integral equation with a degenerated kernel:

$$\varphi(x) + \frac{1}{\pi} \int_0^{2\pi} [\sin 3(x-u) + 2 \cos(x+u)]\varphi(u)du = 8 \cos^3 x. \quad (3.37)$$

Using formula (3.27) we have the solution

$$\varphi(x) = 8 \cos^3 x - \sin 3x - \cos 3x - 4 \cos x.$$

In fact, we can obtain the above solution by the degenerated kernel method.

Example 3.2. Find the solution of a linear integral equation with a non-degenerated kernel:

$$\varphi(x) + \frac{1}{\pi} \int_0^{2\pi} \left[\frac{\sqrt{2}}{\sin(x-u)-3} + \frac{\sqrt{3}}{\cos(x+u)+2} \right] \varphi(u) du = 2. \quad (3.38)$$

Since $\mathbf{D}_{1(n)} = \mathbf{D}_{2(n)} = 0$ for every $n \geq 1$, we can use formula (3.27) to obtain solution $\varphi(x) = \mathbf{D}_1(0)/\mathbf{D}(0) = 1$.

On the other hand, the degenerated kernel method cannot be applied in this case.

Example 3.3. Find the solution of a linear integral equation with a non-degenerated kernel:

$$\varphi(x) + \frac{1}{\pi} \int_0^{2\pi} \left[\sin(x-u) + \frac{\sqrt{3}}{\cos(x+u)+2} \right] \varphi(u) du = (5-2\sqrt{3}) \sin x - \cos x. \quad (3.39)$$

Since $\mathbf{D}_1(n) = \mathbf{D}_2(n) = 0$ for every $n \geq 2$, by formula (3.27) we have the exact unique solution

$$\varphi(x) = \frac{\mathbf{D}_1(0)}{\mathbf{D}(0)} + \frac{\mathbf{D}_1(1)}{\mathbf{D}(1)} \cos(x) + \frac{\mathbf{D}_2(1)}{\mathbf{D}(1)} \cos(-x) = \sin x.$$

Again, the degenerated kernel method cannot be applied in this case.

Proposition 3.1 below can be proved immediately.

Proposition 3.1. Let f has the integrable second derivative on interval $[0, 2\pi]$. Then

$$\begin{aligned} \mathcal{H}_1\{f'(x)\} &= -n\mathcal{H}_2\{f(x)\} + \frac{1}{2\pi}[f(2\pi) - f(0)], \\ \mathcal{H}_2\{f'(x)\} &= n\mathcal{H}_1\{f(x)\} + \frac{1}{2\pi}[f(2\pi) - f(0)], \\ \mathcal{H}_1\{f''(x)\} &= -n^2\mathcal{H}_1\{f(x)\} + \frac{1}{2\pi}[f'(2\pi) - f'(0) - nf(2\pi) + nf(0)], \\ \mathcal{H}_2\{f''(x)\} &= -n^2\mathcal{H}_2\{f(x)\} + \frac{1}{2\pi}[f'(2\pi) - f'(0) + nf(2\pi) - nf(0)]. \end{aligned}$$

Examples 3.4–3.6 are served as the illustrations for three typical types of partial-differential equations on finite domains. Other applications of the Fourier and Hartley transforms on entire axis \mathbb{R} can be found in many works, for instance [21,19].

Example 3.4 (*Heat Conduction Problem in a Finite Domain with the Dirichlet Data at the Boundary*). Obtain the temperature distribution $u(x, t)$ of the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2\pi, t > 0,$$

knowing the boundary and initial conditions

$$u(0, t) = u(2\pi, t), \quad u_x(0, t) = u_x(2\pi, t), \quad t > 0, \quad u(x, 0) = f(x), \quad 0 < x < 2\pi.$$

Let

$$U(n, t) = \frac{1}{2\pi} \int_0^{2\pi} u(x, t) \cos(nx) dx.$$

We have

$$\frac{\partial U}{\partial t} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial t} \cos(nx) dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 u}{\partial x^2} \cos(nx) dx \quad (3.40)$$

$$= -n^2 \frac{1}{2\pi} \int_0^{2\pi} u(x, t) \cos(nx) dx = -n^2 U. \quad (3.41)$$

It implies $U(n, t) = A(n)e^{-n^2 t}$. Putting $t = 0$, we get

$$A(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx = \tilde{f}_1(n).$$

Hence $U(n, t) = \tilde{f}_1(n)e^{-n^2 t}$. Thus, the solution is

$$u(x, t) = \sum_{n \in \mathbb{Z}} \tilde{f}_1(n) e^{-n^2 t} \operatorname{cas}(nx).$$

Example 3.5 (*Free Transverse Vibrations of an Elastic String of Finite Length*). We consider the free vibration of a string of length 2π . The free transverse displacement function $v(x, t)$ satisfies the wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < 2\pi, t > 0,$$

with the boundary and initial conditions

$$\begin{aligned} v(0, t) &= v(2\pi, t), & v_x(0, t) &= v_x(2\pi, t), & t > 0, \\ v(x, 0) &= f(x), & v_x(x, 0) &= g(x), & 0 < x < 2\pi. \end{aligned}$$

In the same way as in Example 3.1, we obtain the solution

$$v(x, t) = \sum_{n \in \mathbb{Z}} \left[\tilde{f}_1(n) \cos(nt) + \frac{\tilde{g}_1(n)}{n} \sin(nt) \right] \operatorname{cas}(nx).$$

Example 3.6. Obtain the solutions $w(x, t)$ of the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad 0 < x < 2\pi, 0 < y < a,$$

with the boundary and initial conditions

$$\begin{aligned} w(0, y) &= w(2\pi, y), & w_x(0, y) &= w_x(2\pi, y), & 0 < y < a, \\ w(x, 0) &= f(x), & w_x(x, a) &= g(x), & 0 < x < 2\pi. \end{aligned}$$

This is the problem of the steady distribution of heat in a finite strip with the edges kept at given temperatures. We also obtain

$$w(x, y) = \sum_{n \in \mathbb{Z}} \left[\tilde{f}_1(n) \cosh(ny) + \frac{\tilde{g}_1(n) - \tilde{f}_1(n) \cosh(na)}{\sinh(na)} \sinh(ny) \right] \operatorname{cas}(nx).$$

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