



Wave breaking for a generalized weakly dissipative two-component Camassa–Holm system

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ABSTRACT

In this paper we study a generalized weakly dissipative two-component Camassa–Holm system. We show that this system can exhibit the wave-breaking phenomenon. We also determine the exact blow-up rate of such solutions.

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1. Introduction

In recent years, the Camassa–Holm equation [1,2]

$$u_t - u_{xx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R} \quad (1.1)$$

which models propagation of shallow water waves has attracted the attention of a large number of researchers, and they find two remarkable properties of (1.1). The first one is the equation possesses the solutions in the form of peaked solitons or 'peakons' [1,3]. The peakon $u(t, x) = ce^{-|x-ct|}$, $c \neq 0$ is smooth except at its crest and the tallest among all waves of fixed energy. It is a feature observed for the traveling waves of largest amplitude which solve the governing equations for water waves [4–11]. Another remarkable property is the equation has breaking waves [1,12], that is, the solution remains bounded while its slope becomes unbounded in finite time. After wave breaking the solutions can be continued uniquely as either global conservative [13] or global dissipative solutions [14].

The Camassa–Holm equation also admits many integrable multicomponent generalizations. The most popular one is

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho \rho_x = 0 \\ \rho_t + (\rho u)_x = 0 \\ m = u - u_{xx}. \end{cases} \quad (1.2)$$

Notice that the C–H equation can be obtained via the obvious reduction $\rho \equiv 0$ and $A = 0$. System (1.2) was derived first in [15], where $\rho(t, x)$ is related to the free surface elevation from equilibrium (or scalar density), and $A \geq 0$ characterizes a linear underlying shear flow. Recently, Constantin–Ivanov [16] and Ivanov [17] have established a rigorous justification of the derivation of system (1.2). Mathematical properties of the system have been also studied further in many works, for example [18–23]. Chen, Liu and Zhang [19] established a reciprocal transformation between the two-component Camassa–Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld, and Yin [20] investigated

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local well-posedness for the two-component Camassa–Holm system with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$ with $s \geq 2$ by applying Kato's theory [24] and provided some precise blow-up scenarios for strong solutions to the system. The local well-posedness is improved by Gui and Liu [25] to the Besov Spaces (especially in the Sobolev space $H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$), and they showed that the finite time blow-up is determined by either the slope of the first component u or the slope of the second component ρ (also see [16,20]). The blow-up criterion is made more precise in [26] where the authors showed that the wave breaking in finite time only depends on the slope of u . In other words, the wave breaking in u must occur before that in ρ . This blow-up criterion is further improved in [27] to the lowest Sobolev spaces $H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$.

In general, it is difficult to avoid energy dissipation mechanisms in a real world. We are interested in the effect of the weakly dissipative term on the two-component Camassa–Holm equation. Wu, Escher and Yin have investigated the blow-up phenomena, blow-up rate of the strong solutions of the weakly dissipative CH equation [28] and DP equation [29]. Inspired by the above results, in this paper, we investigate the following generalized weakly dissipative two-component Camassa–Holm system

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho\rho_x = 0 \\ \rho_t + (\rho u)_x = 0 \end{cases} \quad (1.3)$$

or equivalently,

$$\begin{cases} m_t - Au_x + \sigma(um_x + 2u_xm) + 3(1 - \sigma)uu_x + \lambda m + \rho\rho_x = 0 \\ \rho_t + (\rho u)_x = 0 \\ m = u - u_{xx} \end{cases} \quad (1.4)$$

with $u \rightarrow 0, \rho \rightarrow 1$ as $|x| \rightarrow \infty$, where $\lambda m = \lambda(I - \partial_{xx})u$ is the weakly dissipative term, $\lambda \geq 0$ and A are constants, σ is a new free parameter. When $A = 0, \lambda = 0$ and $\rho = 1$, Guan and Yin have obtained a new result of the existence of the strong solution and some new blow-up results (see [30]). Meanwhile, they have proved the global existence of the weak solution about the two-component CH equation (see [31]). In [32] Henry investigates the infinite propagation speed of the solution for a two-component CH equation.

Similar to [16,20], we can use the method of Besov spaces together with the transport equation theory to show that system (1.4) is locally well-posed in $H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$. The two equations for u and ρ are of a transport structure $\partial_t f + v\partial_x f = g$. It is well known that most of the available estimates require v to have some level of regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as v belongs to $L^1(0, T; Lip)$. More specially, u and ρ are “transported” along directions of σu and u respectively. Then, the solution can be estimated in a Gronwall way involving $\|u_x\|_{L^\infty}$. Hence, one can use these estimates to derive a criterion which says if $\int_0^T \|u_x(\tau)\|_{L^\infty} d\tau < \infty$, then solutions can be extended further in time. Compared with the result in [33], we find that Eq. (1.4) has the same blow-up rate when the blow-up occurs. This fact shows that the blow-up rate of Eq. (1.4) is not affected by the weakly dissipative term. But the occurrence of blow-up of Eq. (1.4) is affected by the dissipative parameter λ . When $\lambda = 0$ and $\rho = 1$, the result in this paper can go back to the originally paper [27].

The basic elementary framework is as follows. Section 2 gives the local well-posedness of system (1.4) and a wave-breaking criterion, which implies that the wave breaking only depends on the slope of u , not the slope of ρ . Section 3 improves the blow-up criterion with a more precise conditions. Section 4 determine the exact blow-up rate of strong solutions of system (1.4).

2. Formation of singularities for $\sigma \neq 0$

We consider the following generalized weakly dissipative two-component Camassa–Holm system:

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho\rho_x = 0, & t > 0, x \in R \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in R \\ u(0, x) = u_0(x), & x \in R \\ \rho(0, x) = \rho_0(x), & x \in R \end{cases} \quad (2.1)$$

where $\lambda \geq 0$ and A are constants, σ is a new free parameter.

The system (2.1) can be written in the following “transport” form:

$$\begin{cases} u_t + \sigma uu_x = -\partial_x p * \left(-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 \right) - \lambda u \\ \rho_t + (\rho u)_x = 0 \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x) \end{cases} \quad (2.2)$$

where $p(x) := \frac{1}{2}e^{-|x|}$, $x \in R$, and $*$ indicates the convolution operator.

Applying transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [25] to obtain the following local well-posedness result for the system (2.1). The proof is very similar to that of Theorem 1.1 in [25] and hence is omitted.

Theorem 2.1. Assume $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$, then there exist a maximal time $T = T(\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution (u, ρ) of Eq. (2.1) in $C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$ with initial data (u_0, ρ_0) . Moreover, the solution depends continuously on the initial data, and T is independent of S .

Lemma 2.1 (See [22]). Let $0 < s < 1$. Suppose that $f_0 \in H^s, g \in L^1([0, T]; H^s)$ and $v, v_x \in L^1([0, T]; L^\infty)$ and that $f \in L^\infty([0, T]; H^s) \cap C([0, T]; S')$ solves the one-dimensional linear transport equation

$$\begin{cases} \partial_t f + v \partial_x f = g \\ f(0, x) = f_0(x) \end{cases}$$

then $f \in C([0, T]; H^s)$. More precisely, there exists a constant C depending only on s such that the following estimate holds:

$$\|f(t)\|_{H^s} \leq \|f_0\|_{H^s} + C \left(\int_0^t \|g(\tau)\|_{H^s} d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau \right)$$

then,

$$\|f(t)\|_{H^s} \leq e^{CV(t)} \left(\|f_0\|_{H^s} + C \int_0^t \|g(\tau)\|_{H^s} d\tau \right)$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) d\tau$.

We may use Lemma 2.1 derived in [27] to handle the regularity propagation of solutions to (2.1). In addition, Lemma 2.1 was proved using Littlewood–Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument as in [27], we can obtain the following blow-up criterion.

Theorem 2.2. Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$, and T be the maximal time of existence. Then

$$T < \infty \Rightarrow \int_0^t \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \quad (2.3)$$

In concern of the finite time blow-up, we consider the trajectory equation of the system (2.1).

$$\begin{cases} \frac{dq(t, x)}{dt} = u(t, q(t, x)), & t \in [0, T] \\ q(0, x) = x, & x \in R \end{cases} \quad (2.4)$$

where $u \in C^1([0, T]; H^{s-1})$ is the first component of the solution (u, ρ) to (2.1) with initial data $(u_0, \rho_0) \in H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$, and $T > 0$ is the maximal time of existence. Applying Theorem 2.1, we know that $q(t, \cdot) : R \rightarrow R$ is the diffeomorphism for every $t \in [0, T]$, and

$$q_x(t, x) = e^{\int_0^t u_x(\tau, q(\tau, x)) d\tau} > 0, \quad \forall (t, x) \in [0, T] \times R. \quad (2.5)$$

Hence, the L^∞ -norm of any function $v(t, \cdot) \in L^\infty(R)$, $t \in [0, T]$ is preserved under the diffeomorphism $q(t, \cdot)$ with $t \in [0, T]$, that is, $\|v(t, \cdot)\|_{L^\infty(R)} = \|v(t, q(t, \cdot))\|_{L^\infty(R)}$.

Lemma 2.2 (See [34]). Let $T > 0$ and $v \in C^1([0, T]; H^1(R))$, then for every $t \in [0, T]$, there exists at least one point $\xi(t) \in R$ with $m(t) := \inf_{x \in R} [v_x(t, x)] = v_x(t, \xi(t))$. The function $m(t)$ is absolutely continuous on $(0, T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, T).$$

Lemma 2.3. Assume $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$ with $s > \frac{3}{2}$, and (u, ρ) be the solution of system (2.1), then $\|(u, \rho - 1)\|_{H^1 \times L^2}^2 \leq \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2$.

Proof. Multiplying the first equation in (2.1) by u and integrating by parts, then

$$\frac{d}{dt} \int_R (u^2 + u_x^2) dx + 2\lambda \int_R (u^2 + u_x^2) dx + 2 \int_R u \rho \rho_x dx = 0.$$

Rewriting the second equation in (2.1) to the form $(\rho - 1)_t + \rho_x u + \rho u_x = 0$, and then multiplying by $(\rho - 1)$ and integrating by parts, we have

$$\frac{d}{dt} \int_R (\rho - 1)^2 dx + 2 \int_R u \rho \rho_x dx - 2 \int_R u \rho_x dx + 2 \int_R u_x \rho^2 dx - 2 \int_R u_x \rho dx = 0.$$

Combining the above equalities, we have

$$\frac{d}{dt} \int_R (u^2 + u_x^2 + (\rho - 1)^2) dx + 2\lambda \int_R (u^2 + u_x^2) dx = 0.$$

Then

$$\frac{d}{dt} \int_R \left(u^2 + u_x^2 + (\rho - 1)^2 + 2\lambda \int_0^t (u^2 + u_x^2) d\tau \right) dx = 0.$$

So

$$\int_R \left(u^2 + u_x^2 + (\rho - 1)^2 + 2\lambda \int_0^t (u^2 + u_x^2) d\tau \right) dx = \int_R (u_0^2 + u_{0x}^2 + (\rho_0 - 1)^2) dx = \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2.$$

Since $2\lambda \int_0^t (u^2 + u_x^2) d\tau \geq 0$, therefore

$$\|(u, \rho - 1)\|_{H^1 \times L^2}^2 = \int_R (u^2 + u_x^2 + (\rho - 1)^2) dx \leq \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2.$$

Hence, we complete the proof of [Lemma 2.3](#). \square

Applying [Lemma 2.2](#) and the method of characteristics, we may carry out the estimates along the characteristics $q(t, x)$ which captures $\sup_{x \in R} u_x(t, x)$ and $\inf_{x \in R} u_x(t, x)$.

Lemma 2.4. Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$, and T be the maximal time of existence.

(1) When $\sigma > 0$, then

$$\sup_{x \in R} u_x(t, x) \leq \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}. \quad (2.6)$$

(2) When $\sigma < 0$, then

$$\inf_{x \in R} u_x(t, x) \geq -\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} \quad (2.7)$$

where the constants are defined as follows:

$$C_1 = \sqrt{\frac{1 + A^2 + |\sigma| + 2|3 - \sigma|}{2}} \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2} \quad (2.8)$$

$$C_2 = \sqrt{2 + \frac{8 + A^2 - 3\sigma}{2}} \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2. \quad (2.9)$$

Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Here we may assume that $u_0 \neq 0$. Otherwise, the results become trivial.

Differentiating the first equation in (2.2) with respect to x and using the identity $-\partial_x^2 p * f = f - p * f$, we have

$$u_{tx} + \sigma u u_{xx} + \frac{\sigma}{2} u_x^2 = \frac{1}{2} \rho^2 + \frac{3 - \sigma}{2} u^2 + A \partial_x^2 p * u - p * \left(\frac{\sigma}{2} u_x^2 + \frac{3 - \sigma}{2} u^2 + \frac{1}{2} \rho^2 \right) - \lambda u_x. \quad (2.10)$$

(1) When $\sigma > 0$, using [Lemma 2.2](#) and the fact that

$$\sup_{x \in R} [v_x(t, x)] = -\inf_{x \in R} [-v_x(t, x)].$$

We can consider $\bar{m}(t)$ and $\eta(t)$ as follows:

$$\bar{m}(t) := u_x(t, \eta(t)) = \sup_{x \in R} (u_x(t, x)), \quad t \in [0, T). \quad (2.11)$$

Hence

$$u_{xx}(t, \eta(t)) = 0 \quad \text{a.e. on } t \in [0, T). \quad (2.12)$$

Take the trajectory $q(t, x)$ defined in (2.4). We know that $q(t, \cdot) : R \rightarrow R$ is a diffeomorphism for every $t \in [0, T)$, then there exists $x_1(t) \in R$ such that

$$q(t, x_1(t)) = \eta(t), \quad t \in [0, T). \quad (2.13)$$

Let

$$\bar{\zeta}(t) = \rho(t, q(t, x_1)), \quad t \in [0, T). \quad (2.14)$$

Then along the trajectory $q(t, x_1(t))$, Eq. (2.10) and the second equation of (2.1) become

$$\begin{cases} \bar{m}'(t) = -\frac{\sigma}{2}\bar{m}^2(t) - \lambda\bar{m}(t) + \frac{1}{2}\bar{\zeta}^2(t) + f(t, q(t, x_1)) \\ \bar{\zeta}'(t) = -\bar{\zeta}(t)\bar{m}(t) \end{cases} \quad (2.15)$$

where

$$f = \frac{3-\sigma}{2}u^2 + A\partial_x^2 p * u - p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2 \right). \quad (2.16)$$

Since $\partial_x^2 p * u = \partial_x p * \partial_x u$, then

$$\begin{aligned} f &= \frac{3-\sigma}{2}u^2 + A\partial_x p * \partial_x u - p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 \right) - \frac{1}{2}p * 1 - p * (\rho - 1) - \frac{1}{2}p * (\rho - 1)^2 \\ &\leq \frac{3-\sigma}{2}u^2 + A|\partial_x p * \partial_x u| + \left| p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 \right) \right| - \frac{1}{2} + |p * (\rho - 1)|. \end{aligned}$$

Since

$$\begin{aligned} A|\partial_x p * \partial_x u| &\leq A\|p_x\|_{L^2}\|u_x\|_{L^2} = \frac{A}{2}\|u_x\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}A^2\|u_x\|_{L^2}^2 \\ |p * (\rho - 1)| &\leq \|p\|_{L^2}\|\rho - 1\|_{L^2} = \frac{1}{2}\|\rho - 1\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}\|\rho - 1\|_{L^2}^2 \\ \frac{3-\sigma}{2}u^2 &\leq \frac{|3-\sigma|}{4} \int_R (u^2 + u_x^2) dx \\ \left| p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 \right) \right| &\leq \frac{1}{2} \left\| \frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 \right\|_{L^1} \leq \int_R \left(\frac{|3-\sigma|}{4}u^2 + \frac{|\sigma|}{4}u_x^2 \right) dx. \end{aligned}$$

From the above inequalities and Lemma 2.3, we can get the upper bound of f ,

$$\begin{aligned} f &\leq \frac{1}{4}\|\rho - 1\|_{L^2}^2 + \frac{|3-\sigma|}{2}\|u\|_{L^2}^2 + \frac{A^2 + |3-\sigma| + |\sigma|}{4}\|u_x\|_{L^2}^2 \\ &\leq \frac{1 + A^2 + 2|3-\sigma| + |\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_1^2. \end{aligned} \quad (2.17)$$

Similarly, we get the lower bound of f ,

$$\begin{aligned} -f &\leq \frac{\sigma-3}{2}u^2 + A|\partial_x p * \partial_x u| + \left| p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 \right) \right| + \frac{1}{2} + |p * (\rho - 1)| + \frac{1}{2}p * (\rho - 1)^2 \\ &\leq 1 + \frac{1}{2}\|\rho - 1\|_{L^2}^2 + \frac{|3-\sigma|}{2}\|u\|_{L^2}^2 + \frac{A^2 + |3-\sigma| + |\sigma|}{4}\|u_x\|_{L^2}^2 \\ &\leq 1 + \frac{2 + A^2 + 2|3-\sigma| + |\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2. \end{aligned} \quad (2.18)$$

Combining (2.17) and (2.18), we get

$$|f| \leq 1 + \frac{2 + A^2 + 2|3-\sigma| + |\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2. \quad (2.19)$$

Since $s \geq 3$, then $u \in C_0^1(R)$. Therefore,

$$\inf_{x \in R} u_x(t, x) \leq 0, \quad \sup_{x \in R} u_x(t, x) \geq 0, \quad t \in [0, T). \quad (2.20)$$

Hence,

$$\bar{m}(t) > 0 \quad \text{for } t \in [0, T]. \quad (2.21)$$

From the second equation of (2.15), we have

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau) d\tau}. \quad (2.22)$$

Then

$$|\rho(t, q(t, x_1))| = |\bar{\zeta}(t)| \leq |\bar{\zeta}(0)| \leq \|\rho_0\|_{L^\infty}.$$

For any given $x \in R$, define

$$P_1(t) = \bar{m}(t) - \|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}.$$

Notice that $P_1(t)$ is a C^1 -differentiable function in $[0, T)$ and satisfies

$$P_1(0) = \bar{m}(0) - \|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}} \leq \bar{m}(0) - \|u_{0x}\|_{L^\infty} \leq 0.$$

Next, we claim

$$P_1(t) \leq 0 \quad \text{for } t \in [0, T). \quad (2.23)$$

If not, then suppose there is a $t_0 \in [0, T)$ such that $P_1(t_0) > 0$. Define $t_1 = \max\{t < t_0 : P_1(t) = 0\}$, then $P_1(t_1) = 0$, $P_1'(t_1) \geq 0$. That is,

$$\bar{m}(t_1) = \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}, \quad \bar{m}'(t_1) = P_1'(t_1) \geq 0.$$

On the other hand, we have a contraction.

$$\begin{aligned} \bar{m}'(t_1) &= -\frac{\sigma}{2}\bar{m}^2(t_1) - \lambda\bar{m}(t_1) + \frac{1}{2}\bar{\zeta}^2(t_1) + f(t_1, q(t_1, x_1)) \\ &\leq -\frac{\sigma}{2}\left(\|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}} + \frac{\lambda}{\sigma}\right)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2}\|\rho_0\|_{L^\infty}^2 + \frac{1}{2}C_1^2 < 0. \end{aligned}$$

Therefore, $P_1(t) \leq 0$ for $t \in [0, T)$. Since x is chosen arbitrarily, we obtain (2.6).

(2) When $\sigma < 0$, we have a finer estimate

$$\begin{aligned} -f &\leq 1 + \frac{1}{2}\|\rho - 1\|_{L^2}^2 + \frac{3 - \sigma}{2}\|u\|_{L^2}^2 + \frac{3 + A^2 - 2\sigma}{4}\|u_x\|_{L^2}^2 \\ &\leq 1 + \frac{8 + A^2 - 3\sigma}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_2^2. \end{aligned} \quad (2.24)$$

We consider the functions $m(t)$ and $\xi(t)$ in Lemma 2.2,

$$m(t) := \inf_{x \in R} [u_x(t, x)], \quad t \in [0, T).$$

Then $u_{xx}(t, \xi(t)) = 0$ a.e. on $t \in [0, T)$.

Choose $x_2(t) \in R$, such that $q(t, x_2(t)) = \xi(t)$, $t \in [0, T)$.

Let $\zeta(t) = \rho(t, q(t, x_2))$, $t \in [0, T)$. Along the trajectory $q(t, x_2)$, Eq. (2.10) and the second equation of (2.1) become

$$\begin{cases} m'(t) = -\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{1}{2}\zeta^2(t) + f(t, q(t, x_2)) \\ \zeta'(t) = -\zeta(t)m(t). \end{cases}$$

Let $P_2(t) = m(t) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}$, $\forall x \in R$. Then $P_2(t)$ is a C^1 -differentiable function in $[0, T)$ and satisfies

$$P_2(0) = m(0) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} \geq m(0) + \|u_{0x}\|_{L^\infty} \geq 0.$$

Now we claim that

$$P_2(t) \geq 0 \quad \text{for } t \in [0, T]. \quad (2.25)$$

Assume there is a $\bar{t}_0 \in [0, T]$ such that $P_2(\bar{t}_0) < 0$. Define $t_2 = \max\{t < \bar{t}_0 : P_2(t) = 0\}$, then $P_2(t_2) = 0$, $P'_2(t_2) \leq 0$. That is,

$$m(t_2) = -\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}, \quad m'(t_2) = P'_2(t_2) \leq 0.$$

In addition,

$$\begin{aligned} m'(t_2) &= -\frac{\sigma}{2}m^2(t_2) - \lambda m(t_2) + \frac{1}{2}\zeta^2(t_2) + f(t_2, q(t_2, x_2)) \\ &\geq -\frac{\sigma}{2} \left(-\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2 > 0. \end{aligned}$$

Then we have a contradiction.

Therefore, we have $P_2(t) \geq 0$ for $t \in [0, T]$. Since x is chosen arbitrarily, we obtain (2.7). \square

Next, we may get the following estimates for $\|\rho\|_{L^\infty(R)}$, if σu_x is bounded from below.

Lemma 2.5 (See [33]). Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$, and T be the maximal time of existence. If there is an $M \geq 0$ such that $\inf_{(t,x) \in [0,T) \times R} \sigma u_x \geq -M$. Then

$$(1) \text{ If } \sigma > 0, \|\rho(t, \cdot)\|_{L^\infty(R)} \leq \|\rho_0\|_{L^\infty(R)} e^{Mt/\sigma}.$$

$$(2) \text{ If } \sigma < 0, \|\rho(t, \cdot)\|_{L^\infty(R)} \leq \|\rho_0\|_{L^\infty(R)} e^{Nt},$$

where $N = \|u_{0x}\|_{L^\infty} + (C_2/\sqrt{-\sigma})$ and C_2 is given in (2.24).

From the above results, we can get the necessary and sufficient condition for the blow-up of solutions.

Theorem 2.3 (Wave-Breaking Criterion for $\sigma \neq 0$). Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$, and T be the maximal time of existence. Then the solution blows up in finite time if and only if

$$\lim_{t \rightarrow T^-} \inf_{x \in R} \sigma u_x(t, x) = -\infty. \quad (2.26)$$

Proof. Assume that $T < \infty$ and (2.26) is not valid, then there is some positive number $M > 0$, such that $\sigma u_x(t, x) \geq -M$, $\forall (t, x) \in [0, T) \times R$. From above lemmas, we have $|u_x(t, x)| \leq C$, where $C = C(A, M, \sigma, \lambda, \|u_0, \rho_0 - 1\|_{H^s \times H^{s-1}})$. Therefore, Theorem 2.2 implies that the maximal existence time $T = \infty$, which contradicts the assumption $T < \infty$.

On the other hand, the Sobolev embedding theorem $H^s(R) \hookrightarrow L^\infty(R)$ with $s > \frac{1}{2}$ implies that if (2.26) holds, the corresponding solution blows up in finite time. Then we complete the proof of theorem. \square

3. Blow-up scenarios

Now we give the following series of theorems that provide some cases for wave breaking in finite time.

Theorem 3.1. Let $\sigma > 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$, and T be the maximal time of existence. Assume that there is some $x_0 \in R$ such that $\rho_0(x_0) = 0$, $u_{0x}(x_0) = \inf_{x \in R} u_{0x}(x)$ and

$$\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 < \left(\frac{1}{8} - \frac{\lambda^2}{2\sigma} \right) \frac{4}{1 + 2A^2 + 2|3 - \sigma| + \sigma} \quad (3.1)$$

then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T with

$$0 < T \leq \frac{2}{\sigma - \lambda} + \frac{8\sigma + 8\sigma |u_{0x}(x_0)|}{\sigma - 4\lambda^2 - 2\sigma(1 + 2A^2 + 2|3 - \sigma| + \sigma) \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2} \quad (3.2)$$

such that $\liminf_{t \rightarrow T^-} (\inf_{x \in R} u_x(t, x)) = -\infty$.

Proof. Here we also consider $s \geq 3$. We still consider along the trajectory $q(t, x_2)$ defined as before. In this way, we can write the transport equation of ρ in (2.1) along the trajectory of $q(t, x_2)$ as

$$\frac{d\rho(t, \xi(t))}{dt} = -\rho(t, \xi(t))u_x(t, \xi(t)). \quad (3.3)$$

By the assumption, we obtain

$$m(0) = u_x(0, \xi(0)) = \inf_{x \in R} u_{0x}(x) = u_{0x}(x_0).$$

Now we choose $\xi(0) = x_0$ and we have $\rho(\xi(0)) = \rho_0(x_0) = 0$. Then by (3.3), we can obtain

$$\rho(t, \xi(t)) = 0, \quad \forall t \in [0, T]. \quad (3.4)$$

Evaluating the result at $x = \xi(t)$ and combining (3.4) with $u_{xx}(t, \xi(t)) = 0$, we have

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{3-\sigma}{2}u^2(t, \xi(t)) + A(p_x * u_x)(t, \xi(t)) - p * \left(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2 \right)(t, \xi(t)) \\ &= -\frac{\sigma}{2}m^2(t) - \lambda m(t) + f(t, q(t, x_2)) = -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t, x_2)). \end{aligned} \quad (3.5)$$

Modifying the estimate $A|p_x * u_x| \leq \frac{1}{2}A\|u_x\|_{L^2} \leq \frac{1}{8} + \frac{1}{2}A^2\|u_x\|_{L^2}^2$. Similarly, we get the upper bound of f , $f \leq -\frac{1}{8} + \frac{1+2A^2+2|3-\sigma|+\sigma}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 := -C_3$.

By the assumption (3.1), we get $\frac{\lambda^2}{2\sigma} - C_3 < 0$. Then

$$m'(t) \leq -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - C_3 \leq \frac{\lambda^2}{2\sigma} - C_3 < 0, \quad t \in [0, T]. \quad (3.6)$$

So $m(t)$ is strictly decreasing in $[0, T]$. If the solution (u, ρ) of (2.1) exists globally in time, that is $T = \infty$, we will show that it leads to a contradiction.

Let $t_1 = \frac{2\sigma(1+|u_{0x}(x_0)|)}{2\sigma C_3 - \lambda^2}$, integrating (3.6) over $[0, t_1]$, then

$$m(t_1) = m(0) + \int_0^{t_1} m'(t)dt \leq |u_{0x}(x_0)| + \left(\frac{\lambda^2}{2\sigma} - C_3 \right) t_1 = -1. \quad (3.7)$$

Hence, for $t \in [t_1, T]$, we have $m(t) \leq m(t_1) \leq -1$.

From (3.6), we have

$$m'(t) \leq -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2. \quad (3.8)$$

Integrating (3.8) over $[t_1, T]$ and by (3.7),

$$-\frac{1}{m(t) + \frac{\lambda}{\sigma}} + \frac{1}{\frac{\lambda}{\sigma} - 1} \leq -\frac{1}{m(t) + \frac{\lambda}{\sigma}} + \frac{1}{m(t_1) + \frac{\lambda}{\sigma}} \leq -\frac{\sigma}{2}(t - t_1), \quad t \in [t_1, T].$$

Then, $m(t) \leq \frac{1}{\frac{\sigma}{2}(t-t_1) + \frac{\lambda}{\sigma} - 1} - \frac{\lambda}{\sigma} \rightarrow -\infty$ as $t \rightarrow t_1 + \frac{2}{\sigma - \lambda}$.

So, $T \leq t_1 + \frac{2}{\sigma - \lambda}$, which is a contradiction to $T = \infty$.

Then we obtain that $T < \infty$ and complete the proof of Theorem 3.1. \square

Theorem 3.2. Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$, and T be the maximal time of existence.

(1) When $\sigma > 0$, assume that there is some $x_0 \in R$ such that $\rho_0(x_0) = 0$, $u_{0x}(x_0) = \inf_{x \in R} u_{0x}(x)$ and $u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2} + \frac{C_1^2}{\sigma}} - \frac{\lambda}{\sigma}$, where C_1 is defined in (2.8). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T_1 with $0 < T_1 \leq \frac{2(\lambda + \sigma u_{0x}(x_0))}{(\lambda + \sigma u_{0x}(x_0))^2 - (\lambda^2 + \sigma C_1^2)}$, such that

$$\liminf_{t \rightarrow T_1^-} \left\{ \inf_{x \in R} u_x(t, x) \right\} = -\infty.$$

- (2) When $\sigma < 0$, assume that there is some $x_0 \in R$ such that $u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, where C_2 is defined in (2.9). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T_2 with $0 < T_2 \leq -\frac{2(\lambda + \sigma u_{0x}(x_0))}{(\lambda + \sigma u_{0x}(x_0))^2 - (\lambda^2 - \sigma C_2^2)}$, such that

$$\liminf_{t \rightarrow T_2^-} \left\{ \sup_{x \in R} u_x(t, x) \right\} = \infty.$$

Proof. (1) When $\sigma > 0$, using the upper bound of f in (2.17) and (3.4), we have

$$m'(t) \leq -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2}C_1^2, \quad t \in [0, T).$$

By the assumption $m(0) = u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2} + \frac{C_1^2}{\sigma}} - \frac{\lambda}{\sigma}$, we have that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$.

$$\text{Set } \delta = \frac{1}{2} - \frac{1}{\sigma(u_{0x}(x_0) + \frac{\lambda}{\sigma})^2} \left(\frac{\lambda^2}{2\sigma} + \frac{1}{2}C_1^2 \right) \in (0, \frac{1}{2}).$$

Since $m(t) < m(0) = u_{0x}(x_0) < -\frac{\lambda}{\sigma}$, then

$$m'(t) \leq -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2}C_1^2 \leq -\delta\sigma \left(m(t) + \frac{\lambda}{\sigma} \right)^2.$$

A similar argument as in the proof of Theorem 3.1, we get

$$m(t) \leq \frac{\lambda + \sigma u_{0x}(x_0)}{\sigma + (\delta\sigma^2 u_{0x}(x_0) + \lambda\delta\sigma)t} - \frac{\lambda}{\sigma} \rightarrow -\infty \text{ as } t \rightarrow -\frac{1}{\lambda\delta + \delta\sigma u_{0x}(x_0)}.$$

Therefore, $0 < T_1 \leq -\frac{1}{\lambda\delta + \delta\sigma u_{0x}(x_0)}$.

- (2) When $\sigma < 0$, we consider the functions $\bar{m}(t)$ and $\eta(t)$ as defined in (2.11) and take the trajectory $q(t, x_1)$ with x_1 defined in (2.13), then

$$\begin{aligned} \bar{m}'(t) &= -\frac{\sigma}{2} \bar{m}^2(t) - \lambda \bar{m}(t) + \frac{1}{2} \rho^2(t, \eta(t)) + f(t, q(t, x_1)) \\ &\geq -\frac{\sigma}{2} \left(\bar{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t, x_1)). \end{aligned} \quad (3.9)$$

From the lower bound of f in (2.24), then

$$\bar{m}'(t) \geq -\frac{\sigma}{2} \left(\bar{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2, \quad t \in [0, T).$$

By the assumption $\bar{m}(0) \geq u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, we have that $\bar{m}'(0) > 0$ and $\bar{m}(t)$ is strictly increasing over $[0, T)$.

$$\text{Set } \theta = \frac{(\sigma u_{0x}(x_0) + \lambda)^2 - (\lambda^2 - \sigma C_2^2)}{2(\sigma u_{0x}(x_0) + \lambda)^2} \in (0, \frac{1}{2})$$

Since $\bar{m}(t) > \bar{m}(0) \geq u_{0x}(x_0) > -\frac{\lambda}{\sigma}$, then

$$\bar{m}'(t) \geq -\frac{\sigma}{2} \left(\bar{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2 \geq -\theta\sigma \left(\bar{m}(t) + \frac{\lambda}{\sigma} \right)^2.$$

Similarly, we get

$$\bar{m}(t) \geq \frac{\lambda + \sigma u_{0x}(x_0)}{\sigma + (\theta\sigma^2 u_{0x}(x_0) + \lambda\theta\sigma)t} - \frac{\lambda}{\sigma} \rightarrow \infty \text{ as } t \rightarrow -\frac{1}{\lambda\theta + \theta\sigma u_{0x}(x_0)}.$$

Therefore, $0 < T_2 \leq -\frac{1}{\lambda\theta + \theta\sigma u_{0x}(x_0)}$. So, we complete the proof of Theorem 3.2. \square

Remark. If $\sigma = 3$ and $A = 0$, then all solutions of system (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(R) \times H^{s-1}(R)$, $s > \frac{3}{2}$ satisfying $u_0 \neq 0$ and $\rho_0(x_0) = 0$ for some $x_0 \in R$, also blow up in finite time.

4. Blow-up rate

Having established blow up results for system (2.1) under study, attention is given to blow-up rate for solutions to system (2.1).

Theorem 4.1. Let $\sigma \neq 0$. If $T < \infty$ is the blow-up time of the solution (u, ρ) to (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$ satisfying the assumption of Theorem 3.2, then

$$\lim_{t \rightarrow T^-} \left\{ \inf_{x \in \mathbb{R}} u_x(t, x)(T - t) \right\} = -\frac{2}{\sigma}, \quad \sigma > 0 \quad (4.1)$$

$$\lim_{t \rightarrow T^-} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x)(T - t) \right\} = -\frac{2}{\sigma}, \quad \sigma < 0 \quad (4.2)$$

Proof. Similarly, we may assume $s = 3$ to prove the theorem.

(1) When $\sigma > 0$, from (3.5) we have

$$m'(t) = -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t, x)). \quad (4.3)$$

From (2.19) and note

$$M = 1 + \frac{2 + A^2 + |\sigma| + 2|3 - \sigma|}{4} \| (u_0, \rho_0 - 1) \|_{H^1 \times L^2}^2. \quad (4.4)$$

Then

$$-\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 - \frac{\lambda^2}{2\sigma} - M \leq m'(t) \leq -\frac{\sigma}{2} \left(m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + M. \quad (4.5)$$

Now we choose $\varepsilon \in (0, \frac{\sigma}{2})$, since $\lim_{t \rightarrow T^-} \left(m(t) + \frac{\lambda}{\sigma} \right) = -\infty$, then there is some $t_0 \in (0, T)$, such that $m(t_0) + \frac{\lambda}{\sigma} < 0$ and $\left(m(t_0) + \frac{\lambda}{\sigma} \right)^2 > \frac{1}{\varepsilon} \left(\frac{\lambda^2}{2\sigma} + M \right)$. Since m is locally Lipschitz, it follows that m is absolutely continuous. We deduce that m is decreasing on $[t_0, T)$ and

$$\left(m(t) + \frac{\lambda}{\sigma} \right)^2 > \frac{1}{\varepsilon} \left(\frac{\lambda^2}{2\sigma} + M \right), \quad t \in [t_0, T). \quad (4.6)$$

Combining (4.5) with (4.6), we have

$$\frac{\sigma}{2} - \varepsilon \leq \frac{d}{dt} \left(\frac{1}{m(t) + \frac{\lambda}{\sigma}} \right) \leq \frac{\sigma}{2} + \varepsilon, \quad t \in [t_0, T). \quad (4.7)$$

Integrating (4.7) over (t, T) with $t \in [t_0, T)$ and noticing that $\lim_{t \rightarrow T^-} \left(m(t) + \frac{\lambda}{\sigma} \right) = -\infty$, then

$$\left(\frac{\sigma}{2} - \varepsilon \right) (T - t) \leq -\frac{1}{m(t) + \frac{\lambda}{\sigma}} \leq \left(\frac{\sigma}{2} + \varepsilon \right) (T - t).$$

Since $\varepsilon \in (0, \frac{\sigma}{2})$ is arbitrary, in view of the definition of $m(t)$, then $\lim_{t \rightarrow T^-} \{m(t)(T - t) + \frac{\lambda}{\sigma}(T - t)\} = -\frac{2}{\sigma}$, that is, $\lim_{t \rightarrow T^-} \{ \inf_{x \in \mathbb{R}} u_x(t, x)(T - t) \} = -\frac{2}{\sigma}$.

(2) When $\sigma < 0$, we also consider the functions $\bar{m}(t)$ and $\eta(t)$ as defined in (2.11).

From (3.9) and (4.4), we have $\bar{m}'(t) \geq -\frac{\sigma}{2} \left(\bar{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - M$.

Because $\bar{m}(t) \rightarrow \infty$ as $t \rightarrow T^-$, so there is a $t_1 \in (0, T)$, such that $\bar{m}(t_1) > \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{2M}{\sigma}} - \frac{\lambda}{\sigma} > 0$. Therefore, we have that $\bar{m}'(t) > 0$ and $\bar{m}(t)$ is strictly increasing on $[t_1, T)$, and

$$\bar{m}(t) > \bar{m}(t_1) > 0. \quad (4.8)$$

By the transport equation for ρ , we have that

$$\frac{d\rho(t, \eta(t))}{dt} = -\bar{m}(t)\rho(t, \eta(t)).$$

Then

$$\rho(t, \eta(t)) = \rho(t_1, \eta(t_1)) e^{-\int_{t_1}^t \bar{m}(\tau) d\tau}, \quad t \in [t_1, T). \quad (4.9)$$

Combining (4.8) with (4.9), we have

$$\rho^2(t, \eta(t)) \leq \rho^2(t_1, \eta(t_1)), \quad t \in [t_1, T]. \quad (4.10)$$

From (3.9) and (4.10), we have

$$-\frac{\sigma}{2} \left(\bar{m} + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2} \rho^2(t_1, \eta(t_1)) - M \leq \bar{m}' \leq -\frac{\sigma}{2} \left(\bar{m} + \frac{\lambda}{\sigma} \right)^2 - \frac{\lambda^2}{2\sigma} + \frac{1}{2} \rho^2(t_1, \eta(t_1)) + M. \quad (4.11)$$

Choose $\varepsilon \in (0, -\frac{\sigma}{2})$, and also we can pick a $t_2 \in [t_1, T)$, such that

$$\left(\bar{m}(t_2) + \frac{\lambda}{\sigma} \right)^2 > \frac{1}{\varepsilon} \left(\frac{1}{2} \rho^2(t_1, \eta(t_1)) + M - \frac{\lambda^2}{2\sigma} \right). \quad (4.12)$$

From (4.11) and (4.12), we have

$$\frac{\sigma}{2} - \varepsilon \leq \frac{d}{dt} \left(\frac{1}{\bar{m}(t) + \frac{\lambda}{\sigma}} \right) \leq \frac{\sigma}{2} + \varepsilon, \quad t \in [t_2, T). \quad (4.13)$$

Integrating (4.13) over $[t, T)$ with $t \in [t_2, T)$ and $\lim_{t \rightarrow T^-} \bar{m}(t) = \infty$, then

$$\left(\frac{\sigma}{2} - \varepsilon \right) (T - t) \leq -\frac{1}{\bar{m}(t) + \frac{\lambda}{\sigma}} \leq \left(\frac{\sigma}{2} + \varepsilon \right) (T - t).$$

Since $\varepsilon \in (0, -\frac{\sigma}{2})$ is arbitrary, in view of the definition of $\bar{m}(t)$, we have

$$\lim_{t \rightarrow T^-} \left\{ \sup_{x \in \mathbb{R}} u_x(t, x) (T - t) \right\} = -\frac{2}{\sigma}.$$

This completes the proof of Theorem 4.1. \square

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