



Global classical solutions for 3D compressible Navier–Stokes equations with vacuum and a density-dependent viscosity coefficient[☆]



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ABSTRACT

In this paper, we prove the global existence of classical solutions to the three-dimensional (3D) compressible Navier–Stokes equations with a density-dependent viscosity coefficient ($\lambda = \lambda(\rho)$) provided the initial data is of small energy. This in particular implies that the solutions may have large oscillations and contain vacuum states. As a result of the uniform estimates, the large-time behavior of the solution is also studied. The result obtained generalizes those results in Zhang (2011) [39] and Huang et al. (2012) [17] where the non-vacuum initial data and the constant viscosity coefficients are considered, respectively.

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1. Introduction

In this paper, we consider the following 3D compressible Navier–Stokes equations with a density-dependent viscosity coefficient:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & x \in \mathbb{R}^3, t > 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + \nabla((\mu + \lambda(\rho))\operatorname{div} u), \end{cases} \quad (1.1)$$

where the unknowns $\rho, u = (u^1, u^2, u^3)$, and $P(\rho) = A\rho^\gamma$ with $A > 0, \gamma > 1$ are the fluid density, velocity, and pressure, respectively. The dynamic viscosity coefficient $\mu > 0$ is a positive constant, however, the second viscosity coefficient $\lambda = \lambda(\rho)$ is a function of ρ .

We look for the global classical solution (ρ, u) to (1.1) with the initial data

$$\rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3, \quad (1.2)$$

and the far field behavior

$$\rho(x, t) \rightarrow \tilde{\rho} \geq 0, \quad u(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad t > 0, \quad (1.3)$$

where $\tilde{\rho} \geq 0$ is a fixed nonnegative constant.

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In the last decades, there have been a huge number of studies on the compressible Navier–Stokes equations with constant viscosity coefficients. Since the work of Kazhikov and Shelukhin (cf. [22]), the one-dimensional compressible Navier–Stokes equations have been extensively studied. When the initial density is strictly away from vacuum, the local-in-time solvability to various initial and boundary value problems for the multi-dimensional compressible Navier–Stokes equations was proved by Nash [30], Solonnikov [33], and Tani [34]. If the initial vacuum is allowed, the local existence of strong/classical solutions was shown in [5,7,6,31]. The first result about the global theory is due to Matsumura and Nishida [27], who proved the global existence of classical solutions to the initial value problem when the initial data are close to a non-vacuum equilibrium in H^3 . Later, Hoff (cf. [11–13]) extended Matsumura–Nishida’s results [27] to weak solutions with discontinuous initial data. One of the most important breakthrough concerning global theory is the work of Lions [25] (see also Feireisl et al. [9,8]), who first proved the global existence of weak solutions to the initial/boundary value problem of (1.1) with generally large initial data. We also refer to [21,14] for the global existence of weak solutions with large symmetric data.

A lot of attention has also been paid to the compressible Navier–Stokes equations with density-dependent viscosity coefficients due to its physical importance in ocean physics and its close connection to the shallow water equations. For the one-dimensional case, there is an extensive literature; see [20,19,24,28,37,38,41] and among others. Assuming that the viscosity coefficients $\mu = \text{const.}$ and $\lambda(\rho) = \rho^\beta$ with $\beta > 3$, Vaigant and Kazhikov [35] proved the global existence and uniqueness of strong solutions to 2D barotropic compressible Navier–Stokes equations with periodic boundary conditions and large non-vacuum initial data. For the multi-dimensional case, Bresch et al. (cf. [1–3]) studied the global weak solutions for the multi-dimensional equations with the Korteweg stress tensor or the additional quadratic friction term provided the following algebraic relation between $\mu(\rho)$ and $\lambda(\rho)$ holds,

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \quad (1.4)$$

Mellet and Vasseur [29] proved that the Korteweg or friction terms are unnecessary in the proof of compactness for the entropy weak solution. Guo–Jiu–Xin [10] considered the global spherically symmetric solutions to the 3D compressible Navier–Stokes equations with density-dependent viscosity coefficients satisfying (1.4) in a particular form. The nonphysical condition (1.4) is technically needed and plays a key role in the analysis of [1–3,10,29].

Recently, many efforts have been made to improve the global well-posedness theory of the multi-dimensional compressible Navier–Stokes equations. In particular, Chen–Miao–Zhang [4] proved the global existence and uniqueness of solutions with highly oscillating initial velocity for 2D/3D compressible Navier–Stokes equations in the critical functional framework when the viscosity coefficients are constant and the initial data is close to a stable equilibrium. Zhang [39] studied the global existence of classical solutions to the 3D compressible Navier–Stokes equations with a density-dependent viscosity coefficient $\lambda(\rho)$ when the solutions are small in the energy-norm with non-vacuum states. It should be noted that in both [4] and [39], the density is technically required to be strictly away from vacuum, which, as emphasized in [5,7,6,15,16,26,32,36], is one of the major difficulties in the mathematical study of compressible Navier–Stokes equations. In particular, Huang–Li–Xin [17] considered the 3D isentropic compressible Navier–Stokes equations with constant viscosity coefficients, and established the first existence result of classical solutions which may have large oscillations and can contain vacuum states.

In this paper we focus our interest on the compressible Navier–Stokes equations (1.1)–(1.3) with a density-dependent viscosity coefficient $\lambda(\rho)$. The global weak solutions, which are of small energy but may contain vacuum states, to the 2D equations of (1.1)–(1.3) were studied by Zhang [40]. Here, motivated by [17], the main purpose of this paper is to prove the global well-posedness of classical solutions (1.1)–(1.3) with vacuum.

Before stating our main result, we first introduce some notations which will be used in the sequel. For $1 < r \leq \infty$ and $k \in \mathbb{Z}^+$, we shall use the following simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), & D^{k,r} = \{u \in L^1_{\text{loc}}(\mathbb{R}^3) : \|\nabla^k u\|_{L^r} < \infty\}, & \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, & H^k = W^{k,2}, & D^k = D^{k,2}, & D^1 = \{u \in L^6 : \|\nabla u\|_{L^2} < \infty\}. \end{cases}$$

The initial energy of the compressible flows is defined as

$$E_0 = \int \left(\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) dx \quad \text{with } G(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds, \quad (1.5)$$

where $\tilde{\rho} \geq 0$ is given in (1.3). It is easy to see that

$$\begin{cases} G(\rho) = \frac{1}{\gamma-1} P(\rho), & \text{if } \tilde{\rho} = 0, \\ G(\rho) \geq c(\tilde{\rho}, \tilde{\rho})(\rho - \tilde{\rho})^2, & \text{if } \tilde{\rho} > 0, \quad 0 \leq \rho \leq \tilde{\rho} \end{cases} \quad (1.6)$$

for some positive constant $c(\tilde{\rho}, \tilde{\rho})$ depending only on $\tilde{\rho}$, $\tilde{\rho}$, a and γ .

Then our main result in this paper reads as follows.

Theorem 1.1. *Let $\mu = \text{const} > 0$ and $\lambda(\rho)$ be a smooth function of the density ρ satisfying*

$$\lambda(\rho) \in C^3(\mathbb{R}^+), \quad 0 \leq \lambda(\rho) < \infty \quad \text{for all } \rho \in [0, \infty). \quad (1.7)$$

Assume that for given numbers $M > 0$ and $\bar{\rho} \geq \tilde{\rho} + 1$, the initial data (ρ_0, u_0) satisfies

$$\begin{cases} 0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, & (\rho_0 - \tilde{\rho}, P(\rho_0) - P(\tilde{\rho})) \in H^3, \\ |\rho_0|u_0|^2 + G(\rho_0) \in L^1, & u_0 \in D^1 \cap D^3, \quad \|\nabla u_0\|_{L^2}^2 \leq M, \end{cases} \quad (1.8)$$

and the following compatibility condition

$$-\mu \Delta u_0 - \nabla((\mu + \lambda(\rho_0)) \operatorname{div} u_0) + \nabla P(\rho_0) = \rho_0 g, \quad (1.9)$$

holds for some $g \in D^1$ and $\sqrt{\rho_0}g \in L^2$. Then there exists a positive constant $\varepsilon > 0$, depending on $\mu, a, \gamma, \tilde{\rho}, \bar{\rho}$ and M , such that if

$$E_0 \leq \varepsilon, \quad (1.10)$$

the Cauchy problem (1.1)–(1.3) has a unique global classical solution (ρ, u) satisfying

$$0 \leq \rho \leq 2\bar{\rho}, \quad (x, t) \in \mathbb{R}^3 \times (0, T), \quad (1.11)$$

$$\begin{cases} (\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T]; H^3), \\ u \in C([0, T]; D^1 \cap D^3) \cap L^2(0, T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0, T; D^1) \cap L^2(0, T; D^2) \cap L^\infty(\tau, T; D^2) \cap H^1(\tau, T; D^1), \\ \sqrt{\rho}u_t \in L^\infty(0, T; L^2), \end{cases} \quad (1.12)$$

for any $0 < \tau < T < \infty$.

Furthermore, we have the following large-time behavior

$$\lim_{t \rightarrow \infty} \int (|\rho - \tilde{\rho}|^q + \rho^{1/2}|u|^4 + |\nabla u|^2) dx = 0, \quad (1.13)$$

where $q \in (2, \infty)$ if $\tilde{\rho} > 0$ and $q \in (\gamma, \infty)$ if $\tilde{\rho} = 0$.

Theorem 1.1 will be proved by combining the local existence result and the global a priori estimates. The key issue in the proof of global estimates is to derive both the time-independent upper bound for the density and the time-depending higher norm estimates of the solution (ρ, u) . This will be done by modifying the arguments in [17], but the analysis here is more delicate and some new difficulties arise due to the dependence on the density of the viscosity coefficient $\lambda(\rho)$. Indeed, by careful calculations based on the conservation law of mass, we can obtain the time-independent upper bound of density in a similar manner as that in [17] by using the subtle initial layer analysis developed by Hoff [11] and the Zlotnik inequality (cf. Lemma 2.2). However, unlike the case of constant viscosity coefficients considered in [17], to apply the H^m -regularity of the elliptic system to (1.1)₂ in a precise manner and to obtain the higher-order estimates of (ρ, u) , we need to make use of the following elliptic systems:

$$-\Delta u = \nabla \times \omega - \nabla \operatorname{div} u = \nabla \times \omega - \nabla \left(\frac{F + P(\rho) - P(\tilde{\rho})}{2\mu + \lambda(\rho)} \right), \quad (1.14)$$

where F is the so-called “effective viscous flux” and ω is the vorticity, i.e.,

$$F \triangleq (\mu + \lambda(\rho)) \operatorname{div} u - (P(\rho) - P(\tilde{\rho})), \quad \omega = \nabla \times u,$$

and

$$\Delta F = \operatorname{div}(\rho \dot{u}), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u}). \quad (1.15)$$

The higher-order norms of the solutions will be estimated by applying the standard H^m -theory of the elliptic system repeatedly to the elliptic equations (1.14) and (1.15) (see Lemmas 4.3–4.6). Clearly, the estimates of $\rho, u, P(\rho), \omega$ and F are strongly coupled with each other. So, the analysis here is more complicated and delicate than that for the case of constant viscosity coefficients. This is the main difference between our proofs and the ones presented in [17].

This paper is organized as follows. In Section 2, we recall some known facts and inequalities which will be used frequently. In Section 3, we prove the weighted estimates on the gradients and the material derivatives of the velocity as well as the uniform upper bound of the density, all of which suffice to prove the large-time behavior. In Section 4, we estimate the higher order derivatives of (ρ, u) , which are needed to extend the local solutions to all time.

2. Preliminaries

In this section, we state some known facts and elementary inequalities which will be used frequently later. We first recall the following Gagliardo–Nirenberg inequality (see, for example, [23]).

Lemma 2.1. For $p \in [2, 6]$, $q \in (1, \infty)$ and $r \in (3, \infty)$, there exists a positive constant C , depending only on p, q and r , such that for any $f \in H^1$ and $g \in L^q \cap D^{1,r}$,

$$\|f\|_{L^p} \leq C \|f\|_{L^2}^{(6-p)/(2p)} \|\nabla f\|_{L^2}^{(3p-6)/(2p)}, \quad (2.1)$$

$$\|g\|_{L^\infty} \leq C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \quad (2.2)$$

In order to prove the uniform-in-time upper bound of density, we need the Zlotnik inequality, the proof of which can be found in [42].

Lemma 2.2. Assume that the function $y \in W^{1,1}(0, T)$ solves the ODE system:

$$y' = g(y) + b'(t) \quad \text{on } [0, T], \quad y(0) = y_0,$$

where $b \in W^{1,1}(0, T)$ and $g \in C(\mathbb{R})$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \quad (2.3)$$

for all $0 \leq t_1 \leq t_2 \leq T$ with some positive constants N_0 and N_1 , then one has

$$y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad \text{on } [0, T], \quad (2.4)$$

where $\xi^* \in \mathbb{R}$ is a constant such that

$$g(\xi) \leq -N_1 \quad \text{for } \xi \geq \xi^*. \quad (2.5)$$

The following Beale–Kate–Majda's type inequality plays an important role in estimating the gradients of the density and velocity (i.e. $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^6}$). For the details of the proof, we refer the reader to [18,17].

Lemma 2.3. Assume that $\nabla u \in L^2 \cap D^{1,q}$ with some $q \in (3, \infty)$. Then there exists a positive constant C , depending only on q , such that

$$\|\nabla u\|_{L^\infty} \leq C(\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^q}) + C(\|\nabla u\|_{L^2} + 1). \quad (2.6)$$

To extend the local solutions globally in time, we need the following local existence theorem of classical solutions to (1.1)–(1.3). Indeed, the authors [7] studied the local existence of solutions to the 3D compressible isentropic Navier–Stokes equations with constant viscosity coefficients. However, one can adapt the same scheme as that in [7] with minor modifications to deal with the problem (1.1)–(1.3) and prove the local existence of classical solutions. For simplicity, we omit the details.

Lemma 2.4. Assume that the initial data (ρ_0, u_0) satisfy (1.8) and (1.9). Then there exist a small time $T_* > 0$ and a unique classical solution (ρ, u) to the Cauchy problem (1.1)–(1.3) on $\mathbb{R}^3 \times (0, T_*]$ such that

$$\begin{cases} (\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T_*]; H^3), \\ u \in C([0, T_*]; D^1 \cap D^3) \cap L^2(0, T_*; D^4), \\ u_t \in L^\infty(0, T_*; D^1) \cap L^2(0, T_*; D^2), \quad \rho^{1/2} u_{tt} \in L^2(0, T_*; L^2), \\ t^{1/2} u \in L^\infty(0, T_*; D^4), \quad t^{1/2} u_t \in L^\infty(0, T_*; D^2), \quad t^{1/2} u_{tt} \in L^2(0, T_*; D^1), \\ t^{1/2} \sqrt{\rho} u_{tt} \in L^\infty(0, T_*; L^2), \quad t u_t \in L^\infty(0, T_*; D^3), \\ t u_{tt} \in L^\infty(0, T_*; D^1) \cap L^2(0, T_*; D^2), \quad t \sqrt{\rho} u_{ttt} \in L^2(0, T_*; L^2), \\ t^{3/2} u_{tt} \in L^\infty(0, T_*; D^2), \quad t^{3/2} u_{ttt} \in L^2(0, T_*; D^1), \quad t^{3/2} \sqrt{\rho} u_{ttt} \in L^\infty(0, T_*; L^2). \end{cases} \quad (2.7)$$

3. A priori estimates independent of t

This section is concerned with the weighted estimates of the gradients and the material derivative of the velocity and the uniform upper bound of the density. Assume that (ρ, u) is a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$ with some $T > 0$. To derive the desired estimates, we set $\sigma(t) = \min\{1, t\}$ and define

$$\Phi_1(T) = \sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^2}^2) + \int_0^T \sigma \int \rho |\dot{u}|^2 dx dt, \quad (3.1)$$

$$\Phi_2(T) = \sup_{0 \leq t \leq T} \sigma^3 \int \rho |\dot{u}|^2 dx + \int_0^T \sigma^3 \int |\nabla \dot{u}|^2 dx dt, \quad (3.2)$$

and

$$\Phi_3(T) = \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2, \quad (3.3)$$

where $\dot{v} \triangleq v_t + u \cdot \nabla v$ denotes the material derivative.

For simplicity, in this section we denote by C or C_i ($i = 1, 2, \dots$) a generic positive constant which may depend on $\mu, A, \gamma, \tilde{\rho}, \bar{\rho}$ and M , but independent of T . We also sometimes write $C(\alpha)$ to emphasize that C relies on α .

The main purpose of this section is to prove the following key a priori estimates of (ρ, u) under the smallness condition (1.10).

Proposition 3.1. *Let $\mu = \text{const.} > 0$ and $\lambda(\rho)$ satisfy (1.7). For given positive numbers M and $\bar{\rho} \geq \tilde{\rho} + 1$, assume that (ρ_0, u_0) satisfy (1.8). There exist two positive constants ε and K , which may depend on $\mu, A, \gamma, \tilde{\rho}, \bar{\rho}$ and M , but independent of t , such that if the solution (ρ, u) satisfy*

$$\begin{cases} 0 \leq \rho(x, t) \leq 2\bar{\rho} & \text{for all } x \in \mathbb{R}^3, \ t \in [0, T], \\ \Phi_1(T) + \Phi_2(T) \leq 2E_0^{1/2} & \text{and } \Phi_3(\sigma(T)) \leq 3K, \end{cases} \quad (3.4)$$

then one has

$$\begin{cases} 0 \leq \rho(x, t) \leq 7\bar{\rho}/4 & \text{for all } x \in \mathbb{R}^3, \ t \in [0, T], \\ \Phi_1(T) + \Phi_2(T) \leq E_0^{1/2} & \text{and } \Phi_3(\sigma(T)) \leq 2K, \end{cases} \quad (3.5)$$

provided that the initial energy E_0 defined in (1.5) satisfies $E_0 \leq \varepsilon$.

The proof of Proposition 3.1 is based on a series of lemmas. The idea of proof mainly comes from Hoff [11] and Huang–Li–Xin [17]. However, due to the dependence on ρ of the viscosity coefficient $\lambda(\rho)$ and the presence of vacuum states, the analysis here is more delicate. We start with the following standard energy estimate and the preliminary estimates of $\Phi_1(T)$ and $\Phi_2(T)$.

Lemma 3.1. *Let (ρ, u) with $\rho \in [0, 2\bar{\rho}]$ be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,*

$$\int \left(\frac{1}{2} \rho |u|^2 + G(\rho) \right) dx + \int_0^T \int (\mu |\nabla u|^2 + (\mu + \lambda(\rho))(\text{div} u)^2) dx dt \leq E_0, \quad (3.6)$$

$$\Phi_1(T) \leq CE_0 + C \int_0^T \sigma \|\nabla u\|_{L^3}^3 dt, \quad (3.7)$$

and

$$\Phi_2(T) \leq CE_0 + C\Phi_1(T) + C \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 dt. \quad (3.8)$$

Proof. Using the energy estimate, we can easily obtain (3.6), and we omit the details. Next, we start to prove (3.7) and (3.8). Multiplying (1.1)₂ by $\sigma^m \dot{u}$ in L^2 gives

$$\int \sigma^m \rho |\dot{u}|^2 dx = \int \sigma^m [\mu \Delta u + \nabla((\mu + \lambda(\rho)) \text{div} u) - \nabla P(\rho)] \cdot \dot{u} dx \triangleq \sum_{i=1}^3 I_i. \quad (3.9)$$

We obtain after integrating by parts that

$$I_1 \leq -\frac{\mu}{2} \frac{d}{dt} \int \sigma^m |\nabla u|^2 dx + C \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^3}^3. \quad (3.10)$$

By (1.1)₁, we have

$$\begin{aligned} I_2 &= -\frac{1}{2} \frac{d}{dt} \int \sigma^m (\mu + \lambda(\rho)) |\text{div} u|^2 dx + \frac{m}{2} \sigma^{m-1} \sigma' \int (\mu + \lambda(\rho)) |\text{div} u|^2 dx + \frac{1}{2} \sigma^m \int \lambda'(\rho) \rho_t |\text{div} u|^2 dx \\ &\quad + \frac{1}{2} \sigma^m \int \lambda'(\rho) (u \cdot \nabla \rho) |\text{div} u|^2 dx - \sigma^m \int (\mu + \lambda(\rho)) (\text{div} u) \left(\partial_i u^k \partial_k u^i - \frac{1}{2} |\text{div} u|^2 \right) dx \\ &\leq -\frac{1}{2} \frac{d}{dt} \int \sigma^m (\mu + \lambda(\rho)) |\text{div} u|^2 dx + C \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^3}^3, \end{aligned} \quad (3.11)$$

where and whereafter we use the Einstein convention that repeated indices denotes the summation over the indices.

To deal with the last term on the right-hand side of (3.9), we first deduce from (1.1)₁ that

$$P(\rho)_t + \operatorname{div}(P(\rho)u) + (\gamma - 1)P(\rho)\operatorname{div}u = 0, \quad P(\rho) = A\rho^\gamma. \quad (3.12)$$

Thus, integrating by parts and using (3.12), we have

$$\begin{aligned} I_3 &= \int \sigma^m(P(\rho) - P(\tilde{\rho})) (\operatorname{div}u_t + u \cdot \nabla(\operatorname{div}u) + \partial_i u^k \partial_k u^i) dx \\ &= \frac{d}{dt} \int \sigma^m(P(\rho) - P(\tilde{\rho}))(\operatorname{div}u) dx - m\sigma^{m-1}\sigma' \int (P(\rho) - P(\tilde{\rho})) (\operatorname{div}u) dx \\ &\quad - \sigma^m \int (P(\rho)_t + \operatorname{div}(Pu)) (\operatorname{div}u) dx + \sigma^m \int P(\tilde{\rho})(\operatorname{div}u)^2 dx + \sigma^m \int (P(\rho) - P(\tilde{\rho})) (\partial_i u^k \partial_k u^i) dx \\ &\leq \frac{d}{dt} \int \sigma^m(P(\rho) - P(\tilde{\rho}))(\operatorname{div}u) dx + C\sigma^{2(m-1)}\sigma' \|\rho - \tilde{\rho}\|_{L^2}^2 + C\sigma^m \|\nabla u\|_{L^2}^2. \end{aligned} \quad (3.13)$$

So, putting (3.10), (3.11) and (3.13) into (3.9) leads to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \sigma^m (\mu |\nabla u|^2 + (\mu + \lambda(\rho)) |\operatorname{div}u|^2) dx + \sigma^m \int \rho |\dot{u}|^2 dx \\ &\leq C\sigma^{2(m-1)}\sigma' E_0 + C(\sigma^{2(m-1)}\sigma' + \sigma^m) \|\nabla u\|_{L^2}^2 + C\sigma^m \|\nabla u\|_{L^3}^3 + \frac{d}{dt} \int \sigma^m(P(\rho) - P(\tilde{\rho}))(\operatorname{div}u) dx. \end{aligned} \quad (3.14)$$

The last term on the right-hand side of (3.14) can be easily bounded as follows:

$$\left| \int \sigma^m(P(\rho) - P(\tilde{\rho}))(\operatorname{div}u) dx \right| \leq C\sigma^m \|\nabla u\|_{L^2} \|\rho - \tilde{\rho}\|_{L^2} \leq \frac{\mu}{4} \sigma^m \|\nabla u\|_{L^2}^2 + C\sigma^m E_0,$$

and hence, by choosing $m = 1$ in (3.14), integrating it over $(0, T)$ and using (3.6), we immediately arrive at (3.7).

Next, we turn to the proof of (3.8). Operating $\sigma^m \dot{u}^j (\partial_t + \operatorname{div}(u \cdot))$ to both sides of the j -th equation of (1.1)₂, integrating the resulting equations over \mathbb{R}^3 , and summing them up, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \sigma^m \rho |\dot{u}|^2 dx - \frac{m}{2} \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 dx \\ &= -\sigma^m \int \dot{u}^j (\partial_j P_t + \partial_k (u^k \partial_j P)) dx + \mu \sigma^m \int \dot{u}^j [\Delta u_t^j + \partial_k (u^k \Delta u^j)] dx \\ &\quad + \sigma^m \int \dot{u}^j [\partial_j ((\mu + \lambda(\rho)) \operatorname{div}u)_t + \partial_k (u^k \partial_j ((\mu + \lambda(\rho)) \operatorname{div}u))] dx \\ &\triangleq \sum_{i=1}^3 J_i, \end{aligned} \quad (3.15)$$

Using integration by parts and the Cauchy inequality, we infer from (3.12) that

$$\begin{aligned} J_1 &= -\sigma^m \int ((\gamma - 1)P(\operatorname{div}u)(\operatorname{div}\dot{u}) + P\partial_j u^k \partial_k \dot{u}^j) dx \\ &\leq \frac{\mu}{4} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C\sigma^m \|\nabla u\|_{L^2}^2. \end{aligned} \quad (3.16)$$

$$\begin{aligned} J_2 &= -\mu \sigma^m \int (\partial_i \dot{u}^j \partial_i u_t^j + \partial_k \dot{u}^j u^k \Delta u^j) dx \\ &\leq -\frac{3\mu}{4} \sigma^m \int |\nabla \dot{u}|^2 dx + C\sigma^m \int |\nabla u|^4 dx. \end{aligned} \quad (3.17)$$

Direct calculations show that

$$\begin{aligned} J_3 &= \sigma^m \int \dot{u}^j [\partial_j ((\mu + \lambda(\rho)) \operatorname{div}u)_t + \partial_k (u^k \partial_j ((\mu + \lambda(\rho)) \operatorname{div}u))] dx \\ &= -\sigma^m \int (\mu + \lambda(\rho)) (\operatorname{div}u_t)(\operatorname{div}\dot{u}) dx - \sigma^m \int \lambda'(\rho) \rho_t (\operatorname{div}u)(\operatorname{div}\dot{u}) dx \\ &\quad - \sigma^m \int [\partial_j \dot{u}^i \partial_k u^k ((\mu + \lambda(\rho)) \operatorname{div}u) - \partial_k \dot{u}^j \partial_j u^k ((\mu + \lambda(\rho)) \operatorname{div}u)] dx \\ &\quad - \sigma^m \int \lambda'(\rho) (u \cdot \nabla \rho) (\operatorname{div}u)(\operatorname{div}\dot{u}) dx - \sigma^m \int (\mu + \lambda(\rho)) (u \cdot \nabla \operatorname{div}u)(\operatorname{div}\dot{u}) dx. \end{aligned}$$

This, together with (1.1)₁ and $\operatorname{div} u_t + u \cdot \nabla \operatorname{div} u = \operatorname{div} \dot{u} - \partial_i u^k \partial_k u^i$, leads to

$$J_3 \leq -\sigma^m \int (\mu + \lambda(\rho)) (\operatorname{div} \dot{u})^2 dx + \frac{\mu}{4} \sigma^m \int |\nabla \dot{u}|^2 dx + C \sigma^m \int |\nabla u|^4 dx. \quad (3.18)$$

Putting (3.16)–(3.18) into (3.15), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \sigma^m \rho |\dot{u}|^2 dx + \sigma^m \int \left(\frac{\mu}{4} |\nabla \dot{u}|^2 + (\mu + \lambda(\rho)) (\operatorname{div} \dot{u})^2 \right) dx \\ & \leq \frac{m}{2} \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 dx + C \sigma^m \|\nabla u\|_{L^2}^2 + \sigma^m \|\nabla u\|_{L^4}^4. \end{aligned} \quad (3.19)$$

Choosing $m = 3$ in (3.19) and integrating it over $(0, T)$, we immediately obtain (3.8). The proof of Lemma 3.1 is therefore complete. \square

From system (1.15), we have the following lemma which is concerned with the connections among the so-called “effective viscous flux”, the gradient of the velocity, and the pressure. For the details of the proof, the readers can refer to [17].

Lemma 3.2. *Let (ρ, u) with $\rho \in [0, 2\bar{\rho}]$ be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then there exists a positive constant C , depending on $\bar{\rho}$, such that for any $p \in [2, 6]$,*

$$\|\nabla u\|_{L^p} \leq C (\|F\|_{L^p} + \|P(\rho) - P(\bar{\rho})\|_{L^p} + \|\omega\|_{L^p}), \quad (3.20)$$

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C \|\rho \dot{u}\|_{L^p}, \quad (3.21)$$

$$\|\nabla u\|_{L^p} \leq C \|\nabla u\|_{L^2}^{(6-p)/(2p)} (\|\rho \dot{u}\|_{L^2} + \|P(\rho) - P(\bar{\rho})\|_{L^6})^{(3p-6)/(2p)}, \quad (3.22)$$

and

$$\|F\|_{L^p} + \|\omega\|_{L^p} \leq C (\|\nabla u\|_{L^2} + \|P(\rho) - P(\bar{\rho})\|_{L^6})^{(6-p)/(2p)} \|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)}. \quad (3.23)$$

Next, we can close the estimates concerning $\Phi_1(T)$, $\Phi_2(T)$, $\Phi_3(\sigma(T))$ and \dot{u} by some modifications of the method in [17]; thus we omit the details.

Lemma 3.3. *Assume that $2(\rho, u)$ with $\rho \in [0, 2\bar{\rho}]$ is a smooth solution to (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then there exist positive constants K and ε_1 , which may depend only on $\mu, A, \gamma, \bar{\rho}, \tilde{\rho}$ and M , such that*

$$\Phi_3(\sigma(T)) + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx ds \leq 2K, \quad (3.24)$$

$$\Phi_1(T) + \Phi_2(T) \leq E_0^{1/2}, \quad (3.25)$$

and

$$\sup_{0 \leq t \leq T} \sigma \int \rho |\dot{u}|^2 dx + \int_0^T \sigma \int |\nabla \dot{u}|^2 dx dt \leq C(\bar{\rho}, M), \quad (3.26)$$

provided that (ρ, u) satisfies (3.4) and $E_0 \leq \varepsilon_1$.

We are now in a position of proving the uniform upper bound of ρ .

Lemma 3.4. *Assume that (ρ, u) is a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$ satisfying (3.4) with K as in Lemma 3.3. Then there exists a positive constant $\varepsilon > 0$, depending on $\mu, A, \gamma, \bar{\rho}, \tilde{\rho}$ and M , such that*

$$\sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{7\bar{\rho}}{4}, \quad (3.27)$$

provided $E_0 \leq \varepsilon$.

Proof. To apply Lemma 2.2, we first rewrite Eq. (1.1)₁ as

$$D_t \rho = g(\rho) + b'(t),$$

where $D_t \rho = \rho_t + u \cdot \nabla \rho$ and

$$g(\rho) = -\frac{A\rho}{2\mu + \lambda(\rho)} (\rho^\gamma - \tilde{\rho}^\gamma), \quad b(t) = -\int_0^t \frac{\rho F}{2\mu + \lambda(\rho)} dt.$$

We need to estimate $\|b(t)\|_{L^\infty}$. To do this, using [Lemmas 2.1](#) and [3.2](#), [\(3.6\)](#), [\(3.25\)](#) and [\(3.26\)](#), we deduce that for all $0 \leq t_1 < t_2 \leq \sigma(T)$

$$\begin{aligned} |b(t_2) - b(t_1)| &\leq C \int_0^{\sigma(T)} \|F\|_{L^\infty} dt \leq C \int_0^{\sigma(T)} \|F\|_{L^2}^{1/4} \|\nabla F\|_{L^6}^{3/4} dt \\ &\leq C \int_0^{\sigma(T)} \left(\|P(\rho) - P(\tilde{\rho})\|_{L^2}^{1/4} + \|\nabla u\|_{L^2}^{1/4} \right) \|\nabla \dot{u}\|_{L^2}^{3/4} dt \\ &\leq C \int_0^{\sigma(T)} \left(\sigma^{-3/8} E_0^{1/8} + \sigma^{-1/2} (\sigma \|\nabla u\|_{L^2}^2)^{1/8} \right) (\sigma \|\nabla \dot{u}\|_{L^2}^2)^{3/8} dt \\ &\leq C E_0^{1/8} + C \Phi_1^{1/8}(T) \leq C E_0^{1/16}, \end{aligned}$$

provided $E_0 \leq \varepsilon_1$. So, if we choose N_0, N_1 in [\(2.3\)](#) and ξ^* in [\(2.5\)](#) as follows:

$$N_0 = C E_0^{1/16}, \quad N_1 = 0 \quad \text{and} \quad \xi^* = \tilde{\rho},$$

then we readily conclude from [\(2.4\)](#) that (keeping in mind that $0 \leq \rho_0 \leq \tilde{\rho}$)

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max\{\tilde{\rho}, \tilde{\rho}\} + N_0 \leq \tilde{\rho} + C E_0^{1/16} \leq \frac{3}{2} \tilde{\rho}, \quad (3.28)$$

provided E_0 is chosen to be such that

$$E_0 \leq \min\{\varepsilon_1, \varepsilon_2\} \quad \text{with} \quad \varepsilon_2 = \left(\frac{\tilde{\rho}}{2C} \right)^{16}.$$

On the other hand, one has from [Lemmas 3.2](#) and [3.3](#) that for $\sigma(T) \leq t \leq T$,

$$\begin{aligned} |b(t_2) - b(t_1)| &\leq \int_{t_1}^{t_2} \|F\|_{L^\infty} dt \\ &\leq \frac{A}{2\mu} (t_2 - t_1) + C \int_{\sigma(T)}^T \|F\|_{L^\infty}^{8/3} dt \\ &\leq \frac{A}{2\mu} (t_2 - t_1) + C \int_{\sigma(T)}^T \|F\|_{L^2}^{2/3} \|\nabla F\|_{L^6}^2 dt \\ &\leq \frac{A}{2\mu} (t_2 - t_1) + C E_0^{1/6} \int_{\sigma(T)}^T \sigma^3 \|\nabla \dot{u}\|_{L^2}^2 dt \\ &\leq \frac{A}{2\mu} (t_2 - t_1) + C E_0^{2/3}, \end{aligned}$$

provided $E_0 \leq \varepsilon_2$. Thus, if we choose

$$N_0 = C E_0^{2/3}, \quad N_1 = \frac{A}{2\mu} \quad \text{and} \quad \xi^* = \tilde{\rho} + 1$$

then we deduce from [Lemma 2.2](#) and [\(3.28\)](#) that

$$\sup_{\sigma(T) \leq t \leq T} \|\rho\|_{L^\infty} \leq \max\left\{ \frac{3\tilde{\rho}}{2}, \tilde{\rho} + 1 \right\} + N_0 \leq \frac{3\tilde{\rho}}{2} + C E_0^{2/3} \leq \frac{7}{4} \tilde{\rho}, \quad (3.29)$$

provided E_0 satisfies

$$E_0 \leq \varepsilon \triangleq \min\{\varepsilon_2, \varepsilon_3\} \quad \text{with} \quad \varepsilon_3 = \left(\frac{\tilde{\rho}}{4C} \right)^{3/2}.$$

The combination of [\(3.28\)](#) and [\(3.29\)](#) completes the proof of [\(3.27\)](#). \square

Proof of Proposition 3.1. By virtue of [Lemmas 3.2–3.4](#), one immediately proves [Proposition 3.1](#) by choosing K and ε as the ones determined in [Lemmas 3.3–3.4](#). \square

4. Time-dependent higher order estimates

In this section, we prove the higher order estimates of (ρ, u) needed for the existence of classical solutions under the conditions of [Theorem 1.1](#). From now on, we denote by C the various positive constants which may depend on $\rho_0, u_0, g, \mu, A, \gamma, \bar{\rho}, \tilde{\rho}, M$ and T as well.

Lemma 4.1. *For any given $T > 0$, there exists a positive constant C , depending on T , such that*

$$\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho |\dot{u}|^2 dx dt \leq C(T), \quad (4.1)$$

$$\sup_{0 \leq t \leq T} \int \rho |\dot{u}|^2 dx + \int_0^T \int |\nabla \dot{u}|^2 dx ds \leq C(T), \quad (4.2)$$

$$\int_0^T (\|\operatorname{div} u\|_{L^\infty}^2 + \|\nabla \times u\|_{L^\infty}^2) dt \leq C(T). \quad (4.3)$$

Proof. As an immediate result of (3.24) and (3.25), one obtains (4.1). Using (3.6), (4.1), [Lemmas 2.1](#) and [3.2](#), we obtain by taking $m = 0$ in (3.19) that

$$\frac{1}{2} \frac{d}{dt} \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \frac{\mu}{4} \|\nabla \dot{u}\|_{L^2}^2 \leq C + C \|\rho^{1/2} \dot{u}\|_{L^2}^2 \|\rho^{1/2} \dot{u}\|_{L^2}^2, \quad (4.4)$$

which, together with Gronwall's inequality, leads to (4.2). Finally, by [Lemmas 2.2](#) and [3.2](#), we can easily prove (4.3). This finishes the proof of [Lemma 4.1](#). \square

The next lemma is concerned with the estimates of the gradient of the density and velocity, which are crucial for the higher-norm estimates.

Lemma 4.2. *Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,*

$$\sup_{0 \leq t \leq T} (\|(\nabla \rho, \nabla P(\rho))\|_{L^2 \cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty} dt \leq C(T). \quad (4.5)$$

Proof. Differentiating (1.1)₁ with respect to x_i ($i = 1, 2, 3$), multiplying it by $|\partial_i \rho|^{p-2} \partial_i \rho$ with $p \geq 2$, and integrating the resulting equation by parts over \mathbb{R}^3 , we obtain after summing over i from 1 to 3 that

$$\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C (\|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p} + \|\nabla^2 u\|_{L^p}), \quad \forall p \in [2, 6]. \quad (4.6)$$

Since $\nabla \times \omega = \nabla \operatorname{div} u - \Delta u$, we can make use of (1.15) to get that

$$-\Delta u = \nabla \times \omega - \frac{\nabla F + P'(\rho) \nabla \rho}{2\mu + \lambda(\rho)} + \frac{F + P(\rho) - P(\tilde{\rho})}{(2\mu + \lambda(\rho))^2} \lambda'(\rho) \nabla \rho. \quad (4.7)$$

Following from (4.7) and [Lemma 3.2](#), one obtains

$$\begin{aligned} \|\nabla^2 u\|_{L^p} &\leq C (\|\nabla \omega\|_{L^p} + \|\nabla F\|_{L^p} + \|\nabla \rho\|_{L^p} + \|F\|_{L^\infty} \|\nabla \rho\|_{L^p}) \\ &\leq [\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p} + (\|P(\rho) - P(\tilde{\rho})\|_{L^\infty} + \|\operatorname{div} u\|_{L^\infty}) \|\nabla \rho\|_{L^p}] \\ &\leq (\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p} + \|\operatorname{div} u\|_{L^\infty} \|\nabla \rho\|_{L^p}). \end{aligned} \quad (4.8)$$

Therefore, putting (4.8) into (4.6) yields

$$\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C \|\rho \dot{u}\|_{L^p}, \quad \forall p \in [2, 6]. \quad (4.9)$$

Using [Lemma 2.3](#) and by choosing $q = 6$ in (2.6), we first deduce from (4.1) and (4.8) with $p = 6$ that

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq C [1 + (\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \ln(e + \|\nabla \dot{u}\|_{L^2}) + \|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2] \\ &\quad + C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \ln(e + \|\nabla \rho\|_{L^6}). \end{aligned} \quad (4.10)$$

Thus, if we set

$$f(t) = e + \|\nabla \rho\|_{L^6}$$

and

$$g(t) = 1 + (\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \ln(e + \|\nabla \dot{u}\|_{L^2}) + \|\nabla \dot{u}\|_{L^2} + \|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2,$$

then we see from (4.9) with $p = 6$ and (4.10) that

$$(\ln f(t))' \leq Cg(t) + Cg(t) \ln f(t). \quad (4.11)$$

It is easily seen from (4.2) and (4.3) that

$$\int_0^T g(t) dt \leq C \int_0^T (1 + \|\nabla \dot{u}\|_{L^2}^2 + \|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2) dt \leq C.$$

Hence, an application of Gronwall's inequality to (4.11) yields

$$\ln f(t) \leq C \quad \text{and} \quad \|\nabla \rho\|_{L^6} \leq C \quad \text{for all } t \in [0, T], \quad (4.12)$$

which, combined with (4.2), (4.3) and (4.10), also implies that

$$\int_0^T \|\nabla u\|_{L^\infty} ds \leq C. \quad (4.13)$$

Using (4.1) and (4.13), one also infers from (4.9) with $p = 2$ and Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2} \leq C. \quad (4.14)$$

In a manner similar to the derivation of (4.8), using Lemmas 2.1 and 3.2, the Hölder inequality, (4.2), (4.12) and (4.14), we have from (4.7) and the standard L^2 -estimate of elliptic system that

$$\begin{aligned} \|\nabla^2 u\|_{L^2} &\leq C (\|\nabla \omega\|_{L^2} + \|\nabla F\|_{L^2} + \|\nabla \rho\|_{L^2} + \|F \nabla \rho\|_{L^2}) \\ &\leq (\|\rho \dot{u}\|_{L^2} + \|\nabla \rho\|_{L^2} + \|F\|_{L^6} \|\nabla \rho\|_{L^3}) \\ &\leq \left(\|\rho \dot{u}\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla F\|_{L^2} \|\nabla \rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^6}^{1/2} \right) \\ &\leq (1 + \|\rho \dot{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla \rho\|_{L^6}^2) \leq C. \end{aligned}$$

This, together with (4.12)–(4.14), proves Lemma 4.2. \square

By virtue of Lemmas 4.1 and 4.2, we can prove the following.

Lemma 4.3. *Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,*

$$\sup_{0 \leq t \leq T} \int \rho |u_t|^2 dx + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C(T), \quad (4.15)$$

$$\sup_{0 \leq t \leq T} (\|\rho - \tilde{\rho}\|_{H^2} + \|P(\rho) - P(\tilde{\rho})\|_{H^2}) + \int_0^T \|\nabla u\|_{H^2}^2 dt \leq C(T), \quad (4.16)$$

and

$$\sup_{0 \leq t \leq T} (\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2) dt \leq C(T). \quad (4.17)$$

Proof. By (3.6), (4.1), (4.5) and Lemma 2.1, one easily gets (4.15) from (4.2). From (3.12) and (1.1)₁, we have by direct computations and (4.5) that

$$\begin{aligned} \frac{d}{dt} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) &\leq C \|\nabla u\|_{L^\infty} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) + C \|\nabla^3 u\|_{L^2} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}) \\ &\quad + C \|\nabla^2 u\|_{L^3} (\|\nabla \rho\|_{L^6} + \|\nabla P\|_{L^6}) (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}) \\ &\leq C \|\nabla u\|_{L^\infty} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) + C \|\nabla u\|_{H^2} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}). \end{aligned} \quad (4.18)$$

We now need to estimate $\|\nabla u\|_{H^2}$. We first deduce from (4.7) that

$$\begin{aligned}\|\nabla u\|_{H^2} &\leq C \left(\|\nabla u\|_{H^1} + \|\nabla \omega\|_{H^1} + \left\| \nabla \left(\frac{F + P(\rho) - P(\tilde{\rho})}{2\mu + \lambda(\rho)} \right) \right\|_{H^1} \right) \\ &\leq C \left(\|\nabla u\|_{H^1} + \|\nabla \rho\|_{L^2} + \|\nabla |\rho|^2\|_{L^2} + \|F\nabla \rho\|_{L^2} + \|F|\nabla \rho|^2\|_{L^2} \right) \\ &\quad + C \left(\|\nabla \omega\|_{H^1} + \|\nabla F\|_{H^1} + \|\nabla F\nabla \rho\|_{L^2} + \|F\nabla^2 \rho\|_{L^2} \right) + C \left(\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P(\rho)\|_{L^2} \right) \triangleq \sum_{i=1}^3 I_i.\end{aligned}\quad (4.19)$$

By Lemmas 2.1 and 3.2, (4.2) and (4.5), one has

$$\begin{aligned}I_1 &\leq C \left(\|\nabla u\|_{H^1} + \|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{L^4}^2 + \|F\|_{L^6} \|\nabla \rho\|_{L^3} + \|F\|_{L^6} \|\nabla \rho\|_{L^6}^2 \right) \\ &\leq C \left(1 + \|\nabla \rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^6}^{3/2} + \|\rho \dot{u}\|_{L^2} \|\nabla \rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^6}^{1/2} + \|\rho \dot{u}\|_{L^2}^2 \right) \leq C.\end{aligned}$$

To handle the second term on the right-hand side of (4.19), we first notice that

$$\|\nabla F\nabla \rho\|_{L^2} \leq \|\nabla F\|_{L^6} \|\nabla \rho\|_{L^3} \leq C \|\nabla F\|_{H^1}$$

and

$$\|F\nabla^2 \rho\|_{L^2} \leq \|F\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} \leq \|\nabla F\|_{H^1} \|\nabla^2 \rho\|_{L^2}.$$

Hence, it follows from (1.15) and the H^1 -theory of the elliptic system that

$$\begin{aligned}I_2 &\leq C \left(\|\nabla \omega\|_{H^1} + \|\nabla F\|_{H^1} + \|\nabla F\|_{H^1} \|\nabla^2 \rho\|_{L^2} \right) \leq C \|\rho \dot{u}\|_{H^1} (1 + \|\nabla^2 \rho\|_{L^2}) \\ &\leq C \left(\|\rho \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2} + \|\dot{u} \nabla \rho\|_{L^2} \right) (1 + \|\nabla^2 \rho\|_{L^2}) \\ &\leq C (1 + \|\nabla \dot{u}\|_{L^2}) (1 + \|\nabla^2 \rho\|_{L^2}),\end{aligned}$$

where we have also used (4.2), (4.5) and (2.1) as well.

Now, putting the estimates of I_1 and I_2 into (4.19), we get

$$\|\nabla u\|_{H^2} \leq C (1 + \|\nabla \dot{u}\|_{L^2}) (1 + \|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P(\rho)\|_{L^2}), \quad (4.20)$$

which, inserted into (4.18), yields

$$\frac{d}{dt} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) \leq C (1 + \|\nabla u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2}) (1 + \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla P\|_{L^2}^2).$$

This, together with Gronwall's inequality, leads to

$$\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2} \leq C. \quad (4.21)$$

As a result, one also sees from (4.2), (4.20) and (4.21) that

$$\int_0^T \|\nabla u\|_{H^2}^2 dt \leq C,$$

which completes the proof of (4.16).

Due to (4.5) and (4.16), it readily follows from (1.1)₁ and (3.12) that

$$\|\rho_t\|_{L^2} + \|P_t\|_{L^2} \leq C \|u\|_{L^\infty} (\|\nabla \rho\|_{L^2} + \|\nabla P\|_{L^2}) + C \|\nabla u\|_{L^2} \leq C \quad (4.22)$$

and

$$\|\nabla \rho_t\|_{L^2} + \|\nabla P_t\|_{L^2} \leq C \|\nabla^2 u\|_{L^2} + C \|u\|_{L^\infty} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}) + C \|\nabla u\|_{L^3} (\|\nabla \rho\|_{L^6} + \|\nabla P\|_{L^6}) \leq C. \quad (4.23)$$

Moreover, it follows from (3.12) that

$$P_{tt} + u_t \cdot \nabla P + u \cdot \nabla P_t + \gamma P_t \operatorname{div} u + \gamma P \operatorname{div} u_t = 0,$$

so that, using (4.15), (4.16), (4.22) and (4.23), one has

$$\int_0^T \|P_{tt}\|_{L^2}^2 dt \leq C + C \int_0^T (\|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2) dt \leq C.$$

The estimate of $\|\rho_{tt}\|_{L^2(0,T;L^2)}$ can be shown in the same way. This, together with (4.22) and (4.23), proves (4.17). The proof of Lemma 4.3 is therefore complete. \square

The next lemma is a result of (1.15), Lemma 4.3, (4.7) and (4.20), which will be obtained by the standard estimates of the elliptic system and will be used to prove the higher norm estimates of the solutions.

Lemma 4.4. *Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,*

$$\sup_{0 \leq t \leq T} (\|\rho \dot{u}\|_{H^1} + \|\nabla u\|_{H^2}) \leq C(T) (1 + \|\nabla u_t\|_{L^2}), \quad (4.24)$$

$$\sup_{0 \leq t \leq T} \|\nabla u_t\|_{H^1} \leq C(T) (1 + \|\rho u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2}). \quad (4.25)$$

Proof. It follows from (1.1)₂, Lemma 2.1 and (4.16) that

$$\begin{aligned} \|\nabla(\rho \dot{u})\|_{L^2} &\leq C (\|\nabla P(\rho)\|_{H^1} + \|\nabla u\|_{H^2} + \|\nabla \rho \nabla u\|_{L^2} + \|\nabla \rho \nabla^2 u\|_{L^2} + \|\nabla^2 \rho \nabla u\|_{L^2}) \\ &\leq C [1 + \|\nabla u\|_{H^2} + \|\nabla \rho\|_{L^3} (\|\nabla u\|_{L^6} + \|\nabla^2 u\|_{L^6}) + \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}] \\ &\leq C (1 + \|\nabla u\|_{H^2}) \end{aligned}$$

which, together with (4.20), proves (4.24), since it is easy to check from (4.5), (4.16), (4.20) and Sobolev inequalities that

$$\begin{aligned} \|\nabla u\|_{H^2} &\leq C (1 + \|\nabla \dot{u}\|_{L^2}) \leq (\|\nabla u_t\|_{L^2} + \|u \cdot \nabla u\|_{H^1}) \\ &\leq C (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2) \leq C (\|\nabla u_t\|_{L^2} + 1). \end{aligned} \quad (4.26)$$

By the standard H^2 -estimate of the elliptic system and (4.17), we infer from (4.7) in a manner similar to the derivation of (4.18) that

$$\begin{aligned} \|\nabla u_t\|_{H^1} &\leq C \left(\|\nabla u_t\|_{L^2} + \|\omega_t\|_{H^1} + \left\| \partial_t \left(\frac{F + P(\rho) - P(\tilde{\rho})}{2\mu + \lambda(\rho)} \right) \right\|_{H^1} \right) \\ &\leq C (\|\nabla u_t\|_{L^2} + \|\omega_t\|_{H^1} + \|F_t\|_{H^1} + \|\rho_t\|_{H^1} + \|P_t\|_{H^1} + \|F\rho_t\|_{H^1}) \\ &\leq C (1 + \|\nabla u_t\|_{L^2} + \|\omega_t\|_{H^1} + \|F_t\|_{H^1}), \end{aligned} \quad (4.27)$$

since it follows from Lemmas 2.1 and 3.2, (4.5), (4.16), (4.17) and (4.24) that

$$\begin{aligned} \|F\rho_t\|_{H^1} &\leq C (\|F\|_{L^\infty} \|\rho_t\|_{H^1} + \|\nabla F\|_{L^6} \|\rho_t\|_{L^3}) \leq C (\|F\|_{L^2} + \|\nabla F\|_{L^6}) \\ &\leq C (\|\nabla u\|_{L^2} + \|P(\rho) - P(\tilde{\rho})\|_{L^2} + \|\rho \dot{u}\|_{L^6}) \\ &\leq C (1 + \|\rho \dot{u}\|_{H^1}) \leq C (1 + \|\nabla u_t\|_{L^2}). \end{aligned}$$

In order to deal with $\|\omega_t\|_{H^1}$ and $\|F_t\|_{H^1}$, we deduce from (1.15) that

$$\Delta F_t = \operatorname{div}((\rho \dot{u})_t) \quad \text{and} \quad \mu \Delta \omega_t = \nabla \times ((\rho \dot{u})_t),$$

so that, we have by the L^2 -estimate of the elliptic system that

$$\|\nabla F_t\|_{L^2} + \|\nabla \omega_t\|_{L^2} \leq C \|(\rho \dot{u})_t\|_{L^2} \leq C (\|\rho_t \dot{u}\|_{L^2} + \|\rho (\dot{u})_t\|_{L^2}), \quad (4.28)$$

in which the terms on the right-hand side can be estimated as follows, using (4.5), (4.15), (4.17), (4.24), (4.26) and Lemma 2.1:

$$\|\rho_t \dot{u}\|_{L^2} \leq C \|\rho_t\|_{L^3} \|\dot{u}\|_{L^6} \leq C \|\nabla \dot{u}\|_{L^2} \leq C (1 + \|\nabla u_t\|_{L^2})$$

and

$$\begin{aligned} \|\rho (\dot{u})_t\|_{L^2} &\leq C (\|\rho u_{tt}\|_{L^2} + \|\rho u_t \cdot \nabla u\|_{L^2} + \|\rho u \cdot \nabla u_t\|_{L^2}) \\ &\leq C (\|\rho u_{tt}\|_{L^2} + \|\rho u_t\|_{L^2} \|\nabla u\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \\ &\leq C (\|\rho u_{tt}\|_{L^2} + \|\rho u_t\|_{L^2} \|\nabla u\|_{H^2} + \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2}) \\ &\leq C (1 + \|\rho u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2}). \end{aligned}$$

Thus, putting the above two estimates into (4.28), we find

$$\|\nabla F_t\|_{L^2} + \|\nabla \omega_t\|_{L^2} \leq C (1 + \|\rho u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2}), \quad (4.29)$$

which, inserted into (4.27), leads to (4.25). \square

Using Lemmas 4.3 and 4.4 and the compatibility condition (1.9), we can prove the following.

Lemma 4.5. Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,

$$\sup_{0 \leq t \leq T} (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^2} + \|\rho \dot{u}\|_{H^1}) + \int_0^T (\|\rho u_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{H^1}^2) dt \leq C(T). \quad (4.30)$$

Proof. Differentiating (1.1)₂ with respect to t and multiplying the resulting equations by u_{tt} in L^2 , we obtain after integrating by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2) dx + \int \rho |u_{tt}|^2 dx \\ &= \frac{1}{2} \int \lambda'(\rho) \rho_t |\operatorname{div} u_t|^2 dx - \int \lambda'(\rho) \rho_t (\operatorname{div} u) (\operatorname{div} u_{tt}) dx \\ &+ \int P_t (\operatorname{div} u_{tt}) dx - \int \rho_t u \cdot \nabla u \cdot u_{tt} dx - \int \rho (u \cdot \nabla u)_t \cdot u_{tt} dx \\ &\triangleq \sum_{i=1}^5 I_i, \end{aligned} \quad (4.31)$$

where the right-hand side will be estimated term by term in the following. First, by (1.1)₁ and integrating by parts, we have

$$\begin{aligned} I_1 &= -\frac{1}{2} \int \lambda'(\rho) \operatorname{div}(\rho u) |\operatorname{div} u_t|^2 dx \\ &= \int \left(\lambda'(\rho) (\operatorname{div} u_t) \rho u \cdot \nabla \operatorname{div} u_t + \frac{1}{2} \lambda''(\rho) \rho u \cdot \nabla \rho |\operatorname{div} u_t|^2 \right) dx \\ &\leq C (\|u\|_{L^\infty} \|\operatorname{div} u_t\|_{L^2} \|\nabla^2 u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla \rho\|_{L^3} \|\nabla u_t\|_{L^3}^2) \\ &\leq C (\|\nabla u_t\|_{L^2} \|\nabla^2 u_t\|_{L^2} + \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^6}) \leq C \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{H^1} \\ &\leq \delta \|\rho u_{tt}\|_{L^2}^2 + C(\delta) (1 + \|\nabla u_t\|_{L^2}^2), \end{aligned} \quad (4.32)$$

where we have used Lemma 2.1, (4.5) and (4.25). To deal with I_2 , we first write it in the form:

$$\begin{aligned} I_2 &= -\frac{d}{dt} \int \lambda'(\rho) \rho_t (\operatorname{div} u) (\operatorname{div} u_t) dx + \int \lambda''(\rho) \rho_t^2 (\operatorname{div} u) (\operatorname{div} u_t) dx \\ &+ \int \lambda'(\rho) \rho_{tt} (\operatorname{div} u) (\operatorname{div} u_t) dx + \int \lambda'(\rho) \rho_t |(\operatorname{div} u_t)|^2 dx \triangleq \sum_{i=0}^3 I_2^i, \end{aligned}$$

where the second and the third terms can be estimated as follows, using (4.17), (4.24) and (4.25):

$$\begin{aligned} I_2^1 &\leq C \|\nabla u\|_{L^\infty} \|\rho_t\|_{L^4}^2 \|\nabla u_t\|_{L^2} \leq C \|\nabla u\|_{H^2} \|\rho_t\|_{H^1}^2 \|\nabla u_t\|_{L^2} \\ &\leq C (1 + \|\nabla u_t\|_{L^2}^2), \\ I_2^2 &\leq C \|\nabla u\|_{L^\infty} \|\rho_{tt}\|_{L^2} \|\nabla u_t\|_{L^2} \leq C \|\nabla u\|_{H^2} \|\rho_{tt}\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq C (1 + \|\nabla u_t\|_{L^2}^2) \|\rho_{tt}\|_{L^2} \leq C (1 + \|\nabla u_t\|_{L^2}^4 + \|\rho_{tt}\|_{L^2}^2), \end{aligned}$$

and similar to (4.32)

$$I_2^3 \leq C \|\rho_t\|_{L^3} \|\nabla u_t\|_{L^6} \|\nabla u_t\|_{L^2} \leq \delta \|\rho u_{tt}\|_{L^2}^2 + C(\delta) (1 + \|\nabla u_t\|_{L^2}^2),$$

so that

$$I_2 \leq -\frac{d}{dt} \int \lambda'(\rho) \rho_t (\operatorname{div} u) (\operatorname{div} u_t) dx + \delta \|\rho u_{tt}\|_{L^2}^2 + C(\delta) (1 + \|\nabla u_t\|_{L^2}^4 + \|\rho_{tt}\|_{L^2}^2). \quad (4.33)$$

Similarly,

$$I_3 \leq \frac{d}{dt} \int P_t (\operatorname{div} u_t) dx + C (\|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2) \quad (4.34)$$

and

$$I_4 \leq -\frac{d}{dt} \int \rho_t u \cdot \nabla u \cdot u_t dx + C (1 + \|\nabla u_t\|_{L^2}^4 + \|\rho_{tt}\|_{L^2}^2). \quad (4.35)$$

Finally, it is easy to see that

$$\begin{aligned} I_5 &\leq C \int (|u_t| |\nabla u| + |u| |\nabla u_t|) |\rho u_{tt}| dx \\ &\leq C (\|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \|\rho u_{tt}\|_{L^2} \\ &\leq C \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2} \|\rho u_{tt}\|_{L^2} \leq \delta \|\rho u_{tt}\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^2. \end{aligned} \quad (4.36)$$

Thus, putting (4.32)–(4.36) into (4.31) and choosing $\delta > 0$ small enough, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u_t|^2 + (\mu + \lambda(\rho)) |\operatorname{div} u_t|^2) dx + \frac{1}{2} \int \rho |u_{tt}|^2 dx \\ \leq \frac{dM}{dt} + C (1 + \|\nabla u_t\|_{L^2}^4 + \|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2), \end{aligned} \quad (4.37)$$

where

$$M \triangleq - \int (\lambda'(\rho) \rho_t (\operatorname{div} u) (\operatorname{div} u_t) - P_t (\operatorname{div} u_t) + \rho_t u \cdot \nabla u \cdot u_t) dx.$$

By Lemma 2.1, (4.5), (4.17) and the Cauchy–Schwarz inequality, we can bound M as follows:

$$\begin{aligned} |M| &\leq C (\|\rho_t\|_{L^6} \|\nabla u\|_{L^3} + \|P_t\|_{L^2}) \|\nabla u_t\|_{L^2} + C \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6} \\ &\leq C (\|\rho_t\|_{H^1} + \|P_t\|_{L^2}) (1 + \|\nabla u\|_{H^1} + \|\nabla u\|_{H^1}^2) \|\nabla u_t\|_{L^2} \\ &\leq \frac{\mu}{4} \|\nabla u_t\|_{L^2}^2 + C. \end{aligned}$$

Taking this into account and applying Gronwall's inequality, we immediately deduce that

$$\sup_{0 \leq t \leq T} \int |\nabla u_t|^2 dx + \int_0^T \int \rho |u_{tt}|^2 dx dt \leq C, \quad (4.38)$$

since (4.15) together with (4.17) implies that $\|\nabla u_t\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2 \in L^1(0, T)$ and the compatibility condition (1.9) with $g \in D^1$ yields that $\nabla u_t|_{t=0} \in L^2$ is well defined. As a result, combining (4.38) with Lemma 4.4 leads to the desired estimates stated in (4.30). The proof of Lemma 4.5 is therefore finished. \square

The next lemma is concerned with the H^3 -regularity of the density and pressure.

Lemma 4.6. *Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then,*

$$\sup_{0 \leq t \leq T} (\|\rho - \tilde{\rho}\|_{H^3} + \|P(\rho) - P(\tilde{\rho})\|_{H^3}) + \int_0^T \|\nabla u\|_{H^3}^2 dt \leq C(T). \quad (4.39)$$

Proof. By direct computations and integration by parts, we deduce from (3.12) and (1.1)₁ that

$$\frac{d}{dt} (\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 P\|_{L^2}^2) \leq C (1 + \|\nabla^2 u\|_{H^2}^2 + \|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 P\|_{L^2}^2). \quad (4.40)$$

To prove (4.39), it suffices to estimate $\|\nabla^2 u\|_{H^2}$. Thus, by Lemmas 4.3 and 4.5 we apply the H^2 -theory of the elliptic system to infer from (4.7) that

$$\begin{aligned} \|\nabla^2 u\|_{H^2} &\leq C \left(\|\nabla u\|_{H^2} + \|\nabla \omega\|_{H^2} + \left\| \nabla \left(\frac{F + P(\rho) - P(\tilde{\rho})}{2\mu + \lambda(\rho)} \right) \right\|_{H^2} \right) \\ &\leq C (1 + \|\nabla \omega\|_{H^2} + \|\nabla F\|_{H^2}) + R_1 + R_2, \end{aligned} \quad (4.41)$$

where

$$R_1 \triangleq C \|\nabla^2 F \nabla \rho\|_{L^2} + C (\|\nabla F \nabla^2 \rho\|_{L^2} + \|\nabla F |\nabla \rho|^2\|_{L^2}) + C (\|F \nabla^3 \rho\|_{L^2} + \|F \nabla \rho \nabla^2 \rho\|_{L^2} + \|F |\nabla \rho|^3\|_{L^2})$$

and

$$R_2 \triangleq C \|\nabla^3 P\|_{L^2} + C \|\nabla^2 P \nabla \rho\|_{L^2} + C (\|\nabla P \nabla^2 \rho\|_{L^2} + \|\nabla P |\nabla \rho|^2\|_{L^2}) + C (\|\nabla^3 \rho\|_{L^2} + \|\nabla \rho \nabla^2 \rho\|_{L^2} + \||\nabla \rho|^3\|_{L^2}),$$

since, according to the proof of (4.20), (4.16) and (4.30), we only need to handle the terms with the highest-order derivatives. It is easily deduced from (4.16), (4.30) and Lemma 2.1 that

$$R_1 \leq C (1 + \|\nabla^3 \rho\|_{L^2}) \quad (4.42)$$

and

$$R_2 \leq C \left(1 + \|\nabla^3 \rho\|_{L^2} + \|\nabla^3 P\|_{L^2} \right). \quad (4.43)$$

In order to estimate $\|\nabla^3(\omega, F)\|_{L^2}$, we make use of (1.14) and the standard estimate of the elliptic system that

$$\begin{aligned} \|\nabla(\omega, F)\|_{H^2} &\leq C \left(\|\nabla(\omega, F)\|_{H^1} + \|\rho \dot{u}\|_{H^2} \right) \leq C \left(1 + \|\nabla^2(\rho \dot{u})\|_{L^2} \right) \\ &\leq C \left(1 + \|\nabla^2 \rho \dot{u}\|_{L^2} + \|\nabla \rho \nabla \dot{u}\|_{L^2} + \|\rho \nabla^2 \dot{u}\|_{L^2} \right) \\ &\leq C \left(1 + \|\nabla^2 \rho\|_{L^3} \|\dot{u}\|_{L^6} + \|\nabla \rho\|_{L^\infty} \|\nabla \dot{u}\|_{L^2} \right) + C \left(\|\nabla^2 u_t\|_{L^2} + \|\nabla u \nabla^2 u\|_{L^2} + \|u \nabla^3 u\|_{L^2} \right) \\ &\leq C \left(1 + \|\nabla^2 \rho\|_{H^1} \|\nabla \dot{u}\|_{L^2} + \|\nabla^2 u_t\|_{L^2} + \|\nabla u\|_{H^2}^2 \right) \\ &\leq C \left(1 + \|\nabla^3 \rho\|_{L^2} + \|\nabla^2 u_t\|_{L^2} \right), \end{aligned} \quad (4.44)$$

where we have used the following estimate

$$\|\nabla \dot{u}\|_{L^2} \leq C \left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 \right) \leq C.$$

Thus, putting (4.42)–(4.44) into (4.41) gives

$$\|\nabla^2 u\|_{H^2} \leq C \left(1 + \|\nabla^3 \rho\|_{L^2} + \|\nabla^3 P\|_{L^2} + \|\nabla^2 u_t\|_{L^2} \right), \quad (4.45)$$

which, inserted into (4.40) and combined with Gronwall's inequality, yields

$$\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 P\|_{L^2}^2 \leq C.$$

This, together with (4.45) and (4.30), implies that $\|\nabla^2 u\|_{H^2}^2 \in L^1(0, T)$. The proof of Lemma 4.6 is thus complete. \square

Finally, with all the estimates above in hand we can easily obtain the H^2 -estimate of $(u_t, \nabla^2 u)$ as follows.

Lemma 4.7. *Let (ρ, u) be a smooth solution of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T]$. Then for $\tau \in (0, T]$, there exists a positive constant $C(\tau, T)$, depending on τ and T , such that*

$$\sup_{\tau \leq t \leq T} \left(\|\rho^{1/2} u_{tt}\|_{L^2} + \|\nabla u_t\|_{H^1} + \|\nabla^2 u\|_{H^2} \right) + \int_{\tau}^T \int |\nabla u_{tt}|^2 dx dt \leq C(\tau, T). \quad (4.46)$$

With all the estimates above and using the standard arguments based on the local existence theorem (cf. Lemma 2.4), one can easily extend the classical solutions of (ρ, u) to a global one as that in [17]. The proof of Theorem 1.1 is therefore complete.

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