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Multiple solutions for equations involving bilinear, coercive and compact forms with applications to differential equations

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ABSTRACT

The existence of multiple fixed points for the coercive, bilinear, compact forms defined in the cone in the Banach space is proved. Multiple applications to the integral equations derived from BVPs for differential equations are provided.

1. Introduction and motivation

It is well known that the quadratic equation

$$u = au^2 + u_0 \tag{1}$$

can have either none, one or two solutions $u \geq 0$, depending on the data $a > 0$ and $u_0 \geq 0$. For example, if we additionally assume that

$$4au_0 < 1, \tag{2}$$

then the existence of two nonnegative solutions of (1) is guaranteed.

In this paper we would like to show that this simple observation can be generalised if we replace quadratic term au^2 with a bilinear form under suitable conditions. More specifically, we shall consider the equation in the cone P in the Banach space U with the norm $|\cdot|$ in the form

$$u = b(u, u) + u_0 \tag{3}$$

for some given element $u_0 \in P$ and bilinear, coercive and compact form b defined on the product space $P \times P$.

The assumption (2) guaranteeing the existence of two solutions for the quadratic equation (1) has to be adequately rephrased for (3) as

$$4|b|\,|u_0| < 1 \tag{4}$$

where $|b|$ denotes the norm of the bilinear form. However, we shall postpone the proof of this result to the next section.

Let us, however, notice that the Banach Fixed Point Theorem for local contraction was used extensively to prove the existence of at least one fixed point (e.g. for the Navier–Stokes equations in [6] or for a Boltzmann equation in [8]) for the

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bilinear model equation like (3). This fixed point was located in the neighbourhood of zero thus making the contraction approach feasible.

To prove the existence of two solutions we shall use the Krasnoselskii Fixed Point Theorem, cf. [10], which allows us to obtain more solutions if the nonlinear operator has the required property of "crossing" identity twice, i.e. by the cone compression and the expansion on some appropriate subsets of the cone.

A quadratic nonlinearity being the simplest possible example in the nonlinear world poses many questions about the global existence and uniqueness. There are numerous models which possess such a structure, like the Navier–Stokes equation (cf. for the Fujita–Kato–Meyer approach [4,6,16,17]), the Boltzmann equation (cf. [8]), the quadratic reaction diffusion equation (cf. [12]), the Smoluchowski coagulation equation (cf. [19]) or the system modelling chemotaxis and to name but a few. The problem of uniqueness of solutions for these equations attracted a lot of attention and only some partial results are known. In some cases nonuniqueness occurs and the existence of two solutions can be proved. Sometimes one of the solution is a trivial one and then the proof relies on finding a nontrivial one. In these difficult but important models one encounters another problem making our approach not feasible i.e. very common lack of compactness, thus if we would like to make our approach feasible we are forced to consider some truncated baby model compatible with compact (i.e. mapping bounded sets of $P \times P$ into relatively compact ones) bilinear forms.

It should be noted that the problem of existence of multiple solutions of nonlinear equations was addressed by H. Amann in [2] in ordered Banach spaces rather than analysed from topological point of view as in our approach.

2. The main result

To prove the result announced in the previous section we shall use the following theorem [10, Theorem 2.3.4] originating from the works of Krasnoselskii, cf. [13].

Theorem 2.1. *Let E be a Banach space, and let $P \subset E$ be a cone in E. Let Ω_1 and Ω_2 be two bounded, open sets in E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let compact operator $T : P \to P$ satisfy conditions*

$$|Tu| \le |u| \quad \text{for any } u \in P \cap \partial\Omega_1 \quad \text{and} \quad |Tu| \ge |u| \quad \text{for any } u \in P \cap \partial\Omega_2$$

or, alternatively, the following two conditions

$$|Tu| \ge |u| \quad \text{for any } u \in P \cap \partial\Omega_1 \quad \text{and} \quad |Tu| \le |u| \quad \text{for any } u \in P \cap \partial\Omega_2$$

are satisfied. Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2.2. *Assume that, for the given cone $P \subset E$, the bilinear and compact (i.e. mapping bounded subsets of $P \times P$ into relatively compact ones) form $b : P \times P \to P$ satisfies the following coercivity condition*

$$\inf_{|u|=1, u \in P} |b(u, u)| > 0. \tag{5}$$

Then for any $u_0 \in P$ as small as to satisfy (4) the Eq. (3) admits at least two solutions in P.

Proof. Let us define the operator

$$Tu = b(u, u) + u_0 \tag{6}$$

then we shall apply Krasnoselskii Theorem once as a cone-compression in the neighbourhood of zero and secondly as a cone-expansion at infinity. To apply this theorem the compactness of the form b is required, i.e. the assumption that b maps bounded subsets of P into the relatively compact subsets of P. It should be underlined that the coercivity assumption imposed on b relies on the proper definition of the cone P and only on this subset of the Banach space E it holds.

Notice that we have the following estimates

$$\begin{aligned} |Tu| &\le |u_0| + B|u|^2, \\ |Tu| &\ge |u_0| - B|u|^2, \end{aligned} \tag{7}$$

where constant $B = |b| > 0$ denotes the norm of the bilinear form b, i.e. the smallest constant B satisfying, for any $u, v \in P$, the inequality

$$|b(u, v)| \le B|u|\,|v|.$$

If $u_0 = 0$ then $u = 0$ is a solution. Otherwise, if $u_0 \ne 0$ then for sufficiently small $\rho_1 > 0$, i.e. such that $B\rho_1^2 + \rho_1 < |u_0|$ and any $u \in P$ and $|u| = \rho_1$ one has

$$|Tu| \ge |u_0| - B\rho_1^2 > \rho_1 = |u|. \tag{8}$$

Moreover, if we assume that there exists ρ_2 such that

$$|u_0| + B\rho_2^2 < \rho_2 \tag{9}$$

then apparently for any $u \in P$ and $|u| = \rho_2$ one has

$$|Tu| \le |u_0| + B|u|^2 < \rho_2 = |u|. \tag{10}$$

But this can be accomplished if we assume $B\rho_1^2 + \rho_1 < |u_0| < \rho_2 - B\rho_2^2$.

Finally, for sufficiently large values of $\rho_3 > 0$ and any $u \in P$ and $|u| = \rho_3$, due to the coercivity assumption (5), one has

$$|b(u, u)| \ge C|u|^2 \tag{11}$$

implying

$$|Tu| \ge C\rho_3^2 - |u_0| > \rho_3 = |u|. \tag{12}$$

To be more specific ρ_3 has to be so large that $|u_0| < C\rho_3^2 - \rho_3$.

Combining (8) with (10) we get that the intersection of the cone P with the spheres of the radii ρ_1 and ρ_2 (in the $|\cdot|$ norm) is compressed while the one at the radii ρ_2 and ρ_3 is expanded yielding the desired two fixed points in each set. Note that it might be necessary to distinguish between ρ_2 used in both sets as to prevent both fixed points to coincide.

The final observation is that the only assumption which should be made here is to guarantee (10) to hold but this follows readily from (4). □

3. Examples of applications to differential equations

Example 1. Consider the following BVP for ODE for continuous, positive function f

$$-u''(t) = (u(t))^2 + f(t), \tag{13}$$
$$u(0) = u(1) = 0. \tag{14}$$

This problem can be formulated in a required form

$$u = b(u, u) + u_0 \tag{15}$$

where the function

$$u_0(t) = \int_0^1 G(t, s)f(s)\, ds \tag{16}$$

with the symmetric, Green function given by $G(t, s) = t(1-s),\ 0 \le t \le s \le 1$, while the bilinear form b is defined, for any $u, v \in P$, by

$$b(u, v) = \int_0^1 G(t, s)u(s)v(s)\, ds,$$

and the cone P on which coercivity of b holds can be defined as

$$P = \left\{ u \ge 0 : \inf_{t\in[\alpha,\beta]} u(t) \ge \min\{\alpha, 1-\beta\}|u|_\infty \right\} \subset C[0, 1],$$

where α, β are arbitrary, fixed points satisfying $0 < \alpha < \beta < 1$. The compactness of the operator $b(\cdot, \cdot)$ is guaranteed due to the compact embedding of $C^2([0, 1])$ into $C([0, 1])$ implied by the Green function properties. For more applications to this kind of problems see [15,18], while for singular problems one can see [1].

Example 2. Consider the following BVP for PDE, where $\Omega = \{x \in \mathbb{R}^n : \rho < |x| < R\}$ is an annulus in $\mathbb{R}^n,\ n \ge 3$,

$$-\Delta v(x) = (v(x))^2 + g(x), \quad x \in \Omega, \tag{17}$$
$$v|_{\partial\Omega} = 0. \tag{18}$$

Assuming radial symmetry of the g function, i.e. $g(x) = f(|x|) = f(r)$ for some function f, and looking for radial solutions $v(x) = u(|x|)$ we obtain the radial counterpart for the aforementioned boundary value problem reading as

$$-u''(r) - \frac{n-1}{r}u'(r) = (u(r))^2 + f(r), \tag{19}$$
$$u(\rho) = u(R) = 0. \tag{20}$$

Then the above radial problem can be formulated as

$$u = b(u, u) + u_0 \tag{21}$$

where

$$u_0(r) = \int_\rho^R G(r, s) f(s)\, ds \tag{22}$$

for the appropriate Green function G for the Dirichlet problem of the form

$$G(r, s) = \frac{s(R^{n-2} - \max(r, s)^{n-2})(\min(r, s)^{n-2} - \rho^{n-2})}{(n-2) r^{n-2} (R^{n-2} - \rho^{n-2})}.$$

The bilinear form b is defined by

$$b(u, v) = \int_\rho^R G(r, s) u(s) v(s)\, ds$$

and the cone P on which coercivity of b holds, can be expressed as

$$P = \left\{ u \geq 0 : \inf_{r \in [\rho_1, R_1]} u(r) \geq \gamma |u|_\infty \right\},$$

for $\rho < \rho_1 < R_1 < R$ and some positive number γ. This parameter can be evaluated in this case as $\gamma = \left(\frac{\rho}{R_1}\right)^{n-2}$. Once again the compactness is ascertained due to the properties of the Green function. For more applications to this kind of problems one can see [20,22] for problems in annular domains but with another nonlinearities. Also see [11,15] or [21] for problems in exterior domains or [9,20] for the nonlocal problems. It should be underlined that due to the singularity of the Green function for the original problem in x and y variables as x approaches y, neither for the annulus, nor for the ball the required estimate (23) does not hold as $\sup_{x \in U} G(x, y) = \infty$ for $y \in U$. Therefore the symmetry of the problem and also the nonsingular form of the Green function in r and s variables listed above plays the essential role in defining the cone guaranteeing the coercivity of the bilinear form b on $P \times P$. This is not the case if we replace the annulus with the ball, even if the radial symmetry is guaranteed, since the radial Green function still is singular at zero.

Example 3 (*General Case*). Since the crucial assumptions in both examples which guarantee that the cone P is invariant is the following property of the Green function (or in general the kernel of Hammerstein operator involving the bilinear form $b(u, v)$) G

$$\inf_{x \in U} G(x, y) \geq \gamma \sup_{x \in V} G(x, y) \tag{23}$$

where $\gamma = \gamma(U, V) > 0$ depends only on a set V and its subset U such that $\overline{U} \subset V$. Note that this condition holds either if V is interval or annulus but we were not able to prove it for arbitrary domain, e.g. for a ball in higher dimension. If (23) holds without any problems one can prove that a properly defined cone is invariant under the action of the bilinear form, thus making it coercive. It should be noted that the condition ((23)) implies boundedness of the Green function with respect to x and therefore it is not satisfied for the Green function for the ball in dimensions higher than one, but only for one dimensional case or radial case for exterior or annular domains listed in the examples above. The compactness of the $b(\cdot, \cdot)$ (i.e. mapping bounded subsets of $P \times P$ into relatively compact ones) in the proper function space, e.g. $BC(U)$ the space of bounded and continuous function requires the smoothness assumption on the Green function G and moreover its one-dimensional form, implied, e.g., by the radial symmetry of the domain U and its proper form—exterior or annular, guaranteeing (23).

The problem with (23) can be illustrated in the second example, since if we replace an annulus or exterior domain with a ball the Green function is not bounded from above and also in the open problem section for the heat semigroup where the norms cannot be compared. This is the main obstacle in extending this kind of results to physically interesting models like the Navier–Stokes equation [4,6,17,16], the Boltzmann equation [8] or the Smoluchowski coagulation models [19] sharing the property of quadratic nonlinearities.

In the functional analysis approach to the Navier–Stokes equation with the unknown velocity $u = u(t, x)$ the bilinear form equation can be defined with b given by

$$b(u, v) = -\int_0^{\cdot} \exp((t-s)\Delta) \mathbb{P} \nabla \cdot (v \times u)(s) ds$$

where $\mathbb{P}$ stands for some appropriate orthogonal projection operator onto the divergence free vector field while

$$u_0 = S(\cdot) g + \int_0^{\cdot} \exp((t-s)\Delta) \mathbb{P} \nabla \cdot V(s) ds$$

with initial value for u given by g and external force V.

As for the Boltzmann equation it should be underlined that the techniques used here have not much connection with the celebrated DiPerna–Lions' renormalisation technique applied to the Boltzmann equation but the only link to the Boltzmann equation is the quadratic nonlinearity that appears in both contexts. One can see the paper [5] and the references therein to see the functional framework used for the Boltzmann equation while looking for its self-similar solutions, which formally makes the definition of the bilinear form feasible. For more information on the Boltzmann equation one can see for example the pioneering paper of Cercignani [7] and the review paper of Villani [23].

Open problem. Consider for $u = u(x, t)$ the boundary value problem

$$u_t = \Delta u + u^2, \qquad u(x, 0) = f(x). \tag{24}$$

Then one can formulate this problem as

$$u = b(u, u) + u_0 \tag{25}$$

where, for the heat semigroup $S(t)$,

$$u_0 = S(t)f, \tag{26}$$

while the bilinear form b is defined by

$$b(u, v) = \int_0^t S(t-s)u(s)v(s)\, ds. \tag{27}$$

Note that for any t, s and the heat semigroup $S(t)$ one has in L^p norm $|\cdot|_p,\ p \geq 1$

$$C|S(t-s)u(s)^2|_2 \geq |S(t-s)u(s)^2|_1 \geq |u(s)^2|_1 \geq |u(s)|_2^2, \tag{28}$$

hence after integration on $(0, t)$ and taking sup norm with respect to $t \in [0, T]$, the right hand side can be estimated by

$$\int_0^T |u(s)|_2^2\, ds, \tag{29}$$

while taking squared integral with respect to $t \in [0, T]$ of the integrated on $(0, t)$ right hand side can be estimated from below by

$$T^{-1} \int_0^T \int_0^t |u(s)|_2^2\, dsdt. \tag{30}$$

Moreover

$$|S(t)f|_2 \leq C|f|_2, \tag{31}$$

hence the second condition for sup norm in t follows. However, one cannot guarantee the condition for coercivity neither in integrable, nor in supremum setting. Note that due to [12] there are some results guaranteeing nonuniqueness results. Our approach cannot yield this second solution, although some nonuniqueness results are known for semilinear parabolic equations [3,24].

The extension of these results to encompass also equations with fractional Laplacian of sufficiently high order in one dimension can also be treated, but as more subtle estimates for the Green function are required this lies beyond the scope of this paper and it will be treated in the separate paper [14].

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