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Algebraic reflexivity of isometry groups and automorphism groups of some operator structures

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ABSTRACT

We establish the algebraic reflexivity of three isometry groups of operator structures: the group of all surjective isometries on the unitary group, the group of all surjective isometries on the set of all positive invertible operators equipped with the Thompson metric, and the group of all surjective isometries on the general linear group of $B(\mathcal{H})$, the operator algebra over a complex infinite dimensional separable Hilbert space $\mathcal{H}$. We show that those isometry groups coincide with certain groups of automorphisms of corresponding structures and hence we also obtain the reflexivity of some automorphism groups.

1. Introduction and statement of the main result

In the past few decades considerable work has been done related to local maps on operator algebras. The main problem in this area of research is to answer the question whether the local actions of important classes of transformations (e.g., derivations, or automorphisms, or isometries, etc.) on a given operator algebra completely determines the class under consideration.

The originators of investigations of this kind are Kadison, Larson and Sourour [16–18]. Motivated by certain problems concerning the Hochschild cohomology of operator algebras, in [16] Kadison studied local derivations on a von Neumann algebra $\mathcal{R}$. A linear map $\delta : \mathcal{R} \to \mathcal{R}$ is called a local derivation if at each point in the algebra δ coincides with a derivation (that may vary from point to point). More precisely, the assumption is that for every $a \in \mathcal{R}$ there exists a derivation $\delta_a : \mathcal{R} \to \mathcal{R}$ such that $\delta(a) = \delta_a(a)$. It was proved in [16] that in the above setting, every continuous local derivation is a (global) derivation (in fact, the result in [16] was deduced in a more general context). Larson and Sourour proved in [18] that similar conclusion holds for local derivations of the full operator algebra $B(\mathcal{X})$ on a Banach space $\mathcal{X}$ (even without assuming continuity).

Beside derivations, there are other important classes of transformations on operator algebras which also deserve attention from the point of view described above. We mention automorphism groups and isometry groups. The former groups reflect the algebraic properties of the underlying algebras while the latter ones reflect their geometrical features. In [17, Some concluding Remark 5, p. 298] Larson initiated the study of local automorphisms of Banach algebras. The definition

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is straightforward: a local automorphism is a linear map ϕ on a given Banach algebra $\mathcal{A}$ with the property that for every $x \in \mathcal{A}$ there exists an (algebra) automorphism ϕ_x of $\mathcal{A}$ such that $\phi(x) = \phi_x(x)$. In the paper [18], Larson and Sourour proved that if $\mathcal{X}$ is an infinite dimensional Banach space, then every surjective local automorphism of the algebra $\mathrm{B}(\mathcal{X})$ is an automorphism. In [6], Brešar and Šemrl showed that in the case of an infinite dimensional separable Hilbert space $\mathcal{H}$ the assumption of surjectivity can be relaxed, i.e. every local automorphism of $\mathrm{B}(\mathcal{H})$ is an automorphism.

The above mentioned results concern local maps on linear structures and can be put into a general common frame as follows. Given a linear algebraic structure $\mathcal{V}$ (e.g., an operator algebra) and a collection $\mathcal{E}$ of its linear transformations, the local maps ϕ above are transformations on $\mathcal{V}$ which are linear and belong locally to $\mathcal{E}$. This means that for every $x \in \mathcal{V}$ there exists a transformation $\phi_x \in \mathcal{E}$ such that $\phi(x) = \phi_x(x)$. The presented results show that in some important cases, for several particular classes of transformations the corresponding local maps are all 'global', i.e. they in fact belong to the given class of transformations. A number of related papers on this area of research are listed on page 23 in [22].

It has been a natural question how those investigations could be extended to non-linear structures. Probably the most useful idea is due to Šemrl [25] and it is connected to the concepts of 2-locality and 2-local maps. Let $\mathcal{A}$ be any mathematical structure and $\mathcal{E}$ a given class of transformations on $\mathcal{A}$. We say that a map $\phi : \mathcal{A} \to \mathcal{A}$ belongs 2-locally to $\mathcal{E}$ if for any pair $x, y \in \mathcal{A}$ there is an element $\phi_{(x,y)}$ of $\mathcal{E}$ for which $\phi(x) = \phi_{(x,y)}(x)$ and $\phi(y) = \phi_{(x,y)}(y)$. Adopting the notion of algebraic reflexivity for the present setting that has previously been used in the literature in relation with linear (1-)local maps, we call the class $\mathcal{E}$ algebraically reflexive if for every map ϕ that belongs 2-locally to $\mathcal{E}$ we necessarily have $\phi \in \mathcal{E}$. Observe that here we do not assume any sort of linearity.

If, for example, $\mathcal{E}$ is the group of certain automorphisms of $\mathcal{A}$, the maps which belong 2-locally to $\mathcal{E}$ are naturally termed as 2-local automorphisms. In a similar way one can speak of 2-local isometries, 2-local derivations, etc. The main result in [25] says that if $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then every 2-local automorphism of $\mathrm{B}(\mathcal{H})$ (more precisely, every map which belongs 2-locally to the group of all algebra automorphisms of $\mathrm{B}(\mathcal{H})$) is an (algebra) automorphism. This nice and surprising observation has motivated further investigations. One can find corresponding results and references in the book [22, Sections 3.4, 3.5 and see also p. 24 in the Introduction]. For more recent results we refer to the papers [1,4,9,14]. In fact, although one of the main advantages of the concept of 2-locality is that the reflexivity of classes of transformations can be investigated in non-linear settings, in all the latter four papers the authors considered 2-local isometries, 2-local automorphisms, etc. on linear structures. Unlike those investigations, in the present paper we study reflexivity problems on structures of linear operators which are definitely non-linear. These structures are groups or certain substructures of groups of operators acting on a Hilbert space. Our main aim is to explore the reflexivity of isometry and automorphism groups of those structures.

Given a metric space $\mathbb{G}$, by an isometry of $\mathbb{G}$ we mean a distance preserving self-map of $\mathbb{G}$. The set of all surjective isometries of $\mathbb{G}$ forms a group, it is called the isometry group of $\mathbb{G}$ and is denoted by $\mathfrak{I}(\mathbb{G})$. A 2-local isometry on $\mathbb{G}$ is a mapping $\phi : \mathbb{G} \to \mathbb{G}$ such that for every pair $a, b \in \mathbb{G}$ there exists a surjective isometry $\Phi_{(a,b)} : \mathbb{G} \to \mathbb{G}$ such that

$$\phi(a) = \Phi_{(a,b)}(a) \quad \text{and} \quad \phi(b) = \Phi_{(a,b)}(b).$$

The algebraic reflexivity question for the isometry group $\mathfrak{I}(\mathbb{G})$ that we are considering in this paper asks whether every 2-local isometry on $\mathbb{G}$ is a surjective isometry, i.e. if every map that belongs 2-locally to $\mathfrak{I}(\mathbb{G})$ necessarily belongs (globally) to $\mathfrak{I}(\mathbb{G})$.

Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space and $\mathrm{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. Below we establish the algebraic reflexivity of the isometry groups of three important substructures of $\mathrm{B}(\mathcal{H})$:

(a) The unitary group $\mathrm{U}(\mathcal{H})$ of $\mathrm{B}(\mathcal{H})$ with the metric induced by the operator norm;
(b) The set $\mathrm{B}(\mathcal{H})_+^{-1}$ of all positive invertible operators on $\mathcal{H}$ with the Thompson metric; and
(c) The general linear group $\mathrm{GL}(\mathcal{H})$ of $\mathrm{B}(\mathcal{H})$ with the metric induced by the operator norm.

(Observe that by the polar decomposition theorem, the last group is in a sense generated by the previous two structures.) The first and the third metric groups are widely studied in detail, they need no further explanation. As for the second one, we note that the importance of the Thompson metric on $\mathrm{B}(\mathcal{H})_+^{-1}$ arises from the fact that it coincides with the geodesic distance induced by the natural Finsler geometrical structure on $\mathrm{B}(\mathcal{H})_+^{-1}$. In this metric the distance between the elements $A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$ is equal to $\left\| \log \sqrt{A}^{-1} B \sqrt{A}^{-1} \right\|$. The Thompson metric has a wide range of applications in different areas of mathematics (for more information we refer the reader to [23] and the references therein).

Our main theorem can be formulated in short as follows.

Theorem 1.1. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space and let $\mathbb{G}$ be either $\mathrm{U}(\mathcal{H})$, $\mathrm{B}(\mathcal{H})_+^{-1}$, or $\mathrm{GL}(\mathcal{H})$. The isometry group of $\mathbb{G}$ is algebraically reflexive.*

A fundamental requirement for addressing the algebraic reflexivity of those isometry groups is the following characterization of the surjective isometries supported on each setting.

Theorem 1.2 (*See [11, Theorem 8], [23, Theorem 1] and [12, Corollary 2.3]*). *Let $\mathcal{H}$ be a complex Hilbert space.*

(a) *A map $\Phi : \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$ is a surjective isometry if and only if there exist V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that*

$$(\star)\ \Phi(A) = VAW, \quad \forall A \in \mathrm{U}(\mathcal{H}) \quad \textit{or} \quad \Phi(A) = VA^*W, \quad \forall A \in \mathrm{U}(\mathcal{H}).$$

(b) *A map $\Phi : \mathrm{B}(\mathcal{H})_+^{-1} \to \mathrm{B}(\mathcal{H})_+^{-1}$ is a surjective (Thompson) isometry if and only if there exists T a linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ such that*

$$(\star\star)\ \Phi(A) = TAT^*, \quad \forall A \in \mathrm{B}(\mathcal{H})_+^{-1} \quad \textit{or} \quad \Phi(A) = TA^{-1}T^*, \quad \forall A \in \mathrm{B}(\mathcal{H})_+^{-1}.$$

(c) *A map $\Phi : \mathrm{GL}(\mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ is a surjective isometry if and only if there exist V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that*

$$(\star\star\star)\ \Phi(A) = VAW, \quad \forall A \in \mathrm{GL}(\mathcal{H}) \quad \textit{or} \quad \Phi(A) = VA^*W, \quad \forall A \in \mathrm{GL}(\mathcal{H}).$$

It is obviously true that any 2-local isometry of a metric space is an isometry, so one may think that in order to show the algebraic reflexivity of the isometry group we only need to check the surjectivity of all 2-local isometries. However, a short consideration can justify that this is not the proper way to prove such results. First, we remark that the isometry group need not to be reflexive even in such particular cases as represented by general C^*-algebras. In [10] Győry showed that for a non-countable discrete topological space X, the commutative C^*-algebra $C_0(X)$ of all continuous complex functions on X that vanish at infinity has an even linear 2-local isometry which is not a surjective isometry. As we shall see below, the real content of the reflexivity result in Theorem 1.1 is that if a map belongs 2-locally to any of the above described collections $(\star)$, or $(\star\star)$, or $(\star\star\star)$ of transformations, then it necessarily belongs globally to that collection. Let us go a bit further in explanation. We show that the above collections of transformations (hence also the corresponding isometry groups) are in fact groups of automorphisms of certain topological algebraic structures. To see this, first observe that the assignment $(A, B) \mapsto AB^{-1}A$ defines an operation both on $\mathrm{U}(\mathcal{H})$ and on $\mathrm{B}(\mathcal{H})_+^{-1}$ and the correspondence $(A, B) \mapsto AB^*A$ makes $\mathrm{GL}(\mathcal{H})$ an algebraic structure. The next result shows that the groups of all continuous, resp. uniformly continuous automorphisms, coincide with the considered isometry groups of $\mathrm{U}(\mathcal{H})$, $\mathrm{B}(\mathcal{H})_+^{-1}$, resp. $\mathrm{GL}(\mathcal{H})$. Observe that it is not true in general that 2-local morphisms of a (binary) algebraic structure are all morphisms.

Theorem 1.3. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space.*

Let $\Phi : \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$ be a transformation. Then Φ is a continuous bijective map satisfying

$$\text{(i) } \Phi(AB^{-1}A) = \Phi(A)\Phi(B)^{-1}\Phi(A), \quad \forall A, B \in \mathrm{U}(\mathcal{H})$$

if and only if there exist V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that

$$\Phi(A) = VAW, \quad \forall A \in \mathrm{U}(\mathcal{H}) \quad \textit{or} \quad \Phi(A) = VA^{-1}W, \quad \forall A \in \mathrm{U}(\mathcal{H}).$$

The transformation $\Psi : \mathrm{B}(\mathcal{H})_+^{-1} \to \mathrm{B}(\mathcal{H})_+^{-1}$ is a continuous (in the operator norm) bijection for which

$$\text{(ii) } \Psi(AB^{-1}A) = \Psi(A)\Psi(B)^{-1}\Psi(A), \quad \forall A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$$

if and only if there exists T a linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ such that

$$\Psi(A) = TAT^*, \quad \forall A \in \mathrm{B}(\mathcal{H})_+^{-1} \quad \textit{or} \quad \Psi(A) = TA^{-1}T^*, \quad \forall A \in \mathrm{B}(\mathcal{H})_+^{-1}.$$

The map $\Phi : \mathrm{GL}(\mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ is a uniformly continuous bijection satisfying

$$\text{(iii) } \Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A), \quad \forall A, B \in \mathrm{GL}(\mathcal{H})$$

if and only if there exist V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that

$$\Phi(A) = VAW, \quad \forall A \in \mathrm{GL}(\mathcal{H}) \quad \textit{or} \quad \Phi(A) = VA^*W, \quad \forall A \in \mathrm{GL}(\mathcal{H}).$$

It is now obvious that our reflexivity results in Theorem 1.1 originally stated for isometry groups can also be viewed as statements on the algebraic reflexivity of the above described isomorphism groups.

The following sections are devoted to the proofs of our results. Before going into the details we present the notation, some concepts and basic observations we shall use in the rest of the paper. An anti-unitary operator on a Hilbert space $\mathcal{H}$ is a surjective conjugate linear isometry. By a projection on $\mathcal{H}$ we mean a self-adjoint idempotent operator $P \in \mathrm{B}(\mathcal{H})$. We denote by $\mathrm{P}(\mathcal{H})$ the set of all projections on $\mathcal{H}$ and by $\mathrm{P}_1(\mathcal{H})$ the set of all rank-1 elements of $\mathrm{P}(\mathcal{H})$. For any pair $x, y \in \mathcal{H}$ of vectors the symbol $x \otimes y$ stands for the rank at most one operator defined by $(x \otimes y)z = \langle z, y\rangle x$, $z \in \mathcal{H}$. The self-adjoint elements of $\mathrm{U}(\mathcal{H})$ are called symmetries. We denote this collection by $\mathrm{S}(\mathcal{H})$, i.e. $\mathrm{S}(\mathcal{H}) = \{T \in \mathrm{U}(\mathcal{H}) : T^* = T\}$. It is clear that a unitary operator U is a symmetry if and only if it can be written as $U = I - 2P$ with some projection $P \in \mathrm{P}(\mathcal{H})$. The spectrum of any operator $A \in \mathrm{B}(\mathcal{H})$ is denoted by $\sigma(A)$ and tr stands for the usual trace functional.

Observe that the composition of a 2-local isometry with a surjective isometry is also a 2-local isometry. Therefore, without loss of generality, one may assume that a 2-local isometry ϕ fixes any two given elements of the underlying space.

2. The algebraic reflexivity of the isometry group of $\mathrm{U}(\mathcal{H})$

In this section we prove the following statement.

Theorem 2.1. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space. The isometry group of $\mathrm{U}(\mathcal{H})$ is algebraically reflexive.*

According to the basic observation given above, a 2-local isometry of $\mathrm{U}(\mathcal{H})$ can be assumed to fix any two given elements of $\mathrm{U}(\mathcal{H})$. Throughout this section, *ϕ represents a 2-local isometry on $\mathrm{U}(\mathcal{H})$ such that $\phi(I) = I$.*

We first claim that ϕ preservers symmetries.

Lemma 2.2. *We have $\phi(\mathrm{S}(\mathcal{H})) \subseteq \mathrm{S}(\mathcal{H})$.*

Proof. Let T be a symmetry. The 2-local condition on ϕ applied to the pair (I, T) implies the existence of a surjective isometry $\Phi_{(I,T)}$ such that $\Phi_{(I,T)}(I) = \phi(I) = I$ and $\Phi_{(I,T)}(T) = \phi(T)$. An application of Theorem 1.2-(a) assures the existence of a unitary or an anti-unitary operator V such that $\phi(T) = VTV^*$. Therefore $\phi(T) \in \mathrm{S}(\mathcal{H})$. $\square$

It follows from the Lemma 2.2 that ϕ induces a natural map $\Psi : \mathrm{P}(\mathcal{H}) \to \mathrm{P}(\mathcal{H})$ given by

$$\Psi(P) = \frac{I - \phi(I - 2P)}{2}, \quad \forall P \in \mathrm{P}(\mathcal{H}).$$

The next lemma states some properties of Ψ that will be employed in forthcoming steps. In particular, it asserts that the restriction ψ of Ψ onto $\mathrm{P}_1(\mathcal{H})$ is a Wigner transformation meaning that $\psi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$ and $\mathrm{tr}(PQ) = \mathrm{tr}(\psi(P)\psi(Q))$ for any $P, Q \in \mathrm{P}_1(\mathcal{H})$.

By a variant of Wigner's famous theorem on quantum mechanical symmetry transformations (see, e.g., [22, Theorem 2.1.4]) we have the following result.

Theorem 2.3. *Every Wigner transformation $\varphi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$ is of the form*

$$\varphi(P) = UPU^*, \quad \forall P \in \mathrm{P}_1(\mathcal{H})$$

with some linear or conjugate linear isometry U on $\mathcal{H}$.

We now have the following

Lemma 2.4. *If $P \in \mathrm{P}(\mathcal{H})$ is a projection of finite rank, then $\Psi(P)$ is a projection of equal rank. If P and Q are projections of rank one, then $tr(PQ) = tr(\psi(P)\psi(Q))$. Therefore, ψ is a Wigner transformation and hence there exists a linear or conjugate linear isometry U on $\mathcal{H}$ such that*

$$\psi(P) = UPU^*, \quad \forall P \in \mathrm{P}_1(\mathcal{H}).$$

Proof. If P is a projection then the 2-local condition on ϕ applied to the pair $(I, I - 2P)$ implies the existence of a unitary or an anti-unitary operator V such that $\phi(I - 2P) = V(I - 2P)V^*$. Therefore $\Psi(P) = VPV^*$ which has the same rank as P. It clearly follows that $\psi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$.

Since ϕ is a 2-local isometry, it is an isometry. This easily implies that both Ψ and ψ are isometries (with respect to the operator norm). We refer the reader to an argument in [22, p. 127] that shows that $\|P - Q\| = \sqrt{1 - tr(PQ)}$, where P and Q are arbitrary projections of rank one. This implies that ψ preserves the quantity $tr(PQ)$ between projections of rank one, i.e. we have $\mathrm{tr}(PQ) = \mathrm{tr}(\psi(P)\psi(Q))$. Hence ψ is a Wigner transformation and the last statement follows from the above mentioned variant of Wigner's theorem. $\square$

We shall make use of the following simple auxiliary result concerning rank-1 projections.

Lemma 2.5. *Let P_0, P_1 and P_2 be projections on $\mathcal{H}$ of rank one, and let α, β, γ, and η be complex numbers such that $\alpha \cdot \beta \neq 0$. If*

$$\alpha P_0 + \beta P_1 + \gamma P_0 P_1 = \eta P_2, \tag{1}$$

then $P_0 = P_1$.

Proof. There exist unit vectors u, v and w in $\mathcal{H}$ such that $P_0 = u \otimes u$, $P_1 = v \otimes v$ and $P_2 = w \otimes w$. We observe that if u and v are linearly dependent then $P_0 = P_1$. If u and v are linearly independent we choose z, a vector orthogonal to v and not orthogonal to u. Then the equation displayed in (1), applied to z, yields

$$\alpha\langle z, u\rangle u = \eta\langle z, w\rangle w.$$

Since $\alpha \neq 0$ we conclude that u is in the range of P_2 and then $P_0 = P_2$. Therefore (1) reduces to

$$\beta P_1 + \gamma P_0 P_1 = (\eta - \alpha) P_0.$$

If $\alpha \neq \eta$, every nonzero vector z orthogonal to the range of P_1 is also orthogonal to the range of P_0. This implies that $P_0 = P_1$, since P_0 and P_1 are projections of rank one. If $\alpha = \eta$ then $\beta P_1 + \gamma P_0 P_1 = 0$ and $\beta \cdot \gamma \neq 0$. Hence every vector in the range of P_1 is also in the range of P_0 implying $P_0 = P_1$ again. This completes the proof. □

For an easier exposition we shall need the following notation. For any operator A we write $A^{(*)}$ to denote either A or A^*. Similarly, for an invertible A we write $A^{(-1)}$ to denote either A or A^{-1}. We use similar notation for complex numbers, $\lambda^{(*)}$ represents either λ or $\overline{\lambda}$, likewise, if $\lambda \neq 0$, then $\lambda^{(-1)}$ represents either λ or $\frac{1}{\lambda}$.

The next lemma establishes the form of the 2-local isometry ϕ on the most simple elements of $\mathrm{U}(\mathcal{H})$.

Lemma 2.6. *If P is a projection of rank one and λ is a modulus one complex number, then*

$$\phi(P^\perp + \lambda P) = I + (\lambda - 1)^{(*)} \psi(P).$$

Proof. The statement is clearly true for $\lambda = 1$ since $\phi(I) = I$. Assume $\lambda \neq 1$. The 2-local condition on ϕ applied to the pairs $(P^\perp + \lambda P, P^\perp - P)$ and $(P^\perp + \lambda P, I)$ of unitary operators implies the existence of:

V_0 and W_0 both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that

$$\phi(P^\perp + \lambda P) = V_0 (P^\perp + \lambda P)^{(*)} W_0 \quad \text{and} \quad \phi(P^\perp - P) = V_0 (P^\perp - P) W_0; \tag{2}$$

V_1 unitary or anti-unitary operator on $\mathcal{H}$ such that

$$\phi(P^\perp + \lambda P) = V_1 (P^\perp + \lambda P)^{(*)} V_1^*. \tag{3}$$

Thus, from the equations displayed in (2) we get

$$\begin{aligned} \phi(P^\perp + \lambda P)\phi(P^\perp - P)^* &= V_0 (P^\perp + \lambda^{(*)} P)(P^\perp - P) V_0^* \\ &= V_0 (P^\perp - \lambda^{(*)} P) V_0^* \\ &= V_0 (I - (\lambda^{(*)} + 1) P) V_0^* \\ &= I - (\lambda^{(*)} + 1) P_2, \end{aligned}$$

with $P_2 = V_0 P V_0^*$. On the other hand, since $P^\perp - P$ is a symmetry, from (3) we get

$$\begin{aligned} \phi(P^\perp + \lambda P)\phi(P^\perp - P)^* &= \left[V_1 (P^\perp + \lambda P)^{(*)} V_1^* \right] [I - 2\psi(P)] \\ &= \left[I + (\lambda - 1)^{(*)} V_1 P V_1^* \right] [I - 2\psi(P)] \\ &= I - 2\psi(P) + (\lambda - 1)^{(*)} V_1 P V_1^* - 2(\lambda - 1)^{(*)} V_1 P V_1^* \psi(P) \\ &= I - 2P_1 + (\lambda - 1)^{(*)} P_0 - 2(\lambda - 1)^{(*)} P_0 P_1, \end{aligned}$$

with $P_1 = \psi(P)$ and $P_0 = V_1 P V_1^*$. Therefore

$$2P_1 - (\lambda - 1)^{(*)} P_0 + 2(\lambda - 1)^{(*)} P_0 P_1 = (\lambda^{(*)} + 1) P_2.$$

Lemma 2.5 implies that $P_0 = P_1$ and we have

$$\phi(P^\perp + \lambda P) = V_1 (P^\perp + \lambda P)^{(*)} V_1^* = I + (\lambda - 1)^{(*)} V_1 P V_1^* = I + (\lambda - 1)^{(*)} \psi(P).$$

This completes the proof. □

The following lemma will be employed in the proofs of forthcoming statements.

Lemma 2.7. *Let X be an invertible operator in $\mathrm{B}(\mathcal{H})$ and let u, v be vectors in $\mathcal{H}$. Then $Y = X + u \otimes v$ is invertible if and only if $\langle X^{-1} u, v \rangle + 1 \neq 0$.*

Proof. In the first version of the manuscript the assertion was borrowed from [13, see Lemma 2.7 there]. The referee has kindly pointed out that the above statement can be deduced very easily as follows. The operator Y is invertible if and only if $I + X^{-1} u \otimes v$ is invertible. This is equivalent to $-1 \notin \sigma(X^{-1} u \otimes v)$ which is the case if and only if $-1 \neq \langle X^{-1} u, v \rangle$. □

In the sequel we shall need a particular class of unitary operators with diagonal structure relative to a family of pairwise orthogonal projections of rank one with sum equal to I. The construction of this class is as follows.

We first observe that the spectral theorem implies that the set $\mathcal{U}_f$ of all finite spectrum operators in $\mathrm{U}(\mathcal{H})$ is uniformly dense. Let us now consider the set of all elements W of $\mathcal{U}_f$ with the property that $\lambda \in \sigma(W)$ implies $\overline{\lambda} \notin \sigma(W)$ (this set is obviously dense in $\mathcal{U}_f$). For any such W we can write $\sigma(W) = \{c_1, c_2, \ldots, c_k\}$ where $0 < |1 - c_1| < |1 - c_2| < \cdots < |1 - c_k|$. Clearly, W has the form

$$W = \sum_{i=1}^{k} c_i P_i$$

with $\{P_i\}_{i=1,\ldots,k}$ representing a family of pairwise orthogonal projections with sum equal to the identity, i.e. $\sum_{i=1}^{k} P_i = I$. The range of each P_i is a closed subspace of $\mathcal{H}$, hence it has an orthonormal basis. We associate with each basis vector the rank-one projection onto the one-dimensional subspace it spans in $\mathcal{H}$. We enumerate these projections via an index set N_i with cardinality equal to the rank of P_i and obtain $\sum_{j\in N_i} P_j^i = P_i$. We now attach to each P_j^i a modulus one complex number λ_j^i (close enough to c_j) in the following way: For every $i < i'$, and $k_1,\ k_2 \in N_i$ with $k_1 < k_2$ and $l_1,\ l_2 \in N_{i'}$ with $l_1 < l_2$ we have

$$0 < |1-\lambda_{k_1}^i| < |1-\lambda_{k_2}^i| < |1-c_i| < |1-\lambda_{l_1}^{i'}| < |1-\lambda_{l_2}^{i'}| < |1-c_{i'}|.$$

Finally, we define the operator $A \in \mathrm{U}(\mathcal{H})$ by

$$A = \sum_{i=1}^{k}\left(\sum_{j\in N_i}\lambda_j^i P_j^i\right).$$

We denote the class of all unitary operators A obtained in that way by $\mathrm{D}(\mathcal{H})$. It is apparent that $\mathrm{D}(\mathcal{H})$ is dense in $\mathrm{U}(\mathcal{H})$.

In the main step that follows we describe the action of ϕ on the collection $\mathrm{D}(\mathcal{H})$.

Lemma 2.8. *The linear or conjugate linear isometry U that appears in Lemma 2.4 is either a unitary or an anti-unitary operator on $\mathcal{H}$ and $\phi(A) = UA^{(*)}U^*$, for every A in $\mathrm{D}(\mathcal{H})$.*

Proof. Select an operator A from $\mathrm{D}(\mathcal{H})$ which is of the form

$$A = \sum_{i=1}^{k}\left(\sum_{j\in N_i}\lambda_j^i P_j^i\right),$$

where $\{P_j^i\}$ and $\{\lambda_j^i\}$ are as described above.

For the sake of simplicity assume $k = 2$ and index sets N_1 and N_2 are countably infinite. The general case would follow similar steps. So we can write

$$A = \sum_{n=1}^{\infty}\lambda_n P_n + \sum_{n=1}^{\infty}\mu_n Q_n,$$

where $\{P_n, Q_m\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is a family of pairwise orthogonal rank one projections with sum equal to I, $\{\lambda_n\}_n$ and $\{\mu_n\}_n$ are sequences of scalars of modulus 1 close enough to certain c_1 and c_2, respectively. In addition, the following inequalities also hold:

$$0 < |\lambda_1 - 1| < |\lambda_2 - 1| < \cdots < |c_1 - 1| < |\mu_1 - 1| < |\mu_2 - 1| < \cdots < |c_2 - 1|.$$

We represent P_n by $e_n \otimes e_n$ and Q_n by $e'_n \otimes e'_n$, with e_n and e'_n denoting unit vectors in the range of P_n and in the range of Q_n, respectively.

We set $A_1 = I + (\lambda_1 - 1)P_1$. Lemma 2.6 says that

$$\phi(A_1) = I + (\lambda_1 - 1)^{(*)}\psi(P_1).$$

The 2-local condition on ϕ applied to the pair (A, A_1) implies the existence of V_1 and W_1, both unitary or both anti-unitary operators, such that

$$\phi(A) = V_1A^{(*)}W_1 \quad \text{and} \quad \phi(A_1) = V_1A_1^{(*)}W_1.$$

We recall that the superscript $(*)$ over an operator represents either the operator or its adjoint. Moreover, the 2-local condition on ϕ in the above displayed two formulas implies that either $A^{(*)} = A$ in both places or $A^{(*)} = A^*$ in both places. We conclude that

$$\phi(A) - \phi(A_1) = V_1[A - A_1]^{(*)}W_1.$$

Clearly, $A - A_1$ is not invertible which implies that $V_1[A - A_1]^{(*)}W_1$ is also not invertible.

We now apply the 2-local property of ϕ to the pair (A, I) and obtain

$$\phi(A) = V_0A^{(*)}V_0^*, \tag{4}$$

with V_0 a unitary or an anti-unitary operator on $\mathcal{H}$.

From (2) and (4), using Lemma 2.6 we get

$$V_1[A - A_1]^{(*)}W_1 = \phi(A) - \phi(A_1) = V_0[A^{(*)} - I]V_0^* - (\lambda_1 - 1)^{(*)}\psi(P_1).$$

Since $V_1[A - A_1]^{(*)}W_1$ is not invertible and $V_0[A^{(*)} - I]V_0^*$ is invertible, Lemma 2.7 implies that

$$(\lambda_1 - 1)^{(*)}\langle [V_0[A^{(*)} - I]V_0^*]^{-1}v_1, v_1\rangle = 1,$$

with v_1 a unit vector such that $\psi(P_1) = v_1 \otimes v_1$. Equivalently, we write

$$\left\langle \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - 1)^{(*)}} P_n' v_1 + \sum_{n=1}^{\infty} \frac{1}{(\mu_n - 1)^{(*)}} Q_n' v_1,\ v_1 \right\rangle = \frac{1}{(\lambda_1 - 1)^{(*)}},$$

with $P_n' = V_0 P_n V_0^*$ and $Q_n' = V_0 Q_n V_0^*$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda_n - 1)^{(*)}} \|P_n' v_1\|^2 + \sum_{n=1}^{\infty} \frac{1}{(\mu_n - 1)^{(*)}} \|Q_n' v_1\|^2 = \frac{1}{(\lambda_1 - 1)^{(*)}}. \tag{5}$$

Since $\|v_1\| = 1$, $\sum_{n=1}^{\infty} \|P_n' v_1\|^2 + \sum_{n=1}^{\infty} \|Q_n' v_1\|^2 = 1$, $|\lambda_1 - 1| < |\lambda_n - 1|$ for all $n \geq 2$ and $|\lambda_1 - 1| < |\mu_m - 1|$ for all $m \geq 1$, the equation displayed in (5) implies that $P_n' v_1 = 0$ for all $n > 1$, $Q_n' v_1 = 0$ for all $n \geq 1$, and hence we have $P_1' v_1 = \alpha v_1$, with α a scalar of modulus 1. Therefore $\psi(P_1) = P_1'$.

We now consider $A_2 = I + (\lambda_2 - 1)P_2$. Lemma 2.6 implies that

$$\phi(A_2) = I + (\lambda_2 - 1)^{(*)} \psi(P_2).$$

We set $\psi(P_2) = v_2 \otimes v_2$, with v_2 a unit vector in the range of $\psi(P_2)$. Lemma 2.4 implies that $\psi(P_1)$ and $\psi(P_2)$ are mutually orthogonal and hence $\psi(P_1)v_2 = 0$. The proof proceeds as before. The 2-local condition on ϕ applied to (A, A_2) implies the existence of V_2 and W_2, both unitary or both anti-unitary operators, such that

$$V_2[A - A_2]^{(*)} W_2 = \phi(A) - \phi(A_2) = (\phi(A) - I) - (\lambda_2 - 1)^{(*)} \psi(P_2).$$

Since $A - A_2$ is not invertible and $\phi(A) - I = V_0[A^{(*)} - I]V_0^*$ is invertible, Lemma 2.7 implies

$$(\lambda_2 - 1)^{(*)} \langle [V_0[A^{(*)} - I]V_0^*]^{-1} v_2, v_2 \rangle = 1,$$

equivalently

$$\left\langle \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - 1)^{(*)}} P_n' v_2 + \sum_{n=1}^{\infty} \frac{1}{(\mu_n - 1)^{(*)}} Q_n' v_2,\ v_2 \right\rangle = \frac{1}{(\lambda_2 - 1)^{(*)}}.$$

Since $P_1' v_2 = \psi(P_1) v_2 = 0$, the same reasoning applied in the previous step shows that we necessarily have $\psi(P_2) = P_2'$.

Inductively we prove that $\psi(P_n) = P_n'$, for all $n \in \mathbb{N}$. Next, let $B_1 = I + (\mu_1 - 1)Q_1$ and $\psi(Q_1) = u_1 \otimes u_1$. Lemma 2.4 implies that $\psi(P_n)u_1 = 0$, for all n, and from Lemma 2.6 we get $\phi(B_1) = I + (\mu_1 - 1)^{(*)} \psi(Q_1)$. As before we conclude that $\psi(Q_1) = Q_1'$ and inductively we prove that $\psi(Q_n) = Q_n'$ holds for all $n \geq 1$. By its definition the family $\{P_n', Q_m'\}$ of pairwise orthogonal rank-one projections is complete, i.e., its sum is equal to I. Therefore $\{\psi(P_n),\ \psi(Q_m)\} = \{UP_nU^*,\ UQ_mU^*\}$ is also complete implying that U is a unitary or an anti-unitary operator on $\mathcal{H}$.

Finally, since $A = \sum_{n=1}^{\infty} \lambda_n P_n + \sum_{n=1}^{\infty} \mu_n Q_n$, the equation displayed in (4) becomes

$$\begin{aligned}
\phi(A) &= \sum_{n=1}^{\infty} \lambda_n^{(*)} V_0 P_n V_0^* + \sum_{n=1}^{\infty} \mu_n^{(*)} V_0 Q_n V_0^* \\
&= \sum_{n=1}^{\infty} \lambda_n^{(*)} P_n' + \sum_{n=1}^{\infty} \mu_n^{(*)} Q_n' \\
&= \sum_{n=1}^{\infty} \lambda_n^{(*)} \psi(P_n) + \sum_{n=1}^{\infty} \mu_n^{(*)} \psi(Q_n) \\
&= \sum_{n=1}^{\infty} \lambda_n^{(*)} UP_nU^* + \sum_{n=1}^{\infty} \mu_n^{(*)} UQ_nU^* \\
&= U \left(\sum_{n=1}^{\infty} \lambda_n P_n + \sum_{n=1}^{\infty} \mu_n Q_n \right)^{(*)} U^*.
\end{aligned}$$

Then $\phi(A) = UA^{(*)}U^*$, which completes the proof. $\square$

We can now easily complete the proof of the theorem in this section.

Proof of Theorem 2.1. Since ϕ is an isometry and $\mathrm{D}(\mathcal{H})$ is dense in $\mathrm{U}(\mathcal{H})$, it easily follows from Lemma 2.8 that for every $B \in \mathrm{U}(\mathcal{H})$, we have either $\phi(B) = UBU^*$ or $\phi(B) = UB^*U^*$. It remains to show that the appearance of the adjoint does not depend on B. To see this first observe that by the 2-local condition on ϕ, for every $T \in \mathrm{U}(\mathcal{H})$ we have $\phi(iT) = \pm i\phi(T)$. The connectedness of the metric space $\mathrm{U}(\mathcal{H})$ implies that either $\phi(iT) = i\phi(T)$ for every $T \in \mathrm{U}(\mathcal{H})$, or $\phi(iT) = -i\phi(T)$ for every $T \in \mathrm{U}(\mathcal{H})$. We claim that for every $B \in \mathrm{U}(\mathcal{H})$ we have $\phi(B) = UBU^*$ or, for every $B \in \mathrm{U}(\mathcal{H})$ we have $\phi(B) = UB^*U^*$.

The proof would consist of several cases. We present the details for U unitary and $\phi(iT) = i\phi(T)$, for every $T \in \mathrm{U}(\mathcal{H})$. Given $B \in \mathrm{U}(\mathcal{H})$ which is non skew-symmetric ($B^* \neq -B$) we show that $\phi(B) = UBU^*$. To see this, we recall that $\phi(iB) = U(iB)^{(*)}U^*$, and also that $\phi(iB) = i\phi(B)$. If $\phi(iB) = U(iB)^*U^* = -iUB^*U^*$ then $\phi(B) = -UB^*U^*$ which contradicts both possibilities $\phi(B) = UBU^*$ and $\phi(B) = UB^*U^*$. Therefore $\phi(iB) = U(iB)U^*$, and hence $\phi(B) = UBU^*$. Since the set of all non skew-symmetric B's is dense in $\mathrm{U}(\mathcal{H})$ and ϕ is an isometry, we are done in the present case. The remaining cases are: U is unitary and $\phi(iT) = -i\phi(T)$ for every $T \in \mathrm{U}(\mathcal{H})$; U is anti-unitary and $\phi(iT) = i\phi(T)$ for every $T \in \mathrm{U}(\mathcal{H})$; and U is anti-unitary and $\phi(iT) = -i\phi(T)$ for every $T \in \mathrm{U}(\mathcal{H})$. A similar analysis leads to $\phi(B) = UB^*U^*$, for every $B \in \mathrm{U}(\mathcal{H})$ in the first two cases and $\phi(B) = UBU^*$, for every $B \in \mathrm{U}(\mathcal{H})$ in the last case. □

3. The algebraic reflexivity of the Thompson isometry group

In this section we establish the algebraic reflexivity of the group of all Thompson isometries of the space $\mathrm{B}(\mathcal{H})_+^{-1}$ of invertible positive operators on the complex infinite dimensional separable Hilbert space $\mathcal{H}$. The Thompson metric (also called Thompson part metric) can be defined in a rather general setting involving normed linear spaces and certain closed cones, see [26]. This metric has a wide range of applications from non-linear integral equations, linear operator equations, ordinary differential equations to optimal filtering and beyond. Following that general approach, the definition of the Thompson metric d_T for the cone $\mathrm{B}(\mathcal{H})_+^{-1}$ would read

$$d_T(A, B) = \log \max\{M(A/B), M(B/A)\}, \quad A, B \in \mathrm{B}(\mathcal{H})_+^{-1},$$

where $M(X/Y) = \inf\{t > 0 : X \leq tY\}$ for any $X, Y \in \mathrm{B}(\mathcal{H})_+^{-1}$. It is not difficult to see that $d_T(A, B)$ can be rewritten as

$$d_T(A, B) = \left\| \log \left(\sqrt{A}^{-1} B \sqrt{A}^{-1} \right) \right\|, \quad A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$$

(see, e.g., [23]).

In the case of $\mathrm{B}(\mathcal{H})_+^{-1}$, the Thompson metric has important differential geometrical connections. To see this, observe that $\mathrm{B}(\mathcal{H})_+^{-1}$ is an open subset of the Banach space $\mathrm{B}(\mathcal{H})_s$ of all self-adjoint operators $\mathcal{H}$. Therefore it is a differentiable manifold that carries a natural Finsler geometrical structure as follows (for more details and for further reading, see e.g., [2]). At any point $A \in \mathrm{B}(\mathcal{H})_+^{-1}$, the tangent space is identified with the linear space $\mathrm{B}(\mathcal{H})_s$ in which the norm of a vector X is defined as $\left\| \sqrt{A}^{-1} X \sqrt{A}^{-1} \right\|$. It turns out that in the so-obtained Finsler space the geodesic distance $d(A, B)$ between A and $B \in \mathrm{B}(\mathcal{H})_+^{-1}$ can be computed as

$$d(A, B) = \left\| \log \left(\sqrt{A}^{-1} B \sqrt{A}^{-1} \right) \right\|$$

which clearly coincides with the Thompson distance $d_T(A, B)$.

We point out that the differential geometry of the positive cone in operator algebras is an active research area with many applications. Indeed, even in the finite dimensional case, the differential geometry of the space of $n \times n$ positive definite matrices has important applications among others in linear systems, statistics, filters, Lagrangian geometry and quantum systems (see, e.g., [8]).

In this section we prove the following statement.

Theorem 3.1. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space. The Thompson isometry group of $\mathrm{B}(\mathcal{H})_+^{-1}$ is algebraically reflexive.*

Recall that the composition of a 2-local isometry with a surjective isometry is a 2-local isometry. Hence without loss of generality we may assume that it fixes any two given elements of the underlying space. Throughout this section ϕ *denotes a* 2*-local isometry on* $\mathrm{B}(\mathcal{H})_+^{-1}$ *such that* $\phi(I) = I$ *and* $\phi(2I) = 2I$.

We now proceed with the details for the proof of Theorem 3.1.

Lemma 3.2. *If A is an operator in $\mathrm{B}(\mathcal{H})_+^{-1}$, then one of the following holds:*

(a) $\phi(B) \geq \phi(A)$, $\forall\, B \in \mathrm{B}(\mathcal{H})_+^{-1}$ *s.t.* $B \geq A$ *or*
(b) $\phi(B) \leq \phi(A)$, $\forall\, B \in \mathrm{B}(\mathcal{H})_+^{-1}$ *s.t.* $B \geq A$.

Proof. Let B be an operator in $\mathrm{B}(\mathcal{H})_+^{-1}$ such that $B \geq A$. The 2-local condition of ϕ applied to the pair (A, B) implies the existence of a surjective Thompson isometry $\Phi_{(A,B)}$ such that

$$\Phi_{(A,B)}(A) = \phi(A) \quad \text{and} \quad \Phi_{(A,B)}(B) = \phi(B).$$

The form for the surjective isometries described in Theorem 1.2-(b) leads to the following two cases:

1. $\Phi_{(A,B)}(A) = TAT^*$ and $\Phi_{(A,B)}(B) = TBT^*$ which implies $\phi(A) \leq \phi(B)$, and
2. $\Phi_{(A,B)}(A) = TA^{-1}T^*$ and $\Phi_{(A,B)}(B) = TB^{-1}T^*$ which implies $\phi(A) \geq \phi(B)$.

Here T is either linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ and we use the well known fact that $A \le B$ implies $B^{-1} \le A^{-1}$. (On referee's request here is a simple proof: $A \le B$ implies $B^{-1/2}AB^{-1/2} \le I$ which gives us that $I \le (B^{-1/2}AB^{-1/2})^{-1} = B^{1/2}A^{-1}B^{1/2}$ resulting in $B^{-1} \le A^{-1}$.)

This shows that for every $B \ge A$ we have either $\phi(B) \ge \phi(A)$ or $\phi(B) \le \phi(A)$. Let B_0, B_1 be in $\mathrm{B}(\mathcal{H})_+^{-1}$ such that $B_i \ge A$ and $B_i \ne A$, for $i = 0, 1$, and $\phi(B_0) \ge \phi(A)$ and $\phi(B_1) \le \phi(A)$. For $t \in [0, 1]$ we define $B_t = tB_1 + (1-t)B_0$. It is easy to see that $B_t \in \mathrm{B}(\mathcal{H})_+^{-1}$, $B_t \ge A$ and $B_t \ne A$. We clearly have $\phi(B_t) \ne \phi(A)$. The sets

$$\{t \in [0,1] : \phi(B_t) \ge \phi(A)\} \quad \text{and} \quad \{t \in [0,1] : \phi(B_t) \le \phi(A)\}$$

are nonempty, disjoint and their union is $[0, 1]$. Moreover, ϕ is an isometry with respect to the Thompson metric. This metric is known to generate the same topology on $\mathrm{B}(\mathcal{H})_+^{-1}$ as the operator norm (cf. p. 3854 in [23]), hence ϕ is continuous relative to both metrics. It follows that the above sets are closed too. But this contradicts the connectedness of the interval $[0, 1]$ and proves the lemma. □

Lemma 3.3. *For any $A \in \mathrm{B}(\mathcal{H})_+^{-1}$ we have $\sigma(\phi(A)) = \sigma(A)$. This means that ϕ is spectrum preserving.*

Proof. To see this, apply the 2-local property of ϕ to the pairs (A, I) and $(A, 2I)$. We have linear or conjugate linear bounded and invertible operators T, S on $\mathcal{H}$ such that

$$\phi(A) = TA^{(-1)}T^*, \qquad I = \phi(I) = TI^{(-1)}T^* \tag{6}$$

and

$$\phi(A) = SA^{(-1)}S^*, \qquad 2I = \phi(2I) = S(2I)^{(-1)}S^*. \tag{7}$$

If the inverse does not appear in the equalities displayed in (6), then we are done since T is a unitary or anti-unitary operator. The case is similar if the inverse does not appear in the equalities displayed in (7). Then we assume that $\phi(A) = TA^{-1}T^*$, $I = TT^*$, $\phi(A) = SA^{-1}S^*$, and $2I = S(2I)^{-1}S^*$. From the last equality we infer that $\frac{1}{2}S$ is a unitary or anti-unitary operator and then the first and third equalities imply $\sigma(\phi(A)) = \sigma(A^{-1})$, $\sigma(\phi(A)) = 4\sigma(A^{-1})$ which is an obvious contradiction. Therefore ϕ preserves the spectrum of operators in $\mathrm{B}(\mathcal{H})_+^{-1}$. □

Lemma 3.4. *The transformation ϕ is monotone increasing, i.e., for any $A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$ with $A \le B$ we have $\phi(A) \le \phi(B)$. It follows that ϕ is positive homogeneous, too.*

Proof. Pick $A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$ with $A \le B$. Referring to Lemma 3.2, assume that we have case (b), i.e. $\phi(C) \le \phi(A)$ holds for every $C \in \mathrm{B}(\mathcal{H})_+^{-1}$ with $A \le C$. Choosing any positive scalar operator C, by the spectrum preserving property of ϕ we have $\phi(C) = C$. Therefore, we obtain $C \le \phi(A)$ for every positive scalar operator C for which $A \le C$. This is an obvious contradiction, so by Lemma 3.2 we conclude that $\phi(A) \le \phi(B)$.

Let $A \in \mathrm{B}(\mathcal{H})_+^{-1}$ and λ a positive number different from 1. Clearly, by the 2-local condition on ϕ we have either $\phi(\lambda A) = \lambda\phi(A)$ or $\phi(\lambda A) = \frac{1}{\lambda}\phi(A)$. For $\lambda > 1$, using the monotonicity of ϕ, in the latter case we would have

$$\phi(A) \le \phi(\lambda A) = \frac{1}{\lambda}\phi(A),$$

a clear contradiction. If $\lambda < 1$ the proof is similar. Therefore we have $\phi(\lambda A) = \lambda\phi(A)$ and the proof is complete. □

As previously stated we are assuming that $\phi(I) = I$ (and also that $\phi(2I) = 2I$). Given a nontrivial projection $P \in \mathrm{P}(\mathcal{H})$, the 2-local property of ϕ applied to $(I, I + P)$ implies the existence of T a unitary or anti-unitary operator such that either $\phi(I + P) = T(I + P)T^*$ or $\phi(I + P) = T(I + P)^{-1}T^*$. By the spectrum preserving property of ϕ this latter possibility is ruled out. So we have $\phi(I + P) = T(I + P)T^* = I + TPT^*$. Therefore ϕ induces a map on $\mathrm{P}(\mathcal{H})$ given by:

$$\begin{aligned} \Psi : \mathrm{P}(\mathcal{H}) &\to \mathrm{P}(\mathcal{H}) \\ P &\to \phi(I + P) - I. \end{aligned}$$

We observe that Ψ preserves the rank of projections. We denote by $\psi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$ the restriction of Ψ to the set $\mathrm{P}_1(\mathcal{H})$ of all rank-1 projections on $\mathcal{H}$.

Lemma 3.5. *The map $\psi : \mathrm{P}_1(\mathcal{H}) \to \mathrm{P}_1(\mathcal{H})$ defined by $\psi(P) = \phi(I + P) - I$, $P \in \mathrm{P}_1(\mathcal{H})$ is a Wigner transformation and hence there exists a linear or conjugate linear isometry U on $\mathcal{H}$ such that*

$$\psi(P) = UPU^*,$$

for every $P \in \mathrm{P}_1(\mathcal{H})$.

Proof. We show that for every P and Q in $\mathrm{P}_1(\mathcal{H})$, we have $tr(\psi(P)\psi(Q)) = tr(PQ)$. We consider the pair $(I+P, I+Q)$, Theorem 1.2-(b) implies the existence of a linear or conjugate linear bounded and invertible operator T on $\mathcal{H}$ such that

$$\phi(I+P) = T(I+P)^{(-1)}T^* \quad \text{and} \quad \phi(I+Q) = T(I+Q)^{(-1)}T^*.$$

Here in the two appearances of (-1) we either both have -1 or both have 1. In the case where $\phi(I+P) = T(I+P)T^*$ and $\phi(I+Q) = T(I+Q)T^*$ we compute

$$\begin{aligned}\phi(I+P)\phi(I+Q)^{-1} &= T(I+P)(I+Q)^{-1}T^{-1} \\ &= T(I+P)\left(I - \frac{1}{2}Q\right)T^{-1} \\ &= T\left(I + P - \frac{1}{2}Q - \frac{1}{2}PQ\right)T^{-1} \\ &= I + TPT^{-1} - \frac{1}{2}TQT^{-1} - \frac{1}{2}TPQT^{-1}.\end{aligned}$$

If $\phi(I+P) = T(I+P)^{-1}T^*$ and $\phi(I+Q) = T(I+Q)^{-1}T^*$ we similarly have

$$\phi(I+P)\phi(I+Q)^{-1} = I - \frac{1}{2}TPT^{-1} + TQT^{-1} - \frac{1}{2}TPQT^{-1}.$$

On the other hand, we also have

$$\begin{aligned}\phi(I+P)\phi(I+Q)^{-1} &= (I+\psi(P))\left(I - \frac{1}{2}\psi(Q)\right) \\ &= I + \psi(P) - \frac{1}{2}\psi(Q) - \frac{1}{2}\psi(P)\psi(Q).\end{aligned}$$

Hence

$$\psi(P) - \frac{1}{2}\psi(Q) - \frac{1}{2}\psi(P)\psi(Q) = TPT^{-1} - \frac{1}{2}TQT^{-1} - \frac{1}{2}TPQT^{-1}$$

or

$$\psi(P) - \frac{1}{2}\psi(Q) - \frac{1}{2}\psi(P)\psi(Q) = -\frac{1}{2}TPT^{-1} + TQT^{-1} - \frac{1}{2}TPQT^{-1}$$

holds. In either case, taking trace we have $tr(\psi(P)\psi(Q)) = tr(PQ)$. The existence of a linear or conjugate linear isometry U on $\mathcal{H}$ such that $\psi(P) = UPU^*$ holds for every $P \in \mathrm{P}_1(\mathcal{H})$ follows from Theorem 2.3. $\square$

Lemma 3.6. *If $\lambda > 0$ and P is a projection of rank one, then $\phi(I+\lambda P) = I + \lambda\psi(P)$.*

Proof. The statement is trivial for $\lambda = 1$. We assume $\lambda \neq 1, 3$ and we apply the 2-local condition on ϕ to the pair $(I+\lambda P, 2I)$. We have two cases. Either there exists T, a unitary or anti-unitary operator on $\mathcal{H}$, such that

$$\phi(I+\lambda P) = T(I+\lambda P)T^* = I + \lambda TPT^*$$

or there exists a linear or conjugate linear bounded and invertible operator S on $\mathcal{H}$ such that

$$\phi(I+\lambda P) = S(I+\lambda P)^{-1}S^*, 2I = \phi(2I) = S(2I)^{-1}S^*.$$

In the latter case it follows that $\frac{1}{2}S$ is a unitary or anti-unitary operator on $\mathcal{H}$ and we obtain

$$\sigma(\phi(I+\lambda P)) = 4\left\{1, \frac{1}{1+\lambda}\right\}.$$

On the other hand, by the spectrum preserving property of ϕ we have $\sigma(\phi(I+\lambda P)) = \{1, 1+\lambda\}$ and we arrive at $\lambda = 3$, a contradiction. So the first case remains to be addressed. Since λP and P are comparable, it follows from Lemma 3.2 that $\phi(I+\lambda P) = I + \lambda TPT^*$ and $\phi(I+P) = I + \psi(P)$ are also comparable. This implies that λTPT^* and $\psi(P)$ are comparable from which we deduce $TPT^* = \psi(P)$ and hence $\phi(I+\lambda P) = I + \lambda\psi(P)$. This completes the proof for $\lambda \neq 3$. To obtain the conclusion for $\lambda = 3$ one can simply refer to the continuity of ϕ. $\square$

Lemma 3.7. *If $0 < \epsilon < \lambda$ and P is a projection of rank one, then $\phi(\epsilon P^{\perp} + \lambda P) = \epsilon\psi(P)^{\perp} + \lambda\psi(P)$.*

Proof. Lemmas 3.4 and 3.6 imply that

$$\begin{aligned}\phi(\epsilon P^{\perp}+\lambda P) &= \epsilon\phi\left(I+\frac{\lambda-\epsilon}{\epsilon}P\right)\\ &= \epsilon\left[I+\frac{\lambda-\epsilon}{\epsilon}\psi(P)\right]\\ &= \epsilon\psi(P)^{\perp}+\lambda\psi(P).\end{aligned}$$

This completes the proof. □

Similarly to the previous section we now describe a class of positive invertible operators with a diagonal structure relative to a family of pairwise orthogonal projections of rank one with sum equal to I.

The spectral theorem implies that the set $\mathcal{P}_f$ of finite spectrum operators in $\mathrm{B}(\mathcal{H})_+^{-1}$ is dense in $\mathrm{B}(\mathcal{H})_+^{-1}$ relative to the metric induced by the operator norm. Select any element $C \in \mathcal{P}_f$ and write $\sigma(C) = \{c_1, \ldots, c_k\}$ where $c_1 > \cdots > c_k$. Then C has the representation

$$C = \sum_{i=1}^{k} c_i P_i$$

with $\{P_i\}_{i=1,\ldots,k}$ a family of pairwise orthogonal projections such that $\sum_{i=1}^{k} P_i = I$. The range of each P_i is a closed subspace of $\mathcal{H}$, hence it has an orthonormal basis. We associate with each basis vector the orthogonal projection onto its linear span. We denote the so-obtained projections by $\{P_j^i\}_{j\in N_i}$, the index set N_i has cardinality equal to the rank of P_i. We now attach to each P_j^i a positive number λ_j^i (close enough to c_i) in the following way: If $i < i'$, and $k_1, k_2 \in N_i$ with $k_1 < k_2$ and $l_1, l_2 \in N_{i'}$ with $l_1 < l_2$, then

$$\lambda_{k_1}^{i} > \lambda_{k_2}^{i} > c_i > \lambda_{l_1}^{i'} > \lambda_{l_2}^{i'} > c_{i'} > 0.$$

We define the positive invertible operator A by

$$A = \sum_{i=1}^{k}\left(\sum_{j\in N_i}\lambda_j^i P_j^i\right).$$

Denote by $\mathrm{D}(\mathcal{H})_{-1}^{+}$ the class of all operators A obtained in that way. It follows from the construction that $\mathrm{D}(\mathcal{H})_{-1}^{+}$ is dense in $\mathrm{B}(\mathcal{H})_+^{-1}$ relative to the operator norm topology. As we have already referred to it at the end of the proof of Lemma 3.2, that topology on $\mathrm{B}(\mathcal{H})_+^{-1}$ coincides with the Thompson topology. Therefore, the set $\mathrm{D}(\mathcal{H})_{-1}^{+}$ is dense in $\mathrm{B}(\mathcal{H})_+^{-1}$ with respect to Thompson metric.

In the main step we describe the action of ϕ on the set $\mathrm{D}(\mathcal{H})_{-1}^{+}$.

Lemma 3.8. *The operator U in Lemma 3.5 is a unitary or anti-unitary operator on $\mathcal{H}$ and we have $\phi(A) = UAU^*$ for every A in $\mathrm{D}(\mathcal{H})_{-1}^{+}$.*

Proof. Let A be an operator in $\mathrm{D}(\mathcal{H})_{-1}^{+}$ of the above displayed form with scalars λ_j^i and projections P_j^i having all properties listed above.

We continue the proof assuming that $k = 2$ and that the corresponding index sets N_1 and N_2 are countably infinite. The general case follows similar steps. So let

$$A = \sum_{n=1}^{\infty}\lambda_n P_n + \sum_{n=1}^{\infty}\mu_n Q_n,$$

where $\{P_n, Q_m\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is a family of pairwise orthogonal projections with sum equal to I, and for $k < l$ we have

$$\lambda_k > \lambda_l > c_1 > \mu_k > \mu_l > c_2 > 1,$$

where c_1, c_2 are some given positive numbers. By the homogeneity of ϕ we may further suppose that $c_2 > 1$ and can choose ϵ such that $c_2 > \epsilon > 1$. The 2-local condition on ϕ applied to the pair (A, I) implies the existence of a unitary or anti-unitary operator T such that $\phi(A) = TAT^*$ or $\phi(A) = TA^{-1}T^*$. Since $A \geq \epsilon I$ and ϕ preserves the spectrum of A, this second possibility cannot occur. So we have $\phi(A) = TAT^*$ and

$$\phi(A) = T\left[\sum_{n=1}^{\infty}\lambda_n P_n + \sum_{n=1}^{\infty}\mu_n Q_n\right]T^* = \sum_{n=1}^{\infty}\lambda_n P_n' + \sum_{n=1}^{\infty}\mu_n Q_n', \tag{8}$$

where $P_n' = TP_nT^*$ and $Q_n' = TQ_nT^*$. We observe that $\{P_n', Q_m'\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is a family of pairwise orthogonal rank-1 projections with sum equal to I.

We now apply the 2-local condition on ϕ to the pair $(A, \epsilon P_1^{\perp} + \lambda_1 P_1)$. Theorem 1.2 implies the existence of a linear or conjugate linear and invertible operator T_1 such that

$$\phi(A) = T_1 A^{(-1)} T_1^* \quad \text{and} \quad \phi(\epsilon P_1^{\perp} + \lambda_1 P_1) = T_1(\epsilon P_1^{\perp} + \lambda_1 P_1)^{(-1)} T_1^*.$$

Since $\epsilon P_1^{\perp} + \lambda_1 P_1 \leq A$, these two operators are different and ϕ preserves the order, it follows that we necessarily have $\phi(A) = T_1 A T_1^*$ and $\phi(\epsilon P_1^{\perp} + \lambda_1 P_1) = T_1(\epsilon P_1^{\perp} + \lambda_1 P_1)T_1^*$.

We observe that $A - (\epsilon P_1^{\perp} + \lambda_1 P_1)$ has nontrivial kernel, hence

$$\phi(A) - \phi(\epsilon P_1^{\perp} + \lambda_1 P_1) = T_1 \left[A - (\epsilon P_1^{\perp} + \lambda_1 P_1)\right] T_1^*$$

is not invertible. From Lemma 3.7 and the equation displayed in (8) we have

$$\begin{aligned}\phi(A) - \phi(\epsilon P_1^{\perp} + \lambda_1 P_1) &= \sum_{n=1}^{\infty} \lambda_n P_n' + \sum_{n=1}^{\infty} \mu_n Q_n' - \epsilon \psi(P_1)^{\perp} - \lambda_1 \psi(P_1) \\ &= \sum_{n=1}^{\infty} (\lambda_n - \epsilon) P_n' + \sum_{n=1}^{\infty} (\mu_n - \epsilon) Q_n' - (\lambda_1 - \epsilon)\psi(P_1).\end{aligned}$$

Since $\psi(P_1)$ is a projection of rank one, we write $\psi(P_1) = u_1 \otimes u_1$ with u_1 a unit vector in the range of $\psi(P_1)$. Lemma 2.7 yields

$$(\lambda_1 - \epsilon)\langle X^{-1} u_1, u_1\rangle = 1,$$

with $X = \sum_{n=1}^{\infty} (\lambda_n - \epsilon) P_n' + \sum_{n=1}^{\infty} (\mu_n - \epsilon) Q_n'$. Therefore

$$\begin{aligned}1 = (\lambda_1 - \epsilon)\langle X^{-1} u_1, u_1\rangle &= (\lambda_1 - \epsilon)\left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n - \epsilon}\langle P_n' u_1, u_1\rangle + \sum_{n=1}^{\infty} \frac{1}{\mu_n - \epsilon}\langle Q_n' u_1, u_1\rangle\right] \\ &= (\lambda_1 - \epsilon)\left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n - \epsilon}\|P_n' u_1\|^2 + \sum_{n=1}^{\infty} \frac{1}{\mu_n - \epsilon}\|Q_n' u_1\|^2\right].\end{aligned}$$

Since $\sum_{n=1}^{\infty} \|P_n' u_1\|^2 + \sum_{n=1}^{\infty} \|Q_n' u_1\|^2 = 1$ and $\lambda_1 - \epsilon > \lambda_k - \epsilon > c_1 - \epsilon > \mu_l - \epsilon > c_2 - \epsilon > 0$ for all $k > 1$ and $l \geq 1$, we must have $P_n' u_1 = 0$ for all $n > 1$ and $Q_n' u_1 = 0$ for all $n \geq 1$. Therefore we obtain $\psi(P_1) = P_1'$ and

$$\phi(A) = \lambda_1 \psi(P_1) + \sum_{n=2}^{\infty} \lambda_n P_n' + \sum_{n=1}^{\infty} \mu_n Q_n'.$$

We now apply the same analysis to A and $\epsilon P_2^{\perp} + \lambda_2 P_2$. Write $\psi(P_2) = u_2 \otimes u_2$ with u_2 a unit vector in the range of $\psi(P_2)$. It follows from Lemma 3.5 that $\psi(P_1)$ and $\psi(P_2)$ are orthogonal projections. Thus $\psi(P_1)u_2 = 0$. The same reasoning as followed before implies that $\psi(P_2) = P_2'$. Inductively we derive that $\psi(P_n) = P_n'$ holds for every n and then that $\psi(Q_n) = Q_n'$ holds for every n.

By Lemma 3.5 we have $\psi(P_n) = UP_nU^*$ and $\psi(Q_n) = UQ_nU^*$ for all n, where U represents a linear or conjugate linear isometry on $\mathcal{H}$. Therefore $\{P_n', Q_m'\} = \{UP_nU^*, UQ_mU^*\}$ is a family of pairwise orthogonal projections of rank one with sum equal to I. This entails that U is either a unitary or an anti-unitary operator on $\mathcal{H}$. We derive the form of $\phi(A)$ as follows:

$$\begin{aligned}\phi(A) &= \sum_{n=1}^{\infty} \lambda_n P_n' + \sum_{n=1}^{\infty} \mu_n Q_n' \\ &= \sum_{n=1}^{\infty} \lambda_n \psi(P_n) + \sum_{n=1}^{\infty} \mu_n \psi(Q_n) \\ &= \sum_{n=1}^{\infty} \lambda_n UP_nU^* + \sum_{n=1}^{\infty} \mu_n UQ_nU^* \\ &= UAU^*.\end{aligned}$$

This completes the proof. □

Now, the proof of Theorem 3.1 follows easily.

Proof of Theorem 3.1. Since ϕ is a Thompson isometry and, as we have already remarked in the preamble before Lemma 3.8, $\mathrm{D}(\mathcal{H})_{-1}^{+}$ is dense in $\mathrm{B}(\mathcal{H})_{+}^{-1}$ with respect to the Thompson metric, it follows that $\phi(A) = UAU^*$ holds for every $A \in \mathrm{B}(\mathcal{H})_{+}^{-1}$. This shows that the group of all surjective Thompson isometries of $\mathrm{B}(\mathcal{H})_{+}^{-1}$ is algebraically reflexive. □

4. Algebraic reflexivity of the isometry group of the general linear group

We now consider the general linear group $\mathrm{GL}(\mathcal{H})$ on $\mathcal{H}$, the group that consists of all bounded invertible operators on $\mathcal{H}$. In this section we address the algebraic reflexivity question for the isometry group on $\mathrm{GL}(\mathcal{H})$. We prove the following theorem.

Theorem 4.1. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space. The group of isometries on $\mathrm{GL}(\mathcal{H})$ is algebraically reflexive.*

Let $\phi : \mathrm{GL}(\mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ be a 2-local isometry, i.e. a mapping such that given a pair (A, B) of operators in $\mathrm{GL}(\mathcal{H})$ there exists a surjective isometry $\Phi_{(A,B)}$ on $\mathrm{GL}(\mathcal{H})$ such that

$$\phi(A) = \Phi_{(A,B)}(A) \quad \text{and} \quad \phi(B) = \Phi_{(A,B)}(B).$$

As already mentioned, we can assume that ϕ fixes any given pair of elements of the underlying metric space $\mathrm{GL}(\mathcal{H})$. So let us suppose that $\phi(I) = I$.

The structure of all surjective isometries of $\mathrm{GL}(\mathcal{H})$ is described in Theorem 1.2-(c). Therefore the 2-local condition on ϕ applied to (A, I) implies the existence of a unitary or anti-unitary operator V on $\mathcal{H}$ such that

$$\phi(A) = VA^{(*)}V^*.$$

Thus if A is a unitary operator then $\phi(A)$ is also unitary. Consequently, the restriction of ϕ to the unitary group is a 2-local isometry of $\mathrm{U}(\mathcal{H})$. Theorem 2.1 implies that it is necessarily a surjective isometry and hence there is a unitary or anti-unitary operator U on $\mathcal{H}$ such that

$$\phi(A) = UAU^*, \quad \forall A \in \mathrm{U}(\mathcal{H}) \quad \text{or} \quad \phi(A) = UA^*U^*, \quad \forall A \in \mathrm{U}(\mathcal{H}).$$

In the first case considering the map $U^*\phi(\cdot)U$ on $\mathrm{GL}(\mathcal{H})$ while in the second case considering the transformation $U^*\phi(\cdot)^*U$, we have a 2-local isometry of $\mathrm{GL}(\mathcal{H})$ that acts as the identity on $\mathrm{U}(\mathcal{H})$.

Therefore without loss of generality we may and do assume throughout this section that *ϕ represents a 2-local isometry on $\mathrm{GL}(\mathcal{H})$ such that $\phi(A) = A$ for every $A \in \mathrm{U}(\mathcal{H})$*. In what follows we prove that ϕ is the identity on the whole group $\mathrm{GL}(\mathcal{H})$.

We first establish some useful properties of ϕ.

Lemma 4.2. *Let λ be a nonzero real number and A an operator in $\mathrm{GL}(\mathcal{H})$. Then $\phi(\lambda A) = \lambda\phi(A)$.*

Proof. Given λ a real number, we apply the 2-local condition to the pair $(A, \lambda A)$. There exist V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that $\phi(\lambda A) = V(\lambda A)^{(*)}W$ and $\phi(A) = VA^{(*)}W$. Therefore $\phi(\lambda A) = \lambda\phi(A)$. □

Lemma 4.3. *Let P be a projection of rank one and let λ and ϵ be nonzero real numbers. Then $\phi(\epsilon P^\perp + \lambda P) = \epsilon P^\perp + \lambda P$.*

Proof. By the real homogeneity of ϕ we can clearly assume that $\epsilon = 1$. Let $\lambda \neq 1$. We apply the 2-local condition on ϕ to the pair $(I, P^\perp + \lambda P)$. This implies the existence of V_0 a unitary operator or anti-unitary operator on $\mathcal{H}$ such that

$$\phi(P^\perp + \lambda P) = V_0(P^\perp + \lambda P)V_0^*.$$

Thus

$$\phi(P^\perp + \lambda P) = I + (\lambda - 1)P_1, \tag{9}$$

with $P_1 = V_0PV_0^*$. Since $P^\perp - P$ is unitary we have that

$$\phi(P^\perp - P) = P^\perp - P. \tag{10}$$

The 2-local condition on ϕ applied to the pair $(P^\perp + \lambda P, P^\perp - P)$ implies the existence of V_1 and W_1 both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that

$$\phi(P^\perp + \lambda P) = V_1(P^\perp + \lambda P)W_1 \quad \text{and} \quad \phi(P^\perp - P) = V_1(P^\perp - P)W_1.$$

Therefore

$$\begin{aligned}\phi(P^\perp + \lambda P)\phi(P^\perp - P)^* &= V_1(P^\perp + \lambda P)(P^\perp - P)V_1^* \\ &= V_1(P^\perp - \lambda P)V_1^* = I - (\lambda + 1)P_2,\end{aligned}$$

with $P_2 = V_1PV_1^*$. The equations displayed in (9) and (10) now lead to $[I + (\lambda - 1)P_1][P^\perp - P] = I - (\lambda + 1)P_2$. A straightforward computation yields

$$2P - (\lambda - 1)P_1 + 2(\lambda - 1)P_1P = (\lambda + 1)P_2.$$

Lemma 2.5 implies that $P = P_1$. Therefore, by (9) we have $\phi(P^\perp + \lambda P) = P^\perp + \lambda P$. An application of Lemma 4.2 completes the proof. □

We next show that ϕ fixes every operator in $\mathrm{D}_{-1}^{+}(\mathcal{H})$ (definition is given in Section 3), the proof follows a similar approach to the proof provided for the Lemma 3.8.

Lemma 4.4. *For every $A \in \mathrm{D}(\mathcal{H})_{-1}^{+}$ we have $\phi(A) = A$. It follows that ϕ is the identity on $\mathrm{B}(\mathcal{H})_{+}^{-1}$.*

Proof. We present the proof for an operator of the form $A = \sum_{n=1}^{\infty} \lambda_n P_n + \sum_{n=1}^{\infty} \mu_n Q_n$, where $\{P_n, Q_m\}$ is a family of pairwise orthogonal projections of rank one with sum equal to the identity and such that for any $k < l$ we have $\lambda_k > \lambda_l > \lambda > \mu_k > \mu_l > \mu > 0$, where λ, μ are given positive real numbers. We choose ϵ such that $0 < \epsilon < \mu$ and set $A_1 = \epsilon P_1^{\perp} + \lambda_1 P_1$. The 2-local condition on ϕ applied to (A, A_1) implies the existence of V, W unitary or anti-unitary operators such that $\phi(A) = VAW$ and $\phi(A_1) = VA_1W$. Hence $\phi(A) - \phi(A_1) = V[A - A_1]W$ is not invertible since $A - A_1$ is not invertible. The 2-local condition on ϕ applied to the pair (A, I) implies the existence of a unitary or anti-unitary operator V_0 such that $\phi(A) = V_0 A V_0^*$ and thus, using Lemma 4.3,

$$\begin{aligned} \phi(A) - \phi(A_1) &= V_0 A V_0^* - \epsilon P_1^{\perp} - \lambda_1 P_1 \\ &= V_0[A - \epsilon I]V_0^* - (\lambda_1 - \epsilon) P_1 \\ &= \sum_{n=1}^{\infty} (\lambda_n - \epsilon)\, V_0 P_n V_0^* + \sum_{n=1}^{\infty} (\mu_n - \epsilon)\, V_0 Q_n V_0^* - (\lambda_1 - \epsilon) P_1. \end{aligned}$$

Let u_1 be a unit vector such that $P_1 = u_1 \otimes u_1$. We set $P_n' = V_0 P_n V_0^*$ and $Q_n' = V_0 Q_n V_0^*$. Since $V_0[A - \epsilon I]V_0^*$ is invertible, an application of Lemma 2.7 yields

$$(\lambda_1 - \epsilon) \left[\sum_n \frac{1}{\lambda_n - \epsilon} \langle P_n' u_1, u_1 \rangle + \sum_n \frac{1}{\mu_n - \epsilon} \langle Q_n' u_1, u_1 \rangle \right] = 1.$$

A similar reasoning as presented for the proof of Lemma 3.8 implies that $P_1' = P_1$. Then $\phi(A) = \lambda_1 P_1 + \sum_{n=2}^{\infty} \lambda_n P_n' + \sum_n \mu_n Q_n'$. Now, let $A_2 = \epsilon P_2^{\perp} + \lambda_2 P_2$. A similar reasoning gives that $P_2' = P_2$. Inductively we prove that $\phi(A) = A$.

Since ϕ is an isometry and, as we have seen already, $\mathrm{D}(\mathcal{H})_{-1}^{+}$ is dense in $\mathrm{B}(\mathcal{H})_{+}^{-1}$, it follows that ϕ fixes all positive invertible operators. $\square$

In the next step we show that ϕ also fixes the self-adjoint elements of $\mathrm{GL}(\mathcal{H})$.

Lemma 4.5. *For every self-adjoint invertible operator H on $\mathcal{H}$ we have $\phi(H) = H$.*

Proof. Let H be a self-adjoint invertible operator and let B be a positive invertible operator. The 2-local property of ϕ applied to the pair (H, B) implies the existence of V and W unitary or anti-unitary operators on $\mathcal{H}$ such that $\phi(H) = VHW$ and $\phi(B) = VBW$. Thus $\phi(H)B = \phi(H)\phi(B)^* = VHBV^*$. Observe that the spectrum of HB is real. Indeed, since the spectrum of the product of invertible elements is independent of the order of the product, we have $\sigma(HB) = \sigma\left(\sqrt{B}H\sqrt{B}\right)$ and this latter spectrum is clearly real since $\sqrt{B}H\sqrt{B}$ is self-adjoint. Therefore $\sigma(VHBV^*) = \sigma(HB)$ holds even if V is anti-unitary. We conclude $\sigma(\phi(H)B) = \sigma(HB)$ or equivalently

$$\sigma\left(\sqrt{B}\phi(H)\sqrt{B}\right) = \sigma\left(\sqrt{B}H\sqrt{B}\right). \tag{11}$$

Since a rank one projection can be uniformly approximated by positive invertible operators and the spectrum for normal operators is a continuous set-valued function relative to the Hausdorff distance on the compact subsets of $\mathbb{C}$ (cf. [3, Theorem 6.2.1]), it follows from (11) that $\sigma(P\phi(H)P) = \sigma(PHP)$ holds for every rank-1 projection P. This easily implies that

$$\langle\, \phi(H)x,\ x\rangle = \langle Hx, x\rangle, \quad \forall x \in \mathcal{H}.$$

Therefore $\phi(H) = H$ as claimed in the lemma. $\square$

Lemma 4.6. *If u and v are vectors in $\mathcal{H}$ such that $\langle u,\ v\rangle \neq -1$, then $\phi(I + u \otimes v) = I + u \otimes v$.*

Proof. Observe that $I + u \otimes v$ is an element of $\mathrm{GL}(\mathcal{H})$. We first assume that u and v are such that $\langle u,\ v\rangle \notin \mathbb{R}$. We apply the 2-local condition on ϕ to the pair $(I, I + u \otimes v)$, then there exists V_0 a unitary or an anti-unitary operator such that

$$\phi(I + u \otimes v) = V_0(I + u \otimes v)^{(*)} V_0^*.$$

We have two cases to consider: $\phi(I + u \otimes v) = I + V_0(u \otimes v)V_0^*$ and $\phi(I + u \otimes v) = I + V_0(v \otimes u)V_0^*$. For simplicity of notation we use $I + a \otimes b$ representing either case. In what follows we show that $a \otimes b = (u \otimes v)^{(*)}$.

We choose a finite rank self-adjoint operator H such that $-1 \notin \sigma(H)$ (i.e. $I + H$ is invertible). We apply the 2-local condition on ϕ to the pair $(I + u \otimes v, I + H)$. There exist V_1 and W_1, both unitary or anti-unitary operators, such that

$$\phi(I + u \otimes v) = V_1(I + u \otimes v)^{(*)} W_1 \quad \text{and} \quad \phi(I + H) = V_1(I + H)W_1.$$

Therefore

$$\begin{aligned}\phi(I+u\otimes v)\phi(I+H)^* &= V_1(I+u\otimes v)^{(*)}(I+H)V_1^*\\ &= V_1[I+(u\otimes v)^{(*)}](I+H)V_1^*\\ &= I+V_1[(u\otimes v)^{(*)}+H+(u\otimes v)^{(*)}H]V_1^*.\end{aligned}$$

On the other hand, since ϕ fixes the self-adjoint elements of $\mathrm{GL}(\mathcal{H})$, we also have

$$\begin{aligned}\phi(I+u\otimes v)\phi(I+H)^* &= (I+a\otimes b)(I+H)\\ &= I+a\otimes b+H+(a\otimes b)H.\end{aligned}$$

Therefore

$$a\otimes b+H+a\otimes bH = V_1[(u\otimes v)^{(*)}+H+(u\otimes v)^{(*)}H]V_1^*.$$

Applying the trace functional we get

$$tr[a\otimes b+H+a\otimes bH] = tr[V_1[(u\otimes v)^{(*)}+H+(u\otimes v)^{(*)}H]V_1^*].$$

If V_1 is unitary, then we have $tr(V_1CV_1^*) = tr\,C$ for every operator $C \in \mathrm{B}(\mathcal{H})$, while if V_1 is anti-unitary, then we have $tr(V_1CV_1^*) = \overline{tr\,C}$, for every C. Therefore, from the above displayed formula we deduce either

$$\langle a,\ b\rangle + \langle a, Hb\rangle = \langle u, v\rangle + \langle u, Hv\rangle$$

or

$$\langle a,\ b\rangle + \langle a, Hb\rangle = \overline{\langle u, v\rangle + \langle u, Hv\rangle}.$$

Therefore

$$\langle a,\ b\rangle + \langle a, Hb\rangle = (\langle u, v\rangle + \langle u, Hv\rangle)^{(*)} \tag{12}$$

holds for every finite rank self-adjoint operator H such that $-1 \notin \sigma(H)$. The appearance of the conjugation on the right hand side of the Eq. (12) may vary as H changes.

We set $\langle a,\ b\rangle = s_1 + is_2$, $\langle u, v\rangle = t_1 + it_2$, and write $\langle a, Hb\rangle = f_1(H) + if_2(H)$ and $\langle u, Hv\rangle = g_1(H) + ig_2(H)$ where f_1, f_2, g_1, g_2 are continuous real valued and real linear functionals on the space of all finite-rank self-adjoint operators (they are the real and purely complex parts of the functionals $H \mapsto \langle a, Hb\rangle$ and $H \mapsto \langle u, Hv\rangle$, respectively). Then (12) becomes

$$[s_1+f_1(H)] + i[s_2+f_2(H)] = [t_1+g_1(H)] \pm i[t_2+g_2(H)].$$

This implies that

$$s_1+f_1(H) = t_1+g_1(H) \quad \text{and} \quad s_2+f_2(H) = \pm(t_2+g_2(H)), \tag{13}$$

for every finite rank self-adjoint operator H such that $-1 \notin \sigma(H)$ and by the continuity of f_1, f_2, g_1, g_2, the same statement also holds for all finite rank self-adjoint operators. In particular for $H = 0$, (13) yields $s_1 = t_1$ and $s_2 = \pm t_2$. Furthermore, $s_1 = t_1$ implies $f_1(H) = g_1(H)$, for every finite rank self-adjoint operator H. There are two cases to consider: $s_2 = t_2$ and $s_2 = -t_2$. The details are very similar so we present the analysis for $s_2 = t_2$. If $f_2(H) = 0$ then either $g_2(H) = 0$ or $g_2(H) = -2s_2$. Therefore $f_2^{-1}\{0\} \subseteq g_2^{-1}\{0\} \bigcup g_2^{-1}\{-2s_2\}$. The linear subspace and the linear manifold that appear on the right hand side of this inclusion are either equal or disjoint. Therefore we necessarily have $f_2^{-1}\{0\} \subseteq g_2^{-1}\{0\}$. An elementary linear algebraic result implies that we then have $g_2 = \lambda f_2$, for some λ real number. We claim that $\lambda = \pm 1$. If $f_2 = 0$ then $g_2 = 0$. If $f_2 \neq 0$ then there exists H such that $f_2(H) = 1$. Therefore $s_2 + 1 = \pm(s_2+\lambda)$ and hence we have $\lambda = 1$ or $\lambda = -1-2s_2$. If $\lambda = -1-2s_2$ and $s_2 \neq 0$ then let H_0 be such that $f_2(H_0) = 2$. This leads to $s_2 + 2 = \pm(s_2+2\lambda)$ and $\lambda = 1$ or $\lambda = -1-s_2$. The latter equality would contradict $\lambda = -1-2s_2$. Consequently, we obtain $\lambda = \pm 1$ and hence $f_2 = \pm g_2$. Similarly, for $s_2 = -t_2$ we also conclude that $\lambda = \pm 1$ and $f_2 = \pm g_2$.

Therefore we either have $\langle a,\ Hb\rangle = \langle u, Hv\rangle$ for all finite rank self-adjoint operators H, or we have $\langle a,\ Hb\rangle = \overline{\langle u, Hv\rangle}$ for all finite rank self-adjoint operators H. Inserting any rank-1 projection P in the place of H, in the first case we deduce $a\otimes b = u\otimes v$ while in the second case we obtain $a\otimes b = v\otimes u = (u\otimes v)^*$. Let us assume that we have the second case, that is $\phi(I+u\otimes v) = I+v\otimes u$. We show that this leads to a contradiction.

Let $\epsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then, since ϕ is an isometry, we have

$$\|\phi(I+u\otimes v) - \phi(I+\epsilon u\otimes u)\| = \|u\otimes v - \epsilon u\otimes u\|.$$

On the other hand, $I+\epsilon u\otimes u$ is unitary and hence $\phi(I+\epsilon u\otimes u) = I+\epsilon u\otimes u$. This, together with the assumption $\phi(I+u\otimes v) = I+v\otimes u$ imply that

$$\|\phi(I+u\otimes v) - \phi(I+\epsilon u\otimes u)\| = \|v\otimes u - \epsilon u\otimes u\|.$$

Therefore, we obtain

$$\|v \otimes u - \epsilon u \otimes u\| = \|u \otimes v - \epsilon u \otimes u\|.$$

This equation can be written as $\|(v - \epsilon u) \otimes u\| = \|u \otimes (v - \bar{\epsilon} u)\|$. Therefore $\|v - \bar{\epsilon} u\| = \|v - \epsilon u\|$. Straightforward computation gives that $2\Re\bar{\epsilon}\langle u, v\rangle = 2\Re\epsilon\langle u, v\rangle$ from which we deduce that $\langle u, v\rangle = \langle v, u\rangle$. This contradicts our assumption that $\langle u, v\rangle \notin \mathbb{R}$, consequently we have $\phi(I + u \otimes v) = I + u \otimes v$ for any $u, v \in \mathcal{H}$ with $\langle u, v\rangle \notin \mathbb{R}$.

If u and v are such that $\langle u, v\rangle \in \mathbb{R}$, we choose a sequence of vectors $\{v_n\}$ converging to v such that $\langle u, v_n\rangle \notin \mathbb{R}$ $\big($for example, consider $v_n = v + \frac{1}{n} iu\big)$. Since ϕ is continuous, we conclude $\phi(I + u \otimes v) = I + u \otimes v$. This completes the proof. □

We are now in a position to prove the theorem of this section.

Proof of Theorem 4.1. Let A be an operator in $\mathrm{GL}(\mathcal{H})$ such that $-1 \notin \sigma(A)$, i.e. $I+A \in \mathrm{GL}(\mathcal{H})$. We show that $\phi(I+A) = I+A$. We apply the 2-local condition on ϕ to the pair $(I + A, I)$ to derive the following:

$$\phi(I + A) = V(I + A)^{(*)}V^*,$$

with V a unitary or anti-unitary operator on $\mathcal{H}$. This implies that $\sigma(\phi(I + A)) = \sigma(I + A)^{(*)} = 1 + \sigma(A)^{(*)}$ (we use the notation $\sigma(X)^{(*)}$ to represent either the set $\sigma(X)$ or its complex conjugate). Since A is invertible, we conclude $1 \notin \sigma(\phi(I+A))$. Therefore $\phi(I + A) - I \in \mathrm{GL}(\mathcal{H})$.

We select u and v in $\mathcal{H}$ such that $\langle u, v\rangle \neq -1$. Then $I + u \otimes v$ is invertible and the 2-local condition on ϕ applied to the pair $(I + A, I + u \otimes v)$ implies the existence of V and W both either unitary or anti-unitary operators on $\mathcal{H}$ such that

$$\phi(I + A) - \phi(I + u \otimes v) = V[A - u \otimes v]^{(*)}W.$$

Lemma 4.6 applies and we have $\phi(I + A) - I - u \otimes v = V[A - u \otimes v]^{(*)}W$. We set $B = \phi(I + A) - I$. Hence $B - u \otimes v$ is invertible if and only if $A - u \otimes v$ is invertible.

We now consider the following set

$$R = \{(u, v) \in \mathcal{H} \times \mathcal{H} : \langle A^{-1}u, v\rangle \neq 0\}.$$

Let $(u, v) \in R$ then $\langle A^{-1}u, v\rangle = \alpha \neq 0$. This implies that $\left\langle A^{-1}\frac{u}{\alpha}, v\right\rangle = 1$. Lemma 2.7 implies that $A - \frac{u}{\alpha} \otimes v$ is not invertible. Therefore $B - \frac{u}{\alpha} \otimes v$ is also not invertible. Lemma 2.7 yields $\left\langle B^{-1}\frac{u}{\alpha}, v\right\rangle = 1$. Therefore, we have

$$\langle A^{-1}u, v\rangle = \langle B^{-1}u, v\rangle, \quad \forall (u, v) \in R. \tag{14}$$

It is easy to see that the set R is dense in $\mathcal{H} \times \mathcal{H}$ and, by continuity, we have $\langle A^{-1}u, v\rangle = \langle B^{-1}u, v\rangle$ for all $(u, v) \in \mathcal{H} \times \mathcal{H}$. Consequently, $A = B$ and we obtain $\phi(I + A) = I + A$, for all $A \in \mathrm{GL}(\mathcal{H})$ such that $-1 \notin \sigma(A)$. This implies that $\phi(X) = X$ holds whenever $1, 0 \notin \sigma(X)$. If $X \in \mathrm{GL}(\mathcal{H})$ then there is a nonzero real number λ such that $0, \lambda \notin \sigma(X)$. Considering the operator $(1/\lambda)X$ and using the real homogeneity of ϕ we plainly obtain $\phi(X) = X$. The proof of the theorem is now complete. □

5. Automorphisms corresponding to isometries

This section is devoted to the proof of Theorem 1.3 which identifies the isometry groups under consideration with certain groups of automorphisms.

Proof of Theorem 1.3. Observe that the statements in the theorem are all "if and only if" assertions and in all the cases the "if" part is very simple to check. Therefore, in what follows we deal only with the necessity parts of the three statements.

The first assertion is proved in [24, Theorem 2.1].

As for the second one, consider the map $\Psi'(\cdot) = \sqrt{\Psi(I)}^{-1}\Psi(\cdot)\sqrt{\Psi(I)}^{-1}$. It is easy to see that this transformation is a continuous (in the operator norm) bijection of $\mathrm{B}(\mathcal{H})_+^{-1}$ which satisfies the equation (ii) in Theorem 1.3 and, in addition, it is unital, $\Psi'(I) = I$. It follows that $\Psi'(B^{-1}) = \Psi'(B)^{-1}$ and next that $\Psi'(ABA) = \Psi'(A)\Psi'(B)\Psi'(A)$ holds for all $A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$. The continuous bijections of $\mathrm{B}(\mathcal{H})_+^{-1}$ satisfying this equality have been determined in [21, Theorem 1]. Applying that result to Ψ' one can trivially complete the proof of the second statement. It remains to verify the third one.

Let $\Phi : \mathrm{GL}(\mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ be a uniformly continuous bijection satisfying the equation (iii) in Theorem 1.3. It is apparent that $A \in \mathrm{GL}(\mathcal{H})$ is unitary if and only if $AA^*A = A$. It follows that the restriction $\Phi_{|\mathrm{U}(\mathcal{H})} : \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$ is a continuous bijective map for which the first statement of the theorem applies. Therefore, we have V and W both unitary operators or both anti-unitary operators on $\mathcal{H}$ such that either $\Phi_{|\mathrm{U}(\mathcal{H})}(A) = VAW$ holds for all $A \in \mathrm{U}(\mathcal{H})$ or $\Phi_{|\mathrm{U}(\mathcal{H})}(A) = VA^*W$ holds for all $A \in \mathrm{U}(\mathcal{H})$. In the first case consider the map $V^*\Phi(\cdot)W^*$ while in the second case consider the transformation $V^*\Phi(\cdot)^*W^*$. In either case we have a uniformly continuous bijection of $\mathrm{GL}(\mathcal{H})$ that still satisfies the equation (iii) and acts as the identity on $\mathrm{U}(\mathcal{H})$. In what follows let us assume that already Φ has these properties. What we need to show is that Φ is then the identity on the whole group $\mathrm{GL}(\mathcal{H})$. We have $\Phi(I) = I$ implying $\Phi(B^*) = \Phi(B)^*$ and also $\Phi(A^2) = \Phi(A)^2$ for all $A, B \in \mathrm{GL}(\mathcal{H})$. Since the elements of $\mathrm{B}(\mathcal{H})_+^{-1}$ can be characterized as elements of $\mathrm{GL}(\mathcal{H})$ which are of the form $B = A^2$ for some $A \in \mathrm{GL}(\mathcal{H})$ with $A^* = A$, it follows that the restriction $\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}$ is a continuous bijection of $\mathrm{B}(\mathcal{H})_+^{-1}$ satisfying

$$\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(ABA) = \Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(A)\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(B)\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(A)$$

for all $A, B \in \mathrm{B}(\mathcal{H})_+^{-1}$. Therefore, [21, Theorem 1] applies again and there exists a unitary or anti-unitary operator W' on $\mathcal{H}$ such that either $\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(A) = W'AW'^*$ holds for all $A \in \mathrm{B}(\mathcal{H})_+^{-1}$ or $\Phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}(A) = W'A^{-1}W'^*$ holds for all $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. However, by the uniform continuity of Φ this second possibility is ruled out. So, we have $\Phi(A) = W'AW'^*$ for every $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. Let S be any symmetry (self-adjoint unitary) and A an arbitrary element of $\mathrm{B}(\mathcal{H})_+^{-1}$. Clearly, S, A commute if and only if $SAS = A$ which is equivalent to $\Phi(S)\Phi(A)\Phi(S) = \Phi(A)$. Since Φ acts as the identity on $\mathrm{U}(\mathcal{H})$, this is equivalent to $S\Phi(A)S = \Phi(A)$, i.e. to $S\Phi(A) = \Phi(A)S$. Since the symmetries are exactly the operators of the form $S = I - 2P$ with P being a projection, it follows that A and $\Phi(A) = W'AW'^*$ commute with the same projections and hence the commutants and then the second commutants of A and $\Phi(A) = W'AW'^*$ coincide. Choosing any projection P and considering $A = I + P$, it follows that the second commutants of P and $W'PW'^*$ coincide. In particular $W'PW'^*$ belongs to the second commutant of P which consists of operators of the form $\lambda P + \mu(I - P)$, λ, μ are scalars. If P is of finite rank, we easily deduce that in the equality $\lambda P + \mu(I - P) = W'PW'^*$ we necessarily have $\lambda = 1, \mu = 0$. Therefore $W'PW'^* = P$ holds for any finite-rank projection P. This easily implies the same equality for all projections and then by the spectral theorem we obtain that $W'AW'^* = A$ holds also for every $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. Therefore we have $\Phi(A) = A$ for any $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. Consequently, Φ acts as the identity on $\mathrm{U}(\mathcal{H})$ and also on $\mathrm{B}(\mathcal{H})_+^{-1}$. We need to show that Φ is the identity on the whole group $\mathrm{GL}(\mathcal{H})$. If $A \in \mathrm{GL}(\mathcal{H})$ is normal, then in the polar decomposition $A = U|A|$ we have that $U, |A|$ are commuting. Hence we infer $A = \sqrt{|A|}U\sqrt{|A|}$ and it follows that

$$\Phi(A) = \Phi\left(\sqrt{|A|}U^{**}\sqrt{|A|}\right) = \Phi\left(\sqrt{|A|}\right)\Phi(U^*)^*\Phi\left(\sqrt{|A|}\right) = \sqrt{|A|}U^{**}\sqrt{|A|} = A.$$

Since $\Phi : \mathrm{GL}(\mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ is uniformly continuous, it has a unique extension to a uniformly continuous transformation $\Phi' : \overline{\mathrm{GL}(\mathcal{H})} \to \overline{\mathrm{GL}(\mathcal{H})}$ which clearly satisfies the same equation (iii) in the theorem. Obviously, every projection P belongs to $\overline{\mathrm{GL}(\mathcal{H})}$ (for a nice general characterization of the elements of this closure we refer to [5]). Pick an arbitrary element $A \in \mathrm{GL}(\mathcal{H})$. For any rank-one projection P we have that PA^*P is a normal operator and using the continuity of Φ' we obtain $\Phi'(PA^*P) = PA^*P$, $\Phi'(P) = P$. Therefore

$$PA^*P = \Phi'(PA^*P) = \Phi'(P)\Phi(A)^*\Phi'(P) = P\Phi(A)^*P$$

is valid for every rank-one projection P. This easily implies that $\Phi(A) = A$ and the proof of the theorem is complete. □

6. Remarks, further reflexivity results

We conclude the paper with a few remarks. First of all observe that above we considered infinite dimensional spaces. One may naturally ask what happens in finite dimension. As for our reflexivity results, they remain true also in that case. This can be verified either following the proofs presented before (with elementary, rather trivial modifications) or using other techniques. For example, since in finite dimension the unitary group is compact and we have the strong property that every isometry of a compact metric space into itself is automatically surjective [7, Exercise 2.4.1], the corresponding reflexivity result follows immediately. (Observe that automatic surjectivity result fails trivially in noncompact spaces.) Concerning Thompson isometries, a similar approach can be followed: Assuming that $\phi : \mathrm{B}(\mathcal{H})_+^{-1} \to \mathrm{B}(\mathcal{H})_+^{-1}$ is a 2-local isometry with respect to the Thompson metric and supposing $\phi(I) = I$ and $\phi(2I) = 2I$, by Lemma 3.4 we see that ϕ preserves the order and ϕ is positive homogeneous. Therefore, for any λ, μ positive numbers with $\lambda < \mu$ we have that ϕ is an isometry from the operator interval $[\lambda I, \mu I]$ into itself. This interval is compact in the norm metric, so it is compact also in the Thompson metric (recall that the topologies induced by those two metrics coincide on $\mathrm{B}(\mathcal{H})_+^{-1}$). Therefore, we again can apply [7, Exercise 2.4.1] and deduce that ϕ maps $[\lambda I, \mu I]$ onto itself. Since this holds for all scalars $0 < \lambda < \mu$, we easily obtain that ϕ maps $\mathrm{B}(\mathcal{H})_+^{-1}$ onto itself.

Concerning automorphisms, we do not have the first proposition in Theorem 1.3 for the finite dimensional case (see [24]). The proof of the third proposition is based on the first one, hence neither we have it in finite dimension. Nevertheless we conjecture that both statements are true also in that case. Finally, the second statement in Theorem 1.3 is simply not true in finite dimension since the determinant may show up (see [21, Theorem 1]).

Finally, we mention that using the reflexivity results above one could obtain some additional ones relating to other groups of transformations. To demonstrate this, we present the following result on the algebraic reflexivity of the group of order automorphisms and that of the group of surjective isometries of the space of all positive semi-definite operators.

Denote by $\mathrm{B}(\mathcal{H})_+$ the cone of all positive semi-definite operators on $\mathcal{H}$. In the paper [20] we determined the structure of its order automorphisms. We proved that a bijective map $\Phi : \mathrm{B}(\mathcal{H})_+ \to \mathrm{B}(\mathcal{H})_+$ is an order automorphism, (i.e. for any $A, B \in \mathrm{B}(\mathcal{H})_+, A \le B \Leftrightarrow \Phi(A) \le \Phi(B)$) if and only if there exists T a linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ such that $\Phi(A) = TAT^*$ holds for all $A \in \mathrm{B}(\mathcal{H})_+$.

Concerning the surjective isometries of $\mathrm{B}(\mathcal{H})_+$ relative to the metric induced by the operator norm, we recall a nice result of Mankiewicz, namely, [19, Theorem 5] and the follow-up remark which states that if we have a surjective isometry between convex sets in normed real linear spaces with nonempty interiors, then this isometry can be uniquely extended to a surjective affine isometry between the whole spaces. Clearly, the result applies to $\mathrm{B}(\mathcal{H})_+$. Therefore, if $\Phi : \mathrm{B}(\mathcal{H})_+ \to \mathrm{B}(\mathcal{H})_+$ is a surjective isometry, then it can be extended to a surjective affine isometry $\tilde{\Phi} : \mathrm{B}(\mathcal{H})_s \to \mathrm{B}(\mathcal{H})_s$ (recall that $\mathrm{B}(\mathcal{H})_s$ stands for the space of all self-adjoint operators on $\mathcal{H}$). By affinity, Φ sends the unique extremal point 0 of $\mathrm{B}(\mathcal{H})_+$ to itself. It follows that $\tilde{\Phi}$ is in fact a real linear isometry and then the result [15, Theorem 2] can be used to verify that the unique complex linear extension of $\tilde{\Phi}$ to $\mathrm{B}(\mathcal{H})$ is a Jordan *-isomorphism. This implies that Φ is of the form $\Phi(A) = UAU^*$, $A \in \mathrm{B}(\mathcal{H})_+$ with some unitary or anti-unitary operator U on $\mathcal{H}$.

The last result of the paper reads as follows.

Theorem 6.1. *Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space. The group of all order automorphisms and the group of all surjective isometries of $\mathrm{B}(\mathcal{H})_+$ are both algebraically reflexive.*

Proof. We begin with the group of order automorphisms.

Let $\phi : \mathrm{B}(\mathcal{H})_+ \to \mathrm{B}(\mathcal{H})_+$ be a map such that for every pair (A, B) of elements of $\mathrm{B}(\mathcal{H})_+$ we have $T_{(A,B)}$ a linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ such that

$$\phi(A) = T_{(A,B)}AT^*_{(A,B)} \quad \text{and} \quad \phi(B) = T_{(A,B)}BT^*_{(A,B)}.$$

Clearly (see Theorem 1.2) the restriction $\phi_{|\mathrm{B}(\mathcal{H})_+^{-1}}$ is a 2-local Thompson isometry of $\mathrm{B}(\mathcal{H})_+^{-1}$. By Theorem 3.1 it is a surjective Thompson isometry and hence there exists T a linear or conjugate linear bounded and invertible operator on $\mathcal{H}$ such that either $\phi(A) = TAT^*$ for all $A \in \mathrm{B}(\mathcal{H})_+^{-1}$ or $\phi(A) = TA^{-1}T^*$ for all $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. By the local form of ϕ, for any $A, B \in \mathrm{B}(\mathcal{H})_+$ we have $\phi(A) \le \phi(B)$ if and only if $A \le B$. Therefore the second possibility above is ruled out and we have $\phi(A) = TAT^*$ for all $A \in \mathrm{B}(\mathcal{H})_+^{-1}$. Moreover, for any $A \in \mathrm{B}(\mathcal{H})_+$ and $B \in \mathrm{B}(\mathcal{H})_+^{-1}$ we have

$$T^{-1}\phi(A)T^{*-1} \le B \Leftrightarrow \phi(A) \le TBT^* = \phi(B) \Leftrightarrow A \le B.$$

It is easy to conclude that for every $A \in \mathrm{B}(\mathcal{H})_+$ this implies $T^{-1}\phi(A)T^{*-1} = A$ which yields $\phi(A) = TAT^*$. Consequently, ϕ is an order automorphism of $\mathrm{B}(\mathcal{H})_+$.

As for the isometry group, observe that it is a subgroup of the group of order automorphisms. Therefore, if $\phi : \mathrm{B}(\mathcal{H})_+ \to \mathrm{B}(\mathcal{H})_+$ is a 2-local isometry, then it is a 2-local order automorphism. It follows that ϕ is an order automorphism, in particular, ϕ is surjective. On the other hand, ϕ is an isometry, so ϕ is a surjective isometry of $\mathrm{B}(\mathcal{H})_+$ and this completes the proof. □

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References

[1] H. Al-Halees, R.J. Fleming, On 2-local isometries on continuous vector-valued function spaces, J. Math. Anal. Appl. 354 (2009) 70–77.
[2] E. Andruchow, G. Corach, D. Stojanoff, Geometrical significance of the Löwner–Heinz inequality, Proc. Amer. Math. Soc. 128 (2000) 1031–1037.
[3] B. Aupetit, A Primer on Spectral Theory, Springer-Verlag, New York, 1991.
[4] SH.A. Ayupov, K.K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl. 395 (2012) 15–18.
[5] R. Bouldin, Closure of the invertible operators in Hilbert space, Proc. Amer. Math. Soc. 108 (1990) 721–726.
[6] M. Brešar, P. Šemrl, On local automorphisms and mappings that preserve idempotents, Studia Math. 113 (1995) 101–108.
[7] M. Brin, G. Stuck, Introduction to Dynamical Systems, Cambridge Univ. Press, 2002.
[8] G. Corach, A. Maestripieri, D. Stojanoff, Orbits of positive operators from a differentiable viewpoint, Positivity 8 (2004) 31–48.
[9] A. Fošner, 2-local Jordan automorphisms on operator algebras, Studia Math. 209 (2012) 235–246.
[10] M. Győry, 2-local isometries of $C_0(X)$, Acta Sci. Math. (Szeged) 67 (2001) 735–746.
[11] O. Hatori, L. Molnár, Isometries of the unitary group, Proc. Amer. Math. Soc. 140 (2012) 2141–2154.
[12] O. Hatori, K. Watanabe, Isometries between groups of invertible elements in C^*-algebras, Studia Math. 209 (2012) 103–106.
[13] E. Ionascu, Rank one perturbations of diagonal operators, Integral Equations Operator Theory 39 (2001) 421–440.
[14] A. Jiménez-Vargas, M. Villegas-Vallecillos, 2-local isometries on spaces of Lipschitz functions, Canad. Math. Bull. 54 (2011) 680–692.
[15] R.V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. of Math. 56 (1952) 494–503.
[16] R.V. Kadison, Local derivations, J. Algebra 130 (1990) 494–509.
[17] D.R. Larson, Reflexivity, algebraic reflexivity and linear interpolation, Amer. J. Math. 110 (1988) 283–299.
[18] D.R. Larson, A.R. Sourour, Local derivations and local automorphisms of B(X), in: Proc. Sympos. Pure Math., Part 2, Vol. 51, Providence, Rhode Island, 1990, pp. 187–194.
[19] P. Mankiewicz, On extension of isometries in normed linear spaces, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. 20 (1972) 367–371.
[20] L. Molnár, Order-automorphisms of the set of bounded observables, J. Math. Phys. 42 (2001) 5904–5909.
[21] L. Molnár, Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators, Studia Math. 173 (2006) 39–48.

[22] L. Molnár, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, in: Lecture Notes in Mathematics, vol. 1895, Springer-Verlag, Berlin, Heidelberg, 2007.
[23] L. Molnár, Thompson isometries of the space of invertible positive operators, Proc. Amer. Math. Soc. 137 (2009) 3849–3859.
[24] L. Molnár, P. Šemrl, Transformations of the unitary group on a Hilbert space, J. Math. Anal. Appl. 388 (2012) 1205–1217.
[25] P. Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125 (1997) 2677–2680.
[26] A.C. Thompson, On certain contraction mappings in a partially ordered vector space, Proc. Amer. Math. Soc. 14 (1963) 438–443.