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On ground state and nodal solutions of Schrödinger–Poisson equations with critical growth ☆

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A B S T R A C T

In this paper, we study the Schrödinger–Poisson equation with a nonlinearity in the critical growth. By using variational methods, the existence of ground state solutions and nodal solutions is obtained.

1. Introduction

In this paper, we consider the following Schrödinger–Poisson equation:

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)|u|^{p-2}u + u^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (4, 6)$. Eq. (1.1) or the more general one

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x, u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.2}$$

arises while looking for the existence of standing waves for the Schrödinger equation interacting with an electrostatic field. For a more physical background of the problem, the reader may see [4] and the references therein.

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After the work of [4], many papers have been devoted to the existence of solutions for (1.2) under various assumptions on V, K and f. If $V(x) \equiv 1$, $K(x) \equiv 1$ and $f(x,u) = |u|^{p-2}u$, problem (1.2) has been studied sufficiently as p varies. For the case $4 \leq p < 6$, the existence of radial and non-radial solutions is considered in [8,9] and [11] respectively. For the case $p \leq 2$ or $p \geq 6$, the authors in [10] proved that (1.2) has no nontrivial solutions. In [20], the existence and non-existence of nontrivial solutions are also considered for the case $2 < p < 6$. The problem of finding ground state solutions is a very classical problem. Recall that u is a ground state solution of (1.2) if and only if u solves (1.2) and minimizes the functional associated with (1.2) among all nontrivial solutions. In [2], the authors obtained the existence of ground state solutions for the case $3 < p < 6$. The critical case was also considered if $f(x,u) = |u|^{p-2}u + u^5$ with $4 < p < 6$. In [23], we extended the work of [2] to a more general nonlinearity with critical growth. If $V(x)$ is not a constant, $K(x) \equiv 1$ and $f(x,u) = |u|^{p-2}u$, the existence of ground state solutions for (1.2) is obtained in [2] for $4 < p < 6$ and in [24] for $3 < p \leq 4$ respectively. The authors in [2] also considered the critical case. If $V(x) \equiv 1$, $K(x)$ is not a constant and $f(x,u) = a(x)|u|^{p-2}u$ with $4 < p < 6$, by requiring suitable assumptions on $K(x)$ and $a(x)$, the existence of ground state solutions and high energy solutions is investigated in [6]. For other results on the existence and multiplicity of solutions for the problem (1.2), the reader may see [3,5,16–18,21,26] for the subcritical case and [13,25] for the critical case. However, in all these papers mentioned above, nodal solutions are not considered. Recently, the authors in [14] obtained the existence of nodal solutions in the critical case.

One aim of this paper is to consider the existence of ground state solutions for (1.1) in the critical case. The critical exponential growth makes the problem complicated due to the lack of compactness. More important, since the functions $K(x)$ and $a(x)$ are not radial symmetric, it is impossible to work in the radial symmetric space. We need to investigate the influence of the interaction of $K(x)$ and $a(x)$ on the existence of ground state solutions. In this paper, we also consider the existence of nodal solutions for (1.1). Problem (1.1) is rather different from that of [14]. We use a different method to deal with the problem. In order to obtain the existence of ground state and nodal solutions, we introduce the following hypotheses on $K(x)$ and $a(x)$:

- (K_1) There exist $C_1 > 0$ and $a > 0$ such that $0 \leq K(x) \leq C_1 e^{-a|x|}$ for all $x \in \mathbb{R}^3$;
- (K_2) $K(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $K(x) \geq 0$ and $\lim_{|x|\to\infty} K(x) = 0$;
- (a_1) $a(x) \in C(\mathbb{R}^3)$ and $\lim_{|x|\to\infty} a(x) = a_\infty > 0$;
- (a_2) there exist $C_2 > 0$ and $b > 0$ such that $a(x) \geq a_\infty + C_2 e^{-b|x|}$ for all $x \in \mathbb{R}^3$;
- (a_3) $a(x) \geq a_\infty$ for all $x \in \mathbb{R}^3$ and $\text{meas}\{x \in \mathbb{R}^3; a(x) > a_\infty\} > 0$.

Firstly, we consider the existence of ground state solutions.

Theorem 1.1. *Assume (K_1) and (a_1)–(a_2) with $0 < b < a < 2$. Then problem (1.1) admits a positive ground state solution.*

Theorem 1.2. *Assume (K_2), (a_1) and (a_3). Then problem (1.1) admits a positive ground state solution for $\|K\|_2$ small enough.*

Remark 1.1. In Theorem 1.1, the decay rate assumptions (K_1) and (a_2) are the key to the energy estimate. In Theorem 1.2, without any decay rate assumption, we still obtain a ground state solution by requiring $\|K\|_2$ small enough. It is remarkable that (K_2) and (a_3) are more general than (K_1) and (a_2).

Now we consider the existence of nodal solutions.

Theorem 1.3. *Assume (K_1) and (a_1)–(a_2) with $0 < b < a < 1$. Then problem (1.1) admits a positive ground state solution and a nodal solution.*

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The outline of this paper is as follows: in Section 2, we establish some important lemmas. In Section 3, we prove Theorems 1.1–1.2. In Section 4, we prove Theorem 1.3.

Notations:

- $\|u\|_s := \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{\frac{1}{s}}$, $2 \le s \le \infty$.
- Let $H^1(\mathbb{R}^3)$ be the Hilbert space with the norm $\|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x$.
- Let $D^{1,2}(\mathbb{R}^3) := \left\{u \in L^6(\mathbb{R}^3); \nabla u \in L^2(\mathbb{R}^3)\right\}$ be the Sobolev space with the norm $\|u\|^2_{D^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$.
- C denotes a universal positive constant.
- $B_r(y)$ denotes the open ball centered at y with radius $r > 0$.
- S denotes the best Sobolev constant:

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}}}.$$

2. Preliminary lemmas

In this section, we assume (K_2), (a_1) and (a_3). By the Lax–Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta\phi_u = K(x)u^2$. Similar to the argument in [20], we can derive that the function ϕ_u has the following properties.

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, we have*

(i) $\phi_u \ge 0$;
(ii) $\phi_{tu} = t^2\phi_u$, $\forall\ t > 0$;
(iii) $\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x \le C\|u\|^4_\alpha$, *where* $\alpha = \frac{12}{5}$.

For simplicity, we may assume $a_\infty = 1$. Denote $H = H^1(\mathbb{R}^3)$. The functional associated with (1.1) is

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int\limits_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{p}\int\limits_{\mathbb{R}^3} a(x)|u|^p \mathrm{d}x - \frac{1}{6}\int\limits_{\mathbb{R}^3} |u|^6 \mathrm{d}x, \tag{2.1}$$

where $u \in H$. It is easy to check that $I : H \mapsto \mathbb{R}$ is of class C^1 and $(u, \phi) \in H \times D^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H$ is a critical point of I and $\phi = \phi_u$.

Lemma 2.2. *There is a sequence $\{u_n\} \subset H$ such that $\{u_n\}$ is bounded in H, $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I'(u_n) \to 0$.*

Proof. From (a_1) and (a_3), $\forall\ \varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\frac{1}{p}a(x)|u|^p + \frac{1}{6}|u|^6 \le \varepsilon|u|^2 + C(\varepsilon)|u|^6. \tag{2.2}$$

By (2.2) and Sobolev embedding theorem, we can conclude that there exists $r_0 > 0$ such that $I(u) \ge c_0 > 0$ for $\|u\| = r_0$. Set $\varphi \in H$ such that $\varphi \ge 0$, $\varphi \ne 0$. By Lemma 2.1(*ii*), we have $\lim_{t\to+\infty} I(t\varphi) = -\infty$. We also have $I(0) = 0$. Define

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H); \gamma(0) = 0, I(\gamma(1)) < 0\}$. The mountain pass in [1] implies that there is a sequence $\{u_n\} \subset H$ satisfying $I(u_n) \to c \geq c_0$ and $I'(u_n) \to 0$. Similar to Lemma 2.5 in [23], we have $c < \frac{1}{3}S^{\frac{3}{2}}$. By $I(u_n) \to c$ and $I'(u_n) \to 0$,

$$c + o(1)\|u_n\| = I(u_n) - \frac{1}{4}(I'(u_n), u_n) \geq \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in H. $\square$

Define the functional

$$I^{\infty}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int\limits_{\mathbb{R}^3} |u|^p \mathrm{d}x - \frac{1}{6}\int\limits_{\mathbb{R}^3} |u|^6 \mathrm{d}x, \qquad u \in H.$$

Lemma 2.3. *Let $\{u_n\} \subset H$ be a sequence such that $\|u_n\|$ is bounded, $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I'(u_n) \to 0$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, an integer $k \in \mathbb{N} \cup \{0\}$, $v^i \in H$ for $1 \leq i \leq k$ satisfying*

(i) *$u_n \rightharpoonup u$ weakly in H with $I'(u) = 0$,*
(ii) *$v^i \neq 0$ and ${I^{\infty}}'(v^i) = 0$ for $1 \leq i \leq k$,*
(iii) *$c = I(u) + \sum_{i=1}^{k} I^{\infty}(v^i)$,*

where we agree that in the case $k = 0$, the above holds without v^i.

Proof. From $\|u_n\|$ is bounded, we know $u_n \rightharpoonup u$ weakly in H up to a subsequence. Then $I'(u) = 0$ and (i) holds.

Set $v_n^1 = u_n - u$. By the Brezis–Lieb lemma in [22], we obtain that

$$\begin{aligned}
&\|v_n^1\|^2 = \|u_n\|^2 - \|u\|^2 + o(1), \\
&\int\limits_{\mathbb{R}^3} |v_n^1|^6 \mathrm{d}x = \int\limits_{\mathbb{R}^3} |u_n|^6 \mathrm{d}x - \int\limits_{\mathbb{R}^3} |u|^6 \mathrm{d}x + o(1), \\
&\int\limits_{\mathbb{R}^3} a(x)|v_n^1|^p \mathrm{d}x = \int\limits_{\mathbb{R}^3} a(x)|u_n|^p \mathrm{d}x - \int\limits_{\mathbb{R}^3} a(x)|u|^p \mathrm{d}x + o(1).
\end{aligned} \tag{2.3}$$

Combining (a_1), $v_n^1 \to 0$ in $L^p_{loc}(\mathbb{R}^3)$ and third equality of (2.3), we have

$$\int\limits_{\mathbb{R}^3} |v_n^1|^p \mathrm{d}x = \int\limits_{\mathbb{R}^3} a(x)|u_n|^p \mathrm{d}x - \int\limits_{\mathbb{R}^3} a(x)|u|^p \mathrm{d}x + o(1). \tag{2.4}$$

Lemma 2.2 in [24] implies that

$$\int\limits_{\mathbb{R}^3} K(x)\phi_{v_n^1}(v_n^1)^2 \mathrm{d}x = \int\limits_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x + o(1).$$

By $\lim_{|x|\to\infty} K(x) = 0$, we have $\forall\, \varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int\limits_{|x|\geq R(\varepsilon)} K(x)\phi_{v_n^1}(v_n^1)^2 \mathrm{d}x \leq \varepsilon.$$

We also have

$$\lim_{n\to\infty} \int\limits_{|x|\le R(\varepsilon)} K(x)\phi_{v_n^1}(v_n^1)^2 \mathrm{d}x$$
$$\le \lim_{n\to\infty} C \left(\int\limits_{|x|\le R(\varepsilon)} |\phi_{v_n^1}|^6 \mathrm{d}x \right)^{\frac{1}{6}} \left(\int\limits_{|x|\le R(\varepsilon)} |v_n^1|^{\frac{12}{5}} \mathrm{d}x \right)^{\frac{5}{6}} = 0.$$

Thus,

$$\int\limits_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x = o(1). \tag{2.5}$$

Combining (2.3)–(2.5), there holds

$$c - I(u) = I^\infty(v_n^1) + o(1). \tag{2.6}$$

On the other hand, by elliptic estimates in [12], we get $u \in L^\infty(\mathbb{R}^3)$. Since the proof is standard, we omit it here. Then from Lemma 8.9 in [22], we have

$$\left| \int\limits_{\mathbb{R}^3} \left[u_n^5 - u^5 - (v_n^1)^5\right] \varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H. \tag{2.7}$$

Similar to Lemma 8.9 in [22], we also have

$$\left| \int\limits_{\mathbb{R}^3} a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u - |v_n^1|^{p-2}v_n^1)\varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H.$$

The condition (a_1) implies that

$$\left| \int\limits_{\mathbb{R}^3} (a(x) - 1)|v_n^1|^{p-2}v_n^1 \varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H.$$

Then

$$\left| \int\limits_{\mathbb{R}^3} \left[a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u) - |v_n^1|^{p-2}v_n^1\right] \varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H. \tag{2.8}$$

By $\lim_{|x|\to\infty} K(x) = 0$, we can derive that

$$\left| \int\limits_{\mathbb{R}^3} K(x)\phi_{v_n^1} v_n^1 \varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H.$$

Lemma 2.2 in [24] implies that

$$\left| \int\limits_{\mathbb{R}^3} K(x) \left[\phi_{u_n} u_n - \phi_u u - \phi_{v_n^1} v_n^1\right] \varphi \mathrm{d}x \right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H.$$

Thus,

$$\left|\int_{\mathbb{R}^3} K(x)\left[\phi_{u_n}u_n - \phi_u u\right]\varphi \mathrm{d}x\right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H. \tag{2.9}$$

Combining (2.7)–(2.9), there holds

$$\left|(I'(u_n) - I'(u), \varphi) - ({I^{\infty}}'(v_n^1), \varphi)\right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H,$$

which implies that

$${I^{\infty}}'(v_n^1) = o(1). \tag{2.10}$$

We will consider two cases.

Case 1. $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|v_n^1|^2\mathrm{d}x = 0$.

Applying the Lions lemma, we obtain that

$$v_n^1 \to 0 \ \text{ in } \ L^t(\mathbb{R}^3), \quad \forall\, t \in (2,6). \tag{2.11}$$

Combining (2.6), (2.11) and $\left({I^{\infty}}'(v_n^1), v_n^1\right) = o(1)$, there holds $c - I(u) = \frac{1}{2}\|v_n^1\|^2 - \frac{1}{6}\int_{\mathbb{R}^3}|v_n^1|^6\mathrm{d}x + o(1)$ and $\|v_n^1\|^2 - \int_{\mathbb{R}^3}|v_n^1|^6\mathrm{d}x = o(1)$. We assume that $\|v_n^1\|^2 \to l$. Now we prove $l = 0$. In fact, if $l > 0$, then Sobolev embedding theorem implies that $l \geq S^{\frac{3}{2}}$. By $I'(u) = 0$, we have $I(u) \geq 0$. Then $c \geq c - I(u) = \frac{1}{3}l \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction with $c < \frac{1}{3}S^{\frac{3}{2}}$. From $l = 0$, we have $c = I(u)$.

Case 2. There exists $\gamma_1 > 0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|v_n^1|^2\mathrm{d}x \geq \gamma_1 > 0.$$

In this case, there exists $y_n^1 \in \mathbb{R}^3$ with $|y_n^1| \to \infty$ such that $\int_{B_1(y_n^1)}|v_n^1|^2\mathrm{d}x \geq \frac{\gamma_1}{2} > 0$, from which we derive that $v_n^1(.+y_n^1) \rightharpoonup v^1 \neq 0$ weakly in H and

$$\begin{aligned} &c - I(u) = I^{\infty}(v_n^1(.+y_n^1)) + o(1), \\ &{I^{\infty}}'(v_n^1(.+y_n^1)) = o(1). \end{aligned} \tag{2.12}$$

Then ${I^{\infty}}'(v^1) = 0$. Set $v_n^2 = v_n^1(.+y_n^1) - v^1$. We also have $\|v_n^2\|^2 = \|v_n^1\|^2 - \|v^1\|^2 + o(1)$. Together with the first equality of (2.3), there holds

$$\|v_n^2\|^2 = \|u_n\|^2 - \|u\|^2 - \|v^1\|^2 + o(1). \tag{2.13}$$

Similar to the prove of (2.6) and (2.10), we can derive that

$$\begin{aligned} &c - I(u) - I^{\infty}(v^1) + o(1) = I^{\infty}(v_n^2), \\ &{I^{\infty}}'(v_n^2) = o(1). \end{aligned} \tag{2.14}$$

Note that either

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|v_n^2|^2\mathrm{d}x = 0, \tag{2.15}$$

or there exists $\gamma_2 > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int\limits_{B_1(y)} |v_n^2|^2 \mathrm{d}x \geq \gamma_2 > 0. \tag{2.16}$$

If (2.15) holds, similar to the argument of Case 1, we have $c = I(u) + I^\infty(v^1)$. So we may assume (2.16) holds. Then there exists $y_n^2 \in \mathbb{R}^3$ with $|y_n^2| \to \infty$ such that $v_n^2(. + y_n^2) \rightharpoonup v^2 \neq 0$ weakly in H, $I^{\infty'}(v^2) = 0$ and

$$\begin{aligned}
&c - I(u) - I^\infty(v^1) - I^\infty(v^2) = I^\infty(v_n^3) + o(1), \\
&I^{\infty'}(v_n^3) = o(1), \\
&\|v_n^3\|^2 = \|u_n\|^2 - \|u\|^2 - \|v^1\|^2 - \|v^2\|^2 + o(1),
\end{aligned}$$

where $v_n^3 = v_n^2(. + y_n^2) - v^2$. Continuing this process, we obtain $v_n^i \in H$, $y_n^i \in \mathbb{R}^3$ with $|y_n^i| \to \infty$ such that $v_n^i(. + y_n^i) \rightharpoonup v^i \neq 0$ weakly in H, $I^{\infty'}(v^i) = 0$ and

$$\begin{aligned}
&c - I(u) - \sum_{i=1}^{j} I^\infty(v^i) = I^\infty(v_n^{j+1}) + o(1), \\
&I^{\infty'}(v_n^{j+1}) = o(1) \\
&\|v_n^{j+1}\|^2 = \|u_n\|^2 - \|u\|^2 - \sum_{i=1}^{j} \|v^i\|^2 + o(1),
\end{aligned} \tag{2.17}$$

where $v_n^{j+1} = v_n^j(. + y_n^j) - v^j$, $j \in N$. From $\left(I^{\infty'}(v^i), v^i\right) = 0$, (2.2) and Sobolev embedding theorem, we can derive that there exists $\beta > 0$ independent of i such that $\|v^i\|^2 \geq \beta > 0$. Then by the third equality of (2.17), we know $v_n^{j+1} \to 0$ at some $j = k$. Together with the first equality of (2.17), we have $c = I(u) + \sum_{i=1}^{k} I^\infty(v^i)$. □

3. Proof of Theorems 1.1–1.2

Let $m_\infty = \inf_{u\in N_\infty} I^\infty(u)$, where $N_\infty = \{u \in H \setminus \{0\}; (I^{\infty'}(u), u) = 0\}$. Similar to the proof of Theorem 1.7 in [2], we can prove there exists $u_\infty \in H \setminus \{0\}$ satisfying $I^\infty(u_\infty) = m_\infty$ and $I^{\infty'}(u_\infty) = 0$. Since u_∞ is not sign-changing, we may assume $u_\infty \geq 0$ in H. The Maximum Principle implies that u_∞ is positive.

Lemma 3.1. *For any $\delta \in (0, 1)$, there exists $C = C(\delta) > 0$ such that*

$$u_\infty(x) \leq Ce^{-(1-\delta)|x|}.$$

Proof. By elliptic estimates in [12], we can derive that $u_\infty \in L^\infty(\mathbb{R}^3)$ and $u_\infty(x) \to 0$ as $|x| \to \infty$. Since the proof is standard, we omit it here. Then for any $\delta > 0$, there exists $R = R(\delta) > 0$ such that for $|x| \geq R$, $1 - u_\infty^{p-2} - u_\infty^4 \geq (1-\delta)^2$. Thus, we have $-\Delta u_\infty + (1-\delta)^2 u_\infty \leq 0$ for $|x| \geq R$ and there exists $M = M(\delta) > 0$ such that $u_\infty(x) \leq M$ for $|x| = R$. Let $v(x) = Me^{-(1-\delta)(|x|-R)}$. Direct calculation can derive that $-\Delta v + (1-\delta)^2 v \geq 0$ for $x \neq 0$. From the Maximum Principle, we have $u_\infty(x) \leq Me^{-(1-\delta)(|x|-R)}$ for $|x| \geq R$. Then Lemma 3.1 follows easily. □

Proof of Theorem 1.1. From Lemma 2.2, we know there is a bounded sequence $\{u_n\} \subset H$ satisfying $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I'(u_n) \to 0$. From Lemma 2.3, we can derive that I satisfies the Palais–Smale condition at $c \in (0, m_\infty)$. Then if $c \in (0, m_\infty)$, we have $u_n \to u$ in H, $I(u) = c$ and $I'(u) = 0$.

Now we prove $c < m_\infty$. Let $\gamma = (1, 0, 0)$ be a fixed unit vector in $\mathbb{R}^3$. By the definition of c, we have $c \le \sup_{t\ge 0} I(tu_\infty(x - R\gamma))$. From (a_2), Lemma 2.1(iii) and Sobolev embedding theorem,

$$I(u) \le \frac{1}{2}\|u\|^2 + C\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x. \tag{3.1}$$

Then there exists $t' \in (0, 1)$ such that

$$\sup_{0\le t\le t'} I(tu_\infty(x - R\gamma)) \le \frac{1}{2}|t'|^2\|u_\infty\|^2 + C|t'|^4\|u_\infty\|^4 < m_\infty \tag{3.2}$$

independent of $R > 0$. From (3.1), we also have that there exists $t'' > 0$ such that

$$\sup_{t\ge t''} I(tu_\infty(x - R\gamma)) < m_\infty \tag{3.3}$$

independent of $R > 0$. Observe that

$$\begin{aligned} I(tu) &= I^\infty(tu) + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{p}t^p \int_{\mathbb{R}^3} (a(x) - 1)|u|^p \mathrm{d}x \\ &\le I^\infty(tu) + \frac{t^4}{4}\left(\int_{\mathbb{R}^3} |\phi_u|^6 \mathrm{d}x\right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} K(x)^{\frac{6}{5}}|u|^{\frac{12}{5}} \mathrm{d}x\right)^{\frac{5}{6}} \\ &\quad - \frac{1}{p}t^p \int_{\mathbb{R}^3} (a(x) - 1)|u|^p \mathrm{d}x. \end{aligned}$$

By (K_1) and (a_2),

$$\begin{aligned} I(tu_\infty(x - R\gamma)) \le I^\infty(tu_\infty) + Ct^4 &\left(\int_{\mathbb{R}^3} e^{-\frac{6a}{5}|x+R\gamma|}|u_\infty|^{\frac{12}{5}} \mathrm{d}x\right)^{\frac{5}{6}} \\ &- \frac{C_2}{p}t^p \int_{\mathbb{R}^3} e^{-b|x+R\gamma|}|u_\infty|^p \mathrm{d}x. \end{aligned}$$

Set $l(t) = I^\infty(tu_\infty)$, where $t \in (0, \infty)$. Note that $l(t)$ has a unique critical point corresponding to its maximum. Since $l'(1) = 0$, this critical point should be achieved at $t = 1$. Then $\sup_{t\ge 0} l(t) = I^\infty(u_\infty) = m_\infty$. Choose $\delta \in (0, 1 - \frac{a}{2})$. By Lemma 3.1, we have

$$\left(\int_{\mathbb{R}^3} e^{-\frac{6a}{5}|x+R\gamma|}|u_\infty|^{\frac{12}{5}} \mathrm{d}x\right)^{\frac{5}{6}} \le C\left(\int_{\mathbb{R}^3} e^{-\frac{6a}{5}R} e^{[\frac{6a}{5} - \frac{12}{5}(1-\delta)]|x|} \mathrm{d}x\right)^{\frac{5}{6}} \le Ce^{-aR}.$$

We also have

$$\int_{\mathbb{R}^3} e^{-b|x+R\gamma|}|u_\infty|^p \mathrm{d}x \ge e^{-bR}\int_{|x|\le 1} e^{-b|x|}|u_\infty|^p \mathrm{d}x \ge Ce^{-bR}.$$

Thus,

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\beta)) \le m_\infty + Ce^{-aR} - Ce^{-bR}. \tag{3.4}$$

Choose R large enough, we have $\sup_{t' \le t \le t''} I(tu_\infty(x - R\beta)) < m_\infty$. Combining (3.2)–(3.4), we get $c < m_\infty$.

Let $m = \inf\{I(v); v \in H, v \neq 0, I'(v) = 0\}$. Since $I'(u) = 0$, we have $0 \le m \le I(u) < m_\infty$. By the definition of m, there exists $\{v_n\} \subset H$ such that $v_n \neq 0$, $I(v_n) \to m$ and $I'(v_n) = 0$. From $(I'(v_n), v_n) = 0$, (2.2) and Sobolev embedding theorem, we can derive that there exists $\beta > 0$ independent of n such that $\|v_n\| \ge \beta$. Thus,

$$m = I(v_n) - \frac{1}{4}(I'(v_n), v_n) + o(1) \ge \frac{1}{4}\|v_n\|^2 + o(1) \ge \frac{1}{4}\beta^2 + o(1),$$

which implies that $m > 0$. Since I satisfies the Palais–Smale condition at $c \in (0, m_\infty)$, we have $v_n \to v \neq 0$ in H and m is attained by v. It is clear that v is not sign-changing. Thus, we may assume $v \ge 0$ in H. The Maximum Principle implies that v is positive. □

Now we prove Theorem 1.2. Similar argument as Proposition 4.2 in [15] can derive the following result.

Lemma 3.2. *There exists $\gamma \in C([0,1], H)$ such that $\gamma(0) = 0$, $I^\infty(\gamma(1)) < 0$, $u_\infty \in \gamma([0,1])$ and $\max_{t\in[0,1]} I^\infty(\gamma(t)) = I^\infty(u_\infty) = m_\infty$. Moreover, $0 \notin \gamma((0,1])$.*

Define the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x, \qquad u \in H. \tag{3.5}$$

Lemma 3.3. *The functional J admits a nontrivial critical point $w \in H$ satisfying $J(w) \in (0, m_\infty)$ and $\|w\|^2 \le \frac{2pJ(w)}{p-2}$.*

Proof. From Lemma 2.2, we know that there is a sequence $\{w_n\} \subset H$ satisfying $w_n \rightharpoonup w$ weakly in H, $J(w_n) \to \bar{c} \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $J'(w_n) \to 0$, where $\bar{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], H); \gamma(0) = 0, J(\gamma(1)) < 0\}$. Lemma 2.3 implies that J satisfies the Palais–Smale condition at $\bar{c} \in (0, m_\infty)$. Now we claim $\bar{c} \in (0, m_\infty)$. From Lemma 3.2, we know there exists $\gamma \in C([0,1], H)$ such that $\gamma(0) = 0$, $I^\infty(\gamma(1)) < 0$, $0 \notin \gamma((0,1])$ and $\max_{t\in[0,1]} I^\infty(\gamma(t)) = m_\infty$. It is clear that $\gamma \in \Gamma$. By (a_3), we have $\bar{c} \le \max_{t\in[0,1]} J(\gamma(t)) < \max_{t\in[0,1]} I^\infty(\gamma(t)) = m_\infty$. Thus, we have $w_n \to w$ in H, $J(w) = \bar{c} \in (0, m_\infty)$ and $J'(w) = 0$. From

$$J(w) = J(w) - \frac{1}{p}(J'(w), w) \ge \left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^2,$$

we get $\|w\|^2 \le \frac{2pJ(w)}{p-2}$. □

Proof of Theorem 1.2. By Lemma 2.2, there is a bounded sequence $\{u_n\} \subset H$ satisfying $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I'(u_n) \to 0$. We claim that $c < m_\infty$. In fact, from the definition of c, we only need to prove $c \le \sup_{t\ge 0} I(tw) < m_\infty$, where w is obtained in Lemma 3.3.

Set $h(t) = I(tw)$, where $t \in (0, \infty)$. It is obvious that $h(t)$ attains its maximum at $t_0 \in (0, \infty)$. Then $h'(t_0) = 0$, which implies that

$$t_0^2\|w\|^2 \le t_0^p \int_{\mathbb{R}^3} a(x)|w|^p \mathrm{d}x + t_0^6 \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x. \tag{3.6}$$

We claim that $t_0 \geq 1$. In fact, if $t_0 \in (0,1)$, then by (3.6) and $(J'(w), w) = 0$, we have

$$t_0^2\|w\|^2 < t_0^6 \left(\int_{\mathbb{R}^3} a(x)|w|^p \mathrm{d}x + \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x \right) = t_0^6\|w\|^2,$$

a contradiction with $t_0 \in (0,1)$. Then $t_0 \geq 1$. From $t_0 \geq 1$, $h'(t_0) = 0$ and $(J'(w), w) = 0$, we have

$$\begin{aligned} h(t_0) &= \frac{1}{4}t_0^2\|w\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) t_0^p \int_{\mathbb{R}^3} a(x)|w|^p \mathrm{d}x + \frac{1}{12}t_0^6 \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x \\ &\leq t_0^6 \left(J(w) - \frac{1}{4}(J'(w), w) \right) = t_0^6 J(w). \end{aligned}$$

We also have

$$\begin{aligned} & t_0^4 \left(\|w\|^2 + \int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x \right) \\ & \geq t_0^2\|w\|^2 + t_0^4 \int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x = t_0^p \int_{\mathbb{R}^3} a(x)|w|^p \mathrm{d}x + t_0^6 \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x \\ & \geq t_0^p \left(\int_{\mathbb{R}^3} a(x)|w|^p \mathrm{d}x + \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x \right) = t_0^p\|w\|^2, \end{aligned}$$

which implies that $t_0^{p-4} \leq 1 + \frac{\int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x}{\|w\|^2}$. Thus,

$$h(t_0) \leq t_0^6 J(w) \leq \left(1 + \frac{\int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x}{\|w\|^2} \right)^{\frac{6}{p-4}} J(w). \tag{3.7}$$

Observe that

$$\begin{aligned} S\|\phi_w\|_6^2 &\leq \int_{\mathbb{R}^3} |\nabla \phi_w|^2 \mathrm{d}x = \int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x \\ &\leq \|K\|_2 \|\phi_w\|_6 \|w\|_6^2 \leq \frac{1}{S}\|K\|_2 \|\phi_w\|_6 \|w\|^2, \end{aligned}$$

which implies that $\|\phi_w\|_6 \leq \frac{1}{S^2}\|K\|_2\|w\|^2$. Thus,

$$\int_{\mathbb{R}^3} K(x)\phi_w w^2 \mathrm{d}x \leq \frac{1}{S^3}\|K\|_2^2\|w\|^4. \tag{3.8}$$

Combining (3.7)–(3.8) and Lemma 3.3, we have

$$h(t_0) \leq \left(1 + \frac{1}{S^3}\|K\|_2^2\|w\|^2 \right)^{\frac{6}{p-4}} J(w) < m_\infty$$

for $\|K\|_2$ small enough. The rest of the proof is similar to the proof of Theorem 1.1, we omit it here. □

4. Proof of Theorem 1.3

In this section, we consider the existence of nodal solutions for (1.1). We use an idea of [7]. Denote $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. Define the functional $f(u)$ on H by

$$f(u) = \begin{cases} \frac{\int_{\mathbb{R}^3} a(x)|u|^p \mathrm{d}x + \int_{\mathbb{R}^3} |u|^6 \mathrm{d}x}{\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x}, & u \neq 0, \\ 0, & u = 0. \end{cases}$$

Then we define

$$N^* = \{u \in H; f(u^+) = f(u^-) = 1\}$$

and

$$U = \left\{u \in H; |f(u^\pm) - 1| < \frac{1}{2}\right\}.$$

By (a_1)–(a_2), we know there exists $D > 0$ such that $0 \leq a(x) \leq D$. Then for $u \in U$, we have

$$\frac{1}{2}\|u^\pm\|^2 < \int_{\mathbb{R}^3} a(x)|u^\pm|^p \mathrm{d}x + \int_{\mathbb{R}^3} |u^\pm|^6 \mathrm{d}x \leq \int_{\mathbb{R}^3} (D|u^\pm|^p + |u^\pm|^6)\mathrm{d}x.$$

Note that for $p \in (4, 6)$, there hold $\lim_{t\to 0} \frac{|t|^p}{t^2} = 0$ and $\lim_{t\to +\infty} \frac{|t|^p}{t^6} = 0$. Then $\forall\ \varepsilon > 0$, there exist $0 < r_1(\varepsilon) < r_2(\varepsilon)$ such that $|t|^p \leq \varepsilon(t^2 + t^6)$ for $|t| \in [0, r_1(\varepsilon)] \cup [r_2(\varepsilon), +\infty)$. Since $|t|^p \leq \frac{r_2(\varepsilon)^p}{r_1(\varepsilon)^6} t^6$ for $|t| \in [r_1(\varepsilon), r_2(\varepsilon)]$, we derive $|t|^p \leq \varepsilon t^2 + \left(\varepsilon + \frac{r_2(\varepsilon)^p}{r_1(\varepsilon)^6}\right) t^6$. Hence, we obtain that $\forall\ \varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\frac{1}{2}\|u^\pm\|^2 \leq \varepsilon \int_{\mathbb{R}^3} |u^\pm|^2 \mathrm{d}x + C(\varepsilon) \int_{\mathbb{R}^3} |u^\pm|^6 \mathrm{d}x.$$

Set $\varepsilon < \frac{1}{2}$. By Sobolev embedding theorem, we have

$$\left(\int_{\mathbb{R}^3} |u^\pm|^6 \mathrm{d}x\right)^{\frac{1}{3}} \leq C \int_{\mathbb{R}^3} |u^\pm|^6 \mathrm{d}x.$$

Thus, for $u \in U$, there exists $\varrho > 0$ such that $\int_{\mathbb{R}^3} |u^\pm|^6 \mathrm{d}x \geq \varrho > 0$.

Lemma 4.1. *Assume $m < m_\infty$. If $\{u_n\} \subset U$ be a sequence such that $\|u_n\|$ is bounded, $I(u_n) \to c \in (0, m + m_\infty)$ and $I'(u_n) \to 0$, then $u_n \to u$ in H.*

Proof. From $\|u_n\|$ is bounded, we know $u_n \rightharpoonup u$ weakly in H and $I'(u) = 0$. Set $v_n = u_n - u$. Similar to the proof of Lemma 2.3, we have

$$c = I(u) + I^\infty(v_n) + o(1) \tag{4.1}$$

and

$$I^{\infty\prime}(v_n) = o(1). \tag{4.2}$$

If $v_n \to 0$ strongly in H, then Lemma 4.1 holds. So we may assume v_n converges weakly (and not strongly) to 0 in H. Then either v_n^+ converges weakly (and not strongly) to 0 in H, or v_n^- converges weakly (and not strongly) to 0 in H. We will consider three cases.

Case 1. v_n^+ converges weakly (and not strongly) to 0 in H, $v_n^- \to 0$ strongly in H.

We claim that $u_n \rightharpoonup u \neq 0$ weakly in H. In fact, if $u_n \rightharpoonup 0$ weakly in H, then $u_n^- = v_n^- \to 0$ strongly in H, a contradiction with $\int_{\mathbb{R}^3} |u_n^-|^6 \mathrm{d}x \geq \varrho > 0$. So $u_n \rightharpoonup u \neq 0$ weakly in H. From (4.2), we have

$$\|v_n^+\|^2 = \int_{\mathbb{R}^3} |v_n^+|^p \mathrm{d}x + \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x + o(1). \tag{4.3}$$

Observe that there exists $t_n^+ \in (0, \infty)$ such that $t_n^+ v_n^+ \in N_\infty$. Then

$$(t_n^+)^2 \|v_n^+\|^2 = (t_n^+)^p \int_{\mathbb{R}^3} |v_n^+|^p \mathrm{d}x + (t_n^+)^6 \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x. \tag{4.4}$$

Now we prove that $t_n^+ \to 1$. Up to a subsequence, we may assume $\lim_{n\to\infty} \|v_n^+\|$ and $\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x$ exist. From (4.3), we know $\forall\, \varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\|v_n^+\|^2 \leq \varepsilon \int_{\mathbb{R}^3} |v_n^+|^2 \mathrm{d}x + C(\varepsilon) \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x + o(1).$$

Choose $\varepsilon > 0$ small enough, we have

$$\|v_n^+\|^2 \leq C \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x + o(1).$$

Since v_n^+ converges weakly (and not strongly) to 0 in $L^6(\mathbb{R}^3)$, we can derive that $\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x > 0$. Then by (4.4), we have $(t_n^+)^4 \leq \frac{\|v_n^+\|^2}{\int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x}$, from which we get t_n^+ is bounded. Without loss of generality, we may assume $t_n^+ \to t^+ \geq 0$. By (4.3)–(4.4),

$$\left[(t_n^+)^{p-2} - 1\right] \int_{\mathbb{R}^3} |v_n^+|^p \mathrm{d}x + \left[(t_n^+)^4 - 1\right] \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x = o(1).$$

If $t^+ \neq 1$, then

$$\left[(t^+)^{p-2} - 1\right] \lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n^+|^p \mathrm{d}x + \left[(t^+)^4 - 1\right] \lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x = 0,$$

a contradiction with $\lim_{n\to\infty} \int_{\mathbb{R}^3} |v_n^+|^6 \mathrm{d}x > 0$. Then $t_n^+ \to 1$. Thus, by (4.1) and $t_n^+ \to 1$, we have

$$\begin{aligned} c &= I(u) + I^\infty(v_n^+) + o(1) \\ &= I(u) + I^\infty(t_n^+ v_n^+) + o(1) \\ &\geq m + m_\infty + o(1), \end{aligned}$$

a contradiction with $c < m + m_\infty$.

Case 2. $v_n^+ \to 0$ strongly in H, v_n^- converges weakly (and not strongly) to 0 in H.

The proof is similar to Case 1, we omit it.

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Case 3. v_n^+ converges weakly (and not strongly) to 0 in H, v_n^- converges weakly (and not strongly) to 0 in H.

Similar to the proof of Case 1, we can derive that there exists $s_n^\pm \in (0,\infty)$ such that $s_n^\pm v_n^\pm \in N_\infty$ and $s_n^\pm \to 1$. Then by (4.1),

$$\begin{aligned} c &\geq I^\infty(v_n) + o(1) \\ &= I^\infty(v_n^+) + I^\infty(v_n^-) + o(1) \\ &= I^\infty(s_n^+ v_n^+) + I^\infty(s_n^- v_n^-) + o(1) \\ &\geq 2m_\infty + o(1) > m + m_\infty + o(1), \end{aligned}$$

a contradiction with $c < m + m_\infty$. $\square$

Following the idea of [7], we give some definitions. Denote P the cone of non-negative functions in H. Let $Q = [0,1] \times [0,1]$. Define

$$\begin{aligned} \Sigma = \{\, \sigma \in C(Q,H);\ \sigma(t,0) = 0, \sigma(0,s) \in P, \sigma(1,s) \in -P, \\ I(\sigma(t,1)) \leq 0, f(\sigma(t,1)) \geq 2, \forall\, t,s \in [0,1]\}. \end{aligned}$$

Choose $u \in H$ such that $u^\pm \neq 0$. Let $\sigma(t,s) = ks(1-t)u^+ - kstu^-$, where $k > 0$, $t,s \in [0,1]$. It is easy to check that $\sigma \in \Sigma$ for $k > 0$ large enough.

Lemma 4.2. $\inf_{u\in N^*} I(u) = \inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u)$.

Proof. From the definition of Σ, we have $\forall\, \sigma \in \Sigma$, $\forall\, s \in [0,1]$,

$$f(\sigma^+(0,s)) - f(\sigma^-(0,s)) = f(\sigma^+(0,s)) \geq 0$$

and

$$f(\sigma^+(1,s)) - f(\sigma^-(1,s)) = -f(\sigma^-(1,s)) \leq 0.$$

On the other hand, from the definition of Σ, we also have $\forall\, \sigma \in \Sigma$, $\forall\, t \in [0,1]$,

$$f(\sigma^+(t,0)) + f(\sigma^-(t,0)) - 2 = -2 < 0$$

and

$$f(\sigma^+(t,1)) + f(\sigma^-(t,1)) - 2 = f(\sigma(t,1)) - 2 \geq 0.$$

Then from Miranda's theorem in [19], we know $\forall\, \sigma \in \Sigma$, there exists $(\bar{t},\bar{s}) \in Q$ such that

$$f(\sigma^+(\bar{t},\bar{s})) - f(\sigma^-(\bar{t},\bar{s})) = 0 = f(\sigma^+(\bar{t},\bar{s})) + f(\sigma^-(\bar{t},\bar{s})) - 2,$$

which implies that $\sigma(\bar{t},\bar{s}) \in N^*$. Thus,

$$\inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u) \geq \inf_{u\in N^*} I(u). \tag{4.5}$$

Conversely, $\forall\ \bar{u} \in N^*$, choose $\bar{\sigma} \in \Sigma$ such that $\bar{\sigma}(Q) \subset \{\alpha\bar{u}^+ - \beta\bar{u}^-; \alpha \geq 0, \beta \geq 0\}$. Set $g(t) = I(t\bar{u}^\pm)$, where $t \in (0, \infty)$. It is easy to check that $g(t)$ has a unique critical point $\bar{t}^\pm$ corresponding to its maximum. From $\bar{u} \in N^*$, we have $g'(1) = 0$, which implies that $\bar{t}^\pm = 1$. Thus, $\forall\ \bar{u} \in N^*$,

$$\begin{aligned} I(\bar{u}) &= I(\bar{u}^+) + I(\bar{u}^-) \\ &= \sup_{\alpha\geq 0} I(\alpha\bar{u}^+) + \sup_{\beta\geq 0} I(\beta\bar{u}^-) \\ &\geq \sup_{\alpha,\beta\geq 0} \left[I(\alpha\bar{u}^+) + I(\beta\bar{u}^-)\right] \\ &= \sup_{\alpha,\beta\geq 0} I(\alpha\bar{u}^+ - \beta\bar{u}^-) \\ &\geq \sup_{u\in\bar{\sigma}(Q)} I(u) \geq \inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u), \end{aligned}$$

which implies that

$$\inf_{u\in N^*} I(u) \geq \inf_{\sigma\in\Sigma} \sup_{u\in\sigma(Q)} I(u). \tag{4.6}$$

From (4.5)–(4.6), we know Lemma 4.2 holds. □

Lemma 4.3. *There is a sequence $\{u_n\} \subset U$ such that $I(u_n) \to c^* = \inf_{u\in N^*} I(u)$ and $I'(u_n) \to 0$.*

Proof. From Lemma 4.2 and the proof of (4.6), we know there is a sequence $\{\bar{u}_n\} \subset N^*$ and $\bar{\sigma}_n \in \Sigma$ such that

$$\lim_{n\to\infty} \max_{u\in\bar{\sigma}_n(Q)} I(u) = \lim_{n\to\infty} I(\bar{u}_n) = c^*. \tag{4.7}$$

Standard argument can derive that there exists $\{u_n\} \subset H$ such that

$$I(u_n) \to c^*, \quad I'(u_n) \to 0 \ \text{ and } \ \mathrm{dist}(u_n, \bar{\sigma}_n(Q)) \to 0. \tag{4.8}$$

The reader may see [7] for the details of the proof. Now we only need to prove $\{u_n\} \subset U$ for n large enough. By (4.7)–(4.8), there exists a sequence $v_n = \alpha_n\bar{u}_n^+ - \beta_n\bar{u}_n^- \in \bar{\sigma}_n(Q)$ such that

$$I(v_n) \to c^* \ \text{ and } \ \|v_n - u_n\| \to 0. \tag{4.9}$$

Note that for $\bar{u}_n \in N^* \subset U$, we have $\int_{\mathbb{R}^3} |\bar{u}_n^\pm|^6 \mathrm{d}x \geq \varrho > 0$. Then

$$I(\bar{u}_n^\pm) = I(\bar{u}_n^\pm) - \frac{1}{4}(I'(\bar{u}_n^\pm), \bar{u}_n^\pm) \geq \frac{1}{4}\|\bar{u}_n^\pm\|^2 \geq \frac{1}{4} S^2 \varrho^{\frac{2}{3}}.$$

Thus, we may assume $I(\bar{u}_n^+) \to c_1^* > 0$ and $I(\bar{u}_n^-) \to c_2^* = c^* - c_1^* > 0$. By $\bar{u}_n \in N^*$, we also have $I(\bar{u}_n^+) \geq I(\alpha_n\bar{u}_n^+) = I(v_n^+)$ and $I(\bar{u}_n^-) \geq I(\beta_n\bar{u}_n^-) = I(v_n^-)$. Thus,

$$\begin{aligned} c^* &= \lim_{n\to\infty} I(\bar{u}_n) = \lim_{n\to\infty} [I(\bar{u}_n^+) + I(\bar{u}_n^-)] \\ &\geq \lim_{n\to\infty} [I(v_n^+) + I(v_n^-)] = \lim_{n\to\infty} I(v_n) = c^*, \end{aligned}$$

from which we have $\lim_{n\to\infty} I(v_n^+) = \lim_{n\to\infty} I(\bar{u}_n^+) = c_1^*$ and $\lim_{n\to\infty} I(v_n^-) = \lim_{n\to\infty} I(\bar{u}_n^-) = c_2^*$. By (4.9), we get $\|v_n^\pm - u_n^\pm\| \to 0$. Thus, we have $\lim_{n\to\infty} I(u_n^+) = c_1^*$ and $\lim_{n\to\infty} I(u_n^-) = c_2^*$, which implies that $u_n^\pm \neq 0$. Together with $(I'(u_n^\pm), u_n^\pm) = o(1)$, we have $\{u_n\} \subset U$ for n large enough. □

Proof of Theorem 1.3. It is obvious that Theorem 1.1 holds. Thus, problem (1.1) admits a positive ground state solution u_1. Now we prove (1.1) admits a nodal solution. From Lemma 4.3, there is a sequence $\{u_n\} \subset U$ such that $I(u_n) \to c^* = \inf_{u \in N^*} I(u)$ and $I'(u_n) \to 0$. By $\{u_n\} \subset U$, we have $\int_{\mathbb{R}^3} |u_n^{\pm}|^6 \mathrm{d}x \geq \varrho > 0$. Thus, if $u_n \to u$ in H, then $I'(u) = 0$ and $\int_{\mathbb{R}^3} |u^{\pm}|^6 \mathrm{d}x > 0$, which implies that u is a nodal solution of (1.1). Now we prove $u_n \to u$ in H. From Lemmas 4.1–4.2, we only need to prove

$$\sup_{\alpha \geq 0, \beta \in \mathbb{R}} I(\alpha u_1 + \beta u_\infty(. - R\gamma)) < m + m_\infty \tag{4.10}$$

for R large enough, where $\gamma = (1, 0, 0)$ is the fixed unit vector in $\mathbb{R}^3$. By elliptic estimates in [12], we can derive $u \in L^\infty(\mathbb{R}^3)$. Since the proof is standard, we omit it here. Denote $v(x) = u_\infty(x - R\gamma)$. Direct calculation can derive that

$$\begin{aligned} I(\alpha u_1 + \beta v) = {} & I(\alpha u_1) + I^\infty(\beta v) - \frac{|\beta|^p}{p} \int\limits_{\mathbb{R}^3} (a(x) - 1)|v|^p \mathrm{d}x \\ & - \frac{1}{p} \int\limits_{\mathbb{R}^3} a(x) \left(|\alpha u_1 + \beta v|^p - |\alpha u_1|^p - |\beta v|^p - p\alpha\beta |u_1|^{p-2} u_1 v \right) \mathrm{d}x \\ & - \frac{1}{6} \int\limits_{\mathbb{R}^3} \left(|\alpha u_1 + \beta v|^6 - |\alpha u_1|^6 - |\beta v|^6 - 6\alpha\beta |u_1|^4 u_1 v \right) \mathrm{d}x \\ & + \frac{1}{4} \left[\int\limits_{\mathbb{R}^3} K(x) \phi_{\alpha u_1 + \beta v} (\alpha u_1 + \beta v)^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x) \phi_{\alpha u_1} (\alpha u_1)^2 \mathrm{d}x \right] \\ & - \alpha\beta \int\limits_{\mathbb{R}^3} K(x) \phi_{u_1} u_1 v \mathrm{d}x \\ \leq {} & I(\alpha u_1) + I^\infty(\beta u_\infty) - \frac{|\beta|^p}{p} \int\limits_{\mathbb{R}^3} (a(x) - 1)|v|^p \mathrm{d}x \\ & + \frac{1}{4} \left[\int\limits_{\mathbb{R}^3} K(x) \phi_{\alpha u_1 + \beta v} (\alpha u_1 + \beta v)^2 \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x) \phi_{\alpha u_1} (\alpha u_1)^2 \right] \\ & - \alpha\beta \int\limits_{\mathbb{R}^3} K(x) \phi_{u_1} u_1 v \mathrm{d}x. \end{aligned} \tag{4.11}$$

From (K_1), (a_2) and Lemma 2.1(iii),

$$I(\alpha u_1 + \beta v) \leq I(\alpha u_1) + I^\infty(\beta u_\infty) + C\|\alpha u_1 + \beta v\|^4 + C|\alpha\beta|. \tag{4.12}$$

By (4.12), we can derive that there exists $M > 0$ large enough such that

$$\sup_{|\alpha| + |\beta| \geq M} I(\alpha u_1 + \beta v) < m + m_\infty \tag{4.13}$$

independent of $R > 0$. From (K_1), $u_1 \in L^\infty(\mathbb{R}^3)$ and Lemma 3.1, we have

$$\sup_{0 \leq \alpha \leq M, |\beta| \leq M} |\alpha\beta| \left| \int\limits_{\mathbb{R}^3} K(x) \phi_{u_1} u_1 v \mathrm{d}x \right|$$

$$
\begin{aligned}
&\leq M^2\|\phi_{u_1}\|_6 \left[\int_{\mathbb{R}^3} K(x)^{\frac{6}{5}}(u_1 v)^{\frac{6}{5}}\mathrm{d}x\right]^{\frac{5}{6}} \\
&\leq C\left(\int_{\mathbb{R}^3} e^{-\frac{6}{5}a|x+R\gamma|}e^{-\frac{6}{5}(1-\delta)|x|}\mathrm{d}x\right)^{\frac{5}{6}} \\
&\leq C\left(\int_{\mathbb{R}^3} e^{-\frac{6}{5}aR}e^{[\frac{6}{5}a-\frac{6}{5}(1-\delta)]|x|}\mathrm{d}x\right)^{\frac{5}{6}}.
\end{aligned}
\tag{4.14}
$$

Choose $\delta \in (0, 1-a)$, there holds

$$
\sup_{0\leq\alpha\leq M,|\beta|\leq M} |\alpha\beta| \left|\int_{\mathbb{R}^3} K(x)\phi_{u_1}u_1 v\mathrm{d}x\right| \leq Ce^{-aR}. \tag{4.15}
$$

Set

$$
L = \frac{1}{4}\left[\int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1+\beta v}(\alpha u_1+\beta v)^2\mathrm{d}x - \int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1}(\alpha u_1)^2\right].
$$

Then

$$
\begin{aligned}
L = {} & \alpha\beta\int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1}u_1 v\mathrm{d}x + \alpha\beta\int_{\mathbb{R}^3} K(x)\phi_{\beta v}u_1 v\mathrm{d}x \\
& + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\beta v}(\beta v)^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1}(\beta v)^2\mathrm{d}x \\
& + \alpha^2\beta^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)u_1(x)v(x)K(y)u_1(y)v(y)}{|x-y|}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Now we estimate L. Observe that

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)u_1(x)v(x)K(y)u_1(y)v(y)}{|x-y|}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_{\sqrt{u_1 v}}K(x)u_1 v\mathrm{d}x.
$$

Thus, similar to (4.14)–(4.15), we have

$$
\begin{aligned}
&\sup_{0\leq\alpha\leq M,|\beta|\leq M} \alpha^2\beta^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)u_1(x)v(x)K(y)u_1(y)v(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leq Ce^{-aR}, \\
&\sup_{0\leq\alpha\leq M,|\beta|\leq M} |\alpha\beta|\left|\int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1}u_1 v\mathrm{d}x\right| \leq Ce^{-aR}, \\
&\sup_{0\leq\alpha\leq M,|\beta|\leq M} |\alpha\beta|\left|\int_{\mathbb{R}^3} K(x)\phi_{\beta v}u_1 v\mathrm{d}x\right| \leq Ce^{-aR}.
\end{aligned}
\tag{4.16}
$$

Similar to the estimate of $\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x$ in the proof of Theorem 1.1, we also have

$$\sup_{0\le\alpha\le M,|\beta|\le M}\left|\int_{\mathbb{R}^3} K(x)\phi_{\beta v}(\beta v)^2\mathrm{d}x\right| \le Ce^{-aR},$$
$$\sup_{0\le\alpha\le M,|\beta|\le M}\left|\int_{\mathbb{R}^3} K(x)\phi_{\alpha u_1}(\beta v)^2\mathrm{d}x\right| \le Ce^{-aR}. \tag{4.17}$$

Combining (4.11) and (4.15)–(4.17),

$$I(\alpha u_1+\beta v)\le I(\alpha u_1)+I^\infty(\beta v)-\frac{|\beta|^p}{p}\int_{\mathbb{R}^3}(a(x)-1)|v|^p\mathrm{d}x+Ce^{-aR}. \tag{4.18}$$

Since u_1 is a ground state solution of (1.1), we have $\sup_{\alpha\ge 0} I(\alpha u_1)=I(u_1)=m$. Thus, by (4.18) and $a(x)\ge 1$, we derive that there exists $|\beta_0|\in(0,M)$ small enough and $R_1>0$ large enough such that for $R>R_1$,

$$\sup_{|\beta|\le|\beta_0|,|\alpha|+|\beta|\le M} I(\alpha u_1+\beta v)<m+m_\infty. \tag{4.19}$$

We also have $\sup_{\beta\in\mathbb{R}} I^\infty(\beta v)=I^\infty(u_\infty)=m_\infty$. Thus, there exist $\alpha_0\in(0,M)$ small enough and $R_2>0$ such that for $R>R_2$,

$$\sup_{0\le\alpha\le\alpha_0,|\alpha|+|\beta|\le M} I(\alpha u_1+\beta v)<m+m_\infty. \tag{4.20}$$

In view of (4.13) and (4.19)–(4.20), we only need to prove

$$\sup_{\alpha_0\le\alpha\le M,|\beta_0|\le|\beta|\le M} I(\alpha u_1+\beta v)<m+m_\infty \tag{4.21}$$

for R large enough. Similar to the proof of Theorem 1.1, we have

$$\begin{aligned}&\sup_{|\beta_0|\le|\beta|\le M} -\frac{|\beta|^p}{p}\int_{\mathbb{R}^3}(a(x)-1)|v|^p\mathrm{d}x\\ &\le -\frac{C_2|\beta_0|^p}{p}\int_{\mathbb{R}^3} e^{-b|x+R\gamma|}|u_\infty|^p\mathrm{d}x\le -Ce^{-bR}.\end{aligned} \tag{4.22}$$

Combining (4.18) and (4.22),

$$\begin{aligned}\sup_{\alpha_0\le\alpha\le M,|\beta_0|\le|\beta|\le M} I(\alpha u_1+\beta v)&\le \sup_{\alpha\ge 0} I(\alpha u_1)+\sup_{\beta\in\mathbb{R}} I^\infty(\beta u_\infty)+Ce^{-aR}-Ce^{-bR}\\ &=I(u_1)+I^\infty(u_\infty)+Ce^{-aR}-Ce^{-bR}\\ &=m+m_\infty+Ce^{-aR}-Ce^{-bR}.\end{aligned}$$

Choose $R>\max\{R_1,R_2\}$ large enough, we know (4.21) holds. Thus, Theorem 1.3 is proved. □

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