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On super fixed point property and super weak compactness of convex subsets in Banach spaces

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A B S T R A C T

For a nonempty convex set C of a Banach space X, a self-mapping on C is said to a linear (respectively, affine) isometry if it is the restriction of a linear (respectively, affine) isometry defined on the whole space X. By means of super weakly compact set theory established in the recent years, in this paper, we first show that a nonempty closed bounded convex set of a Banach space has super fixed point property for affine (or, equivalently, linear) isometries if and only if it is super weakly compact; and the super fixed point property and the super weak compactness coincide on every closed bounded convex subset of a Banach space under equivalent renorming sense. With the application of Fabian–Montesinos–Zizler's renorming theorem, we finally show that every strongly super weakly compact generated Banach space can be renormed so that every weakly compact convex set has super fixed point property.

1. Introduction

Speaking of fixed points for continuous mappings, the famous Brouwer fixed point theorem states that every continuous mapping from a convex compact subset K of a Euclidean space to K itself has a fixed point. A more general form is known as Schauder fixed point theorem: Every continuous mapping from a convex compact subset K of a locally convex space to K itself has a fixed point [30]. The converse version of Schauder fixed point theorem in Banach spaces was proven by Lin and Sternfeld [27]: If a closed bounded convex set K in a Banach space satisfies that every Lipschitz mapping from K to itself has a fixed point, then K is necessarily compact. This theorem announces that we cannot extend the Schauder fixed point theorem for general continuous self-mappings to a more general class of convex subsets. Combining with Schauder fixed point theorem, Dominguez, Japon Pineda and Prus [11] characterized weak compactness of a closed bounded convex subset in a Banach space by fixed point property for continuous affine mappings.

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The Banach fixed point theorem for contractive mappings says that every α-Lipschitz self-mapping with $0 \le \alpha < 1$ defined on a closed convex subset of a Banach space has a unique fixed point. Since 60s of the last century, mathematicians have focused on searching for fixed point theorems of non-expansive mappings (i.e. 1-Lipschitz mappings) defined on a more general class of closed bounded convex sets of a Banach space: A Banach space is said to have the fixed point property (FPP) provided every non-expansive self-mapping defined on a bounded closed convex subset of the space has a fixed point. Recall that a closed convex subset K of a Banach space X is said to have normal structure provided that for every bounded closed convex subset $C \subset K$ with a positive diameter $\mathrm{diam}(C)$, there is $x \in C$ so that $d_C(x) \equiv \sup_{c\in C} \|x - c\| < \mathrm{diam}(C)$. A remarkable result was proven by Kirk [20]: Every weakly compact convex set of a Banach space with the normal structure has the fixed point property. (See, also, Belluce and Kirk [3].) Browder showed independently that every uniformly convex Banach space has the fixed point property [5].

Mathematicians also often consider super fixed point property of Banach spaces. (See, for instance, [2,21,32,33].) We say that a Banach space Y is finitely representable in another Banach space X, provided for every $\varepsilon > 0$ and for every subspace $F \subset Y$ of finite dimension, there is a finite dimension subspace E of X such that the Banach–Mazur distance $d(E, F) < 1+\varepsilon$. A Banach space X is said to have the super fixed point property if every Banach space Y which is finitely representable in X, has the fixed point property. Maurey (in one of his unpublished paper) first showed that every super reflexive Banach space has the super fixed point property for isometries; Elton, Lin, Odell and Szarek [12] gave Maurey's theorem a different proof. van Dulst and Pach [31] showed the converse part is also true. Thus, the super reflexivity of a Banach space can be characterized by the super fixed point property for isometries, instead of non-expansive mappings. This characterization, Enflo's renorming theorem of convexity [13] and Browder's fixed point theorem [5] together entail that a Banach space can be renormed to have the super fixed point property if and only if X is super reflexive.

In the recent years, parallel to weak compactness of subsets in Banach spaces, a notion of super weakly compact sets was introduced in [7] and further studied in [8,9]. A relatively super weakly compact set of a Banach space acts much like a bounded subset of a super reflexive space. Therefore, super weak compactness of subsets can be regarded as a localized setting of super reflexivity of Banach spaces. It is also shown in [8] that the super weak compactness of a bounded convex set is equivalent to finite-slice-index property (introduced by Raja [29]), and equivalent to finite-dual-index property (introduced by Fabian, Montesinos and Zizler [15]).

By means of super weakly compact set theory, in this paper, we first show that a nonempty closed bounded convex set of a Banach space has super fixed point property for linear isometries if and only if it is super weakly compact. We verify then the super fixed point property and the super weak compactness coincide on every closed bounded convex subset of a Banach space under equivalent renorming sense: For a nonempty closed bounded convex set C of a Banach space X, there is an equivalent norm on X such that C has the super fixed point property with respect to the new norm if and only if C is super weakly compact. With the application of Fabian–Montesinos–Zizler's renorming theorem, we finally prove that every strongly super weakly compact generated Banach space can be renormed to have super weak fixed point property.

In this paper, all notations are standard. The letter X will always be a real Banach space and X^* its dual. B_X and S_X denote the closed unit ball and the unit sphere of X, respectively. For a subset $A \subset X$, $\overline{A}$ and $\mathrm{co}(A)$ present the closure and the convex hull of A, respectively.

2. Preliminaries

In this section, we shall present a series of notions and some properties concerning localized finite representability, slicing indexes, super weak compactness, uniform convexity and smoothness which will be used in Sections 3 and 4.

Recall a Banach space Y is said to be finitely representable in another Banach space X provided for every $\varepsilon > 0$ and for every finite dimensional subspace F of Y, there is a finite dimensional subspace E of X such that the Banach–Mazur distance $d(E, F) < 1 + \varepsilon$. The notion of finite representability is localized to convex subsets by substituting simplexes for finite dimensional subspaces in [7], then it is extended to general subsets of Banach spaces in [8].

An n-simplex S of a linear space is the convex hull $\mathrm{co}(x_0, x_1, \cdots, x_n)$ of $n+1$ affinely independent vectors $x_0, x_1, \cdots, x_n \in X$, i.e. $x_1 - x_0, \cdots, x_n - x_0$ are linearly independent. A 0-simplex is just a singleton $\{x_0\}$. If no confusion arises, we call an n-simplex simply "simplex". We use $\mathrm{aff}(A)$ ($\mathrm{co}(A)$, resp.) to denote the affine (convex, resp.) hull of A, i.e.

$$\mathrm{aff}(A) = \{\sum_{j=1}^{n} \alpha_j x_j : x_j \in A, \alpha_j \in \mathbb{R}, \sum_{j=1}^{n} \alpha_j = 1, 1 \le j \le n, n \in \mathbb{N}\}.$$

Definition 2.1. Let $A \subset X$ and $B \subset Y$ be two subsets of Banach spaces X and Y. We say that B is finitely representable in A if for every $\varepsilon > 0$ and for every simplex S_B with vertices in B there exist a simplex S_A with vertices in A, and an affine mapping $T : \mathrm{aff}(S_B) \to \mathrm{aff}(S_A)$ such that $T(S_B) = S_A$ and such that

$$(1 - \varepsilon)\|x - y\| \le \|Tx - Ty\| \le (1 + \varepsilon)\|x - y\|, \ \forall x, y \in \mathrm{aff}(S_B). \tag{2.1}$$

Definition 2.2. A (weakly closed, resp.) subset A of a Banach space X is said to be relatively super weakly compact (super weakly compact, resp.) provided every subset B of a Banach space Y which is finitely representable in A is relatively weakly compact.

This concept incorporating of [8, Corollary 2.15] entails that in a Banach space X every compact set is super weakly compact; X is super-reflexive if and only if its closed unit ball is super weakly compact; and if X is super reflexive then every bounded set is relatively super weakly compact.

As we have well known, the weak closure of a subset in an infinite dimensional Banach space could be much larger than the subset. Fortunately, we have the following property.

Theorem 2.3. *(See [7, Proposition 3.10].) A subset of a Banach space is relatively super weakly compact is equivalent to that the weak closure of the subset is super weakly compact.*

To avoid difficulty of "weak closure", we often use "relative super weak compactness", instead of "super weak compactness" in the following discussion.

Theorem 2.4. *(See [7, Theorem 4.1].) A subset A of a Banach space X is relatively super weakly compact if (and only if) for every $\varepsilon > 0$ there is a relatively super weakly compact set B such that $A \subset B + \varepsilon B_X$.*

The slicing indices, as variants of the Szlenk index have found many applications in the geometric theory of Banach spaces (see, e.g., [7,15,17,18,22–24,28,29]). The following notion of finite index property can be found in Raja [29], and finite dual index property was introduced by Fabian, Montesinos and Zizler [15].

Let E be a normed space, $F \subset E^*$ be a subspace and let $B \subset E$ be a nonempty bounded set. A $\sigma(E, F)$-slice of B is a subset $S \subset B$ of the form:

$$S = S(B, x^*, \alpha) = \{x \in B : \langle x^*, x\rangle > \alpha\}, \text{ for some } x^* \in F, \ \alpha \in \mathbb{R}. \tag{2.2}$$

In particular, a $\sigma(E, E^*)$-slice is simply called a slice, and a $\sigma(E^*, E)$-slice is said to be a w^*-slice whenever $B \subset E^*$.

Let ϱ be a seminorm defined on E. We denote by $d_\varrho(B)$ the ϱ-diameter of B. Given $\varepsilon > 0$, we define $\sigma(E,F)$–ϱ–ε ($\sigma(E,F)$–ε, resp.)-dentability derivative $[B]'_{(\sigma(E,F),\ \varrho,\ \varepsilon)}$ of B as follows:

$$[B]'_{(\sigma(E,F),\ \varrho,\ \varepsilon)} = \{x \in B : d_\varrho(S) > \varepsilon, \forall \sigma(E,F)\text{-slice } S \text{ of } B \text{ containing } x\}. \tag{2.3}$$

We often omit the letter ϱ in the notations above if it is the norm of E. Please note the three particular cases:

(1) if ϱ is the norm of E and $F = E^*$, then we simply denote the $\sigma(E,E^*)$–ϱ–ε-dentability derivative $[B]'_{(\sigma(E,E^*),\ \varrho,\ \varepsilon)}$ by $[B]'_\varepsilon$, and call it ε-dentability derivative of B;

(2) if $B \subset E^*$ and ϱ is the norm of E^*, then we simply denote the $\sigma(E^*,E)$–ϱ–ε-dentability derivative $[B]'_{(\sigma(E^*,E),\ \varrho,\ \varepsilon)}$ by $[B]'_{(w^*,\ \varepsilon)}$, and call it w^*–ε-dentability derivative of B; and

(3) [Fabian et al. [15]] if $B \subset E^*$ and ϱ is on E^* generated by some bounded set $A \subset E$, i.e.

$$\varrho(x^*) = \sup\{|\langle x^*, x\rangle| : x \in A\}, \text{ for } x^* \in E^*, \tag{2.4}$$

then we simply denote the $\sigma(E^*,E)$–ϱ–ε-dentability derivative $[B]'_{(\sigma(E^*,E),\ \varrho,\ \varepsilon)}$ by $[B]'_{(w^*,\ A,\ \varepsilon)}$, and call it w^*–(A,ε)-dentability derivative of B.

Starting from $[B]'_{(\sigma(E,F),\ \varrho,\ \varepsilon)}$, we can successively define $[B]^{(n)}_{(\sigma(E,F),\ \varrho,\ \varepsilon)}$ for all $n \in \mathbb{N}$. B is said to have finite-$\sigma(E,F)$–ϱ-index property provided for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_{(\sigma(E,F),\ \varrho,\ \varepsilon)} = \emptyset$. We should emphasis that if (in Case (1)) for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_\varepsilon = \emptyset$, then we simply call the set B having **finite-index property**; and if (in Case (3)) $B = B_{X^*}$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[B]^{(n)}_{(w^*,\ A,\ \varepsilon)} = \emptyset$, then we call the set A having **finite-dual-index property**.

Clearly, the finite-index property and the finite-dual-index property of a bounded set are inherited by its subsets. The dual-index property is also inherited by its absolute closed convex hull, but the finite-index property is not inherited by its convex hull [7, Example 5.5].

Theorem 2.5. *(See [7, Theorem 5.6].) For a bounded closed convex set of a Banach space, the finite-index property, the finite-dual-index property and the super weak compactness are equivalent.*

Definition 2.6. (See Fabian et al. [15].) Let M be a nonempty bounded subset of a Banach space X. We say that the norm $\|\cdot\|$ on X is M-uniformly Gâteaux smooth if

$$\lim_n |x_n^* - y_n^*|_M \equiv \lim_n \sup_{x\in M} |\langle x_n^* - y_n^*\rangle| = 0 \tag{2.5}$$

whenever $x_n^*, y_n^* \in S_{X^*}$ with $\|x_n^* + y_n^*\| \to 2$. Or, equivalently,

$$\limsup_{t\downarrow 0, x\in S_X, y\in M} \frac{\|x+ty\| + \|x-ty\| - 2}{t} = 0. \tag{2.6}$$

Theorem 2.7. *(See Fabian et al. [15].) Let X be a Banach space and M be a nonempty bounded subset of X. Then the following assertions are equivalent:*

(i) X admits an equivalent M-uniformly Gâteaux smooth norm;

(ii) M has finite-dual-index property.

The following notions about localized uniform convexity were used in [7–9].

Definition 2.8. A convex function f defined on a convex set C in a Banach space X is said to be uniformly convex provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in C,\ \|x - y\| \ge \varepsilon \implies f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \ge \delta. \tag{2.7}$$

The set C is called uniformly convex if for each $x_0 \in C$,

$$\text{the function } f = \|\cdot - x_0\|^2 \text{ is uniformly convex on } C. \tag{2.8}$$

We say that the set C is uniformly convexifiable if there is an equivalent norm on X such that C is uniformly convex with respect to the new norm.

Theorem 2.9. *(See [8, Theorem 4.12].) A bounded closed convex set of a Banach space is super weakly compact if and only if it is uniformly convexifiable.*

For a convex set C of a Banach space X, $x_0 \in C$, $0 < r \le \operatorname{diam}(C)$ and $\varepsilon > 0$, let

$$S(C, x_0, r, \varepsilon) = \{(x, y) \in C \times C : \|x - x_0\|, \|y - y_0\| \le r, \|x - y\| \ge \varepsilon\}, \tag{2.9}$$

and

$$\delta_C(x_0, r, \varepsilon) = \begin{cases} 1, & \text{if } S(C, x_0, r, \varepsilon) = \emptyset; \\ \inf\{1 - \frac{1}{r}\|\frac{x+y}{2} - x_0\| : (x, y) \in S(C, x_0, r, \varepsilon)\}, & \text{otherwise.} \end{cases} \tag{2.10}$$

We define convexity modulus of C as follows:

$$\delta_C(x_0, \varepsilon) = \inf\{\delta_C(x_0, r, \varepsilon) : r > 0\},\ x_0 \in C. \tag{2.11}$$

It is not difficult to observe that a Banach space X is uniformly convex if and only if $\delta_C(x_0, \varepsilon) > 0$ for all $x_0 \in B_X$ and $\varepsilon > 0$.

Remark 2.10. A similar (but different) notion of convexity modulus was introduced and discussed in [10].

Theorem 2.11. *Suppose that C is a bounded convex set containing at least two points in a Banach space X. Then it is uniformly convex if and only if $\delta_C(x_0, \varepsilon) > 0$ for all $x_0 \in C$ and $\varepsilon > 0$.*

Proof. We first note that for any fixed $x_0, x, y \in C$ the function $1 - \frac{1}{r}\|\frac{x+y}{2} - x_0\|$ is increasing in $r > 0$, and that $\|x - x_0\| \le r,\ \|y - x_0\| \le r$ entail $r \ge \sqrt{\frac{\|x-x_0\|^2+\|y-x_0\|^2}{2}} \equiv r_0(x, y)$. Definition of $\delta_C(x_0, \varepsilon)$ implies

$$\delta_C(x_0, \varepsilon) = \inf\{1 - \frac{1}{r_0(x, y)}\|\frac{x + y}{2} - x_0\| : x, y \in C, \|x - y\| \ge \varepsilon\}. \tag{2.12}$$

Given $\varepsilon > 0$, $x_0, x, y \in C$ with $\|x - y\| \ge \varepsilon$, write $r_0 = r_0(x, y)$. Then

$$r_0 \ge \frac{\|x - y\|}{2} \ge \frac{\varepsilon}{2}. \tag{2.13}$$

Sufficiency. Suppose $\delta_C(x_0, \varepsilon) > 0$. Then by (2.12) and (2.13),

$$\begin{aligned} 0 < \delta_C(x_0, \varepsilon) &\le \delta_C(x_0, \varepsilon)\Big[1 + \frac{1}{r_0}\|\frac{x + y}{2} - x_0\|\Big] \\ &\le \Big[1 - \frac{1}{r_0}\|\frac{x + y}{2} - x_0\|\Big]\Big[1 + \frac{1}{r_0}\|\frac{x + y}{2} - x_0\|\Big] \\ &= 1 - \frac{2}{\|x - x_0\|^2 + \|y - x_0\|^2}\Big\|\frac{x + y}{2} - x_0\Big\|^2 \end{aligned}$$

$$= \frac{1}{r_0^2}\Big[\|x-x_0\|^2 + \|y-x_0\|^2 - 2\|\frac{x+y}{2} - x_0\|^2\Big]$$
$$\le \frac{4}{\varepsilon^2}\Big[\|x-x_0\|^2 + \|y-x_0\|^2 - 2\|\frac{x+y}{2} - x_0\|^2\Big].$$

Therefore,

$$\|x-x_0\|^2 + \|y-x_0\|^2 - 2\|\frac{x+y}{2} - x_0\|^2 \ge \frac{\varepsilon^2}{4}\delta_C(x_0,\varepsilon) \equiv \delta > 0,$$

i.e. C is uniformly convex.

Necessity. Assume that C is uniformly convex. Then there is $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned}
\delta &< \|x-x_0\|^2 + \|y-x_0\|^2 - 2\|\frac{x+y}{2} - x_0\|^2 \\
&= 2r_0^2\Big[1 - \frac{1}{r_0^2}\|\frac{x+y}{2} - x_0\|^2\Big] \\
&= 2r_0^2\Big[1 - \frac{1}{r_0}\|\frac{x+y}{2} - x_0\|\Big]\Big[1 + \frac{1}{r_0}\|\frac{x+y}{2} - x_0\|\Big] \\
&\le 4r_0^2\Big[1 - \frac{1}{r_0}\|\frac{x+y}{2} - x_0\|\Big] \\
&\le 4[\mathrm{diam}(C)]^2\Big[1 - \frac{1}{r_0}\|\frac{x+y}{2} - x_0\|\Big],
\end{aligned}$$

i.e.

$$1 - \frac{1}{r_0}\|\frac{x+y}{2} - x_0\| > \frac{\delta}{4[\mathrm{diam}(C)]^2}.$$

This and inequality (2.12) further imply

$$\delta_C(x_0,\varepsilon) \ge \frac{\delta}{4[\mathrm{diam}(C)]^2}. \qquad \square$$

Let $c_{00} = \{x : \mathbb{N} \to \mathbb{R}$ with finite support$\}$. Now, we conclude this section by the following theorem, which can be found in Brunel and Sucheston [6].

Theorem 2.12. *Every bounded sequence (x_n) in a Banach space X contains a subsequence (e_n) with the following property: For each $a \in c_{00}$ there exists a number $L(a)$ such that for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that*

$$\Big|\|\sum_{i=1}^{\infty} a_i e_{n_i}\| - L(a)\Big| \le \varepsilon, \quad \textit{for all integers } n_i \textit{with } n_0 \le n_1 < n_2 < \cdots. \tag{2.14}$$

Thus,

$$\lim_{\mathcal{N}} \|\sum_{i=1}^{\infty} a_i e_{n_i}\| = L(a) \tag{2.15}$$

is well-defined for all $a \in c_{00}$, where $\mathcal{N}$ denotes the net consisting of all properly increasing subsequence (n_i) of $\mathbb{N}$ ordered by

$$N \equiv (n_i) \ge M \equiv (m_i) \iff n_i \ge m_i \quad \textit{for all } i \in \mathbb{N}.$$

3. Super fixed point property and super weak compactness

In this section, we shall show that for a nonempty closed bounded convex set C of a Banach space X, it has the super fixed point property for self-affine (or, equivalently, linear) isometries if and only if it is super weakly compact. To begin with, we first recall the notion of (linear, affine, resp.) isometry defined on a convex set of a real Banach space.

Definition 3.1. Let $C \subset X$, $D \subset Y$ be two nonempty convex subsets of Banach spaces X, Y, and $T : C \to D$ be a mapping.

(1) T is said to be an isometry (a linear isometry, resp.) on C if there is an isometry (a linear isometry, resp.) $\overline{T} : X \to Y$ so that $\overline{T}|_C = T$;

(2) We call T is affine (continuously affine, resp.) on C if there is an affine (a continuously affine, resp.) mapping $\overline{T} : X \to Y$ so that $\overline{T}|_C = T$.

Remark 3.2. Please note the distinction between these notions and "isometry", "linear isometry" and "affine mapping" defined on a subset in the usual sense. Since, in the usual sense, an isometry (linear isometry, affine mapping, continuous affine mapping, resp.) defined on a subset needs not be the restriction of an isometry (linear isometry, affine mapping, continuously affine mapping, resp.) defined on the whole space, the notions defined above are in the strongest sense.

Definition 3.3. Let C be a bounded closed convex set of a Banach space X, and $\mathcal{M}(C)$ be a class of self-mappings defined on C. We say that C has the fixed point property for $\mathcal{M}(C)$ if every $T \in \mathcal{M}(C)$ has a fixed point.

There are many possibilities, but the following are of main interest to us: (i) $M(C) = \{T : C \to C$ is nonexpansive$\}$; in this case, we say that C has the fixed point property. (ii) $M(C) = \{T : C \to C$ is an isometry$\}$; in this case, C has the fixed point property for isometries. (iii) $M(C) = \{T : C \to C$ is continuous affine$\}$; in this case, C is called to have the fixed point property for affine mappings. And (iv) $M(C) = \{T : C \to C$ is a linear isometry$\}$; in this case, C is said to have the fixed point property for linear isometries.

Definition 3.4. Let C be a nonempty bounded closed convex set of a Banach space X, D also a nonempty bounded closed convex of an arbitrary Banach space Y, and $\mathcal{M}(D)$ be a specific class of self-mappings defined on D. We say that C has the super fixed point property for $\mathcal{M}(D)$ if D has the fixed point property for $\mathcal{M}(D)$ whenever D is finitely representable in C. In particular, if $M(D) = \{T : D \to D$ is nonexpansive$\}$, then C is said to have the super fixed point property.

The following lemma is a core component for showing the main result of this section. The proof is a simplified refinement and generalization of the techniques related to the notion of the so-called spreading model invented by A. Brunel and L. Sucheston [6].

Lemma 3.5. *Suppose that C is a nonempty bounded closed convex subset of a Banach space X. If it is not weakly compact, then there is a convex set D of a Banach space Y such that*

(1) Y is finitely representable in X, and D is finitely representable in C;

(2) D does not have the fixed point property for linear isometries.

Proof. Since $C \subset X$ is not weakly compact, by James' theorem [19], there exist $\alpha > 0$ and two sequences $(x_n) \subset C$ and $(x_n^*) \subset B_{X^*}$ such that

$$\langle x_i^*, x_j \rangle = \begin{cases} \alpha\,, & 1 \le i \le j < \infty; \\ 0\,, & 1 \le j < i < \infty. \end{cases} \tag{3.1}$$

According to Theorem 2.12, there exists a subsequence (e_n) of (x_n) such that for every $a \in c_{00}$ there exists a number $L(a)$ satisfying for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that

$$\Big| \| \sum_{i=1}^{\infty} a_i e_{n_i} \| - L(a) \Big| \le \varepsilon, \quad \text{for all integers } n_i \text{ with } n_0 \le n_1 < n_2 < \cdots. \tag{3.2}$$

Let $(e_n^*) \subset (x_n^*)$ be the subsequence corresponding to the subsequence (e_n). Thus, it also fulfills

$$\langle e_i^*, e_j \rangle = \begin{cases} \alpha\ , & 1 \le i \le j < \infty; \\ 0\ , & 1 \le j < i < \infty. \end{cases} \tag{3.3}$$

Now, let $E = \operatorname{span}(e_n)$ and $|||\cdot||| : E \to \mathbb{R}^+$ be defined by

$$|||\sum_{n=1}^{\infty} a_n e_n||| = L(a), \quad \text{for all } a = (a_n) \in c_{00}. \tag{3.4}$$

It is easy to observe that $|||\cdot|||$ is a seminorm on E with $|||e_n||| = \|e_n\|$ for all $n \in \mathbb{N}$. $|||\cdot|||$ is actually a norm. In fact, assume $x = \sum a_n e_n \in E$ for some $a = (a_n) \in c_{00}$ such that $|||x||| = L(a) = 0$. Then definition of $L(a)$ entails that for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $\| \sum_i a_i e_{n_i} \| < \varepsilon$ for all sequence $(n_i) \subset \mathbb{N}$ with $n_\varepsilon \le n_i < n_2 < \cdots$. Let $m \in \mathbb{N}$ be the largest number such that $a_m \neq 0$. It follows from (3.3)

$$\Big| \sum_{i=j}^{m} \alpha a_i \Big| < \varepsilon, \quad \text{for all } 1 \le j \le m. \tag{3.5}$$

Consequently, $|a_i| < 2\varepsilon/\alpha$ for all $i \in \mathbb{N}$. Arbitrariness of ε implies $a_n = 0$ for all $n \in \mathbb{N}$, i.e. $x = 0$. Therefore, $(E, |||\cdot|||)$ is a normed space and $B \equiv \operatorname{co}(e_n)$ is a bounded convex set of $(E, |||\cdot|||)$.

We define a mapping $T : E \to E$ for $\sum_n a_n e_n$ by

$$T(\sum_n a_n e_n) = \sum_n a_n e_{n+1}. \tag{3.6}$$

Clearly, T is a linear isometry on $(E, |||\cdot|||)$, and its restriction $T|_B$ to B is a self-mapping.

Let Y be a completion of $(E, |||\cdot|||)$, $D = \overline{B} \subset Y$ and $\overline{T}$ be the natural extension of T from E to Y. Then $U \equiv \overline{T}|_D$ is also a self-mapping. We claim that U has no fixed point in D. Suppose, to the contrary, that $x \in D$ satisfies $U(x) = x$. Then $x \in \cap_{n=1}^{\infty} D_n$, where $D_n = \overline{\operatorname{co}}_{|||\cdot|||}(e_j)_{j \ge n}$. Let $(m_n) \subset \mathbb{N}$ be a properly increasing sequence, and

$$x_n = \sum_{j=n}^{m_n} \lambda_{n,j} e_j \in \operatorname{co}(e_j)_{j \ge n} \tag{3.7}$$

so that $x_n \to x$ in the norm $|||\cdot|||$-topology. Then $|||x_n - x_{m_{n+1}}||| \to 0$ as $n \to \infty$. Therefore, for every $\alpha/2 > \varepsilon > 0$ and for all sufficiently large $n \in \mathbb{N}$, we obtain $|||x_n - x_{m_{n+1}}||| < \varepsilon$. Now, for each fixed such $n \in \mathbb{N}$, there is $n_\varepsilon \in \mathbb{N}$ so that

$$\Big| |||\sum_{j=n}^{m_n} \lambda_{n,j} e_j - \sum_{j=m_{n+1}}^{m_{m_{n+1}}} \lambda_{m_{n+1},j} e_j||| - \| \sum_{j=n}^{m_n} \lambda_{n,j} e_{j+n_\varepsilon} - \sum_{j=m_{n+1}}^{m_{m_{n+1}}} \lambda_{m_{n+1},j} e_{j+n_\varepsilon} \| \Big| < \varepsilon.$$

Consequently,

$$\|\sum_{j=n}^{m_n}\lambda_{n,j}e_{j+n_\varepsilon}-\sum_{j=m_{n+1}}^{m_{m_{n+1}}}\lambda_{m_{n+1},j}e_{j+n_\varepsilon}\|<2\varepsilon<\alpha. \tag{3.8}$$

On the other hand, by letting $k=m_{m_{n+1}}+n_\varepsilon$ and applying (3.3), we obtain

$$2\varepsilon>\left|\langle e_k^*,\sum_{j=n}^{m_n}\lambda_{n,j}e_{j+n_\varepsilon}-\sum_{j=m_{n+1}}^{m_{m_{n+1}}}\lambda_{m_{n+1},j}e_{j+n_\varepsilon}\rangle\right|=\alpha. \tag{3.9}$$

This contradiction says that the linear isometry U has no fixed point on $\overline{B}$.

It remains to show that (1) $(D,|||\cdot|||)$ is finitely representable in $(C,\|\cdot\|)$, and (2) $(Y,|||\cdot|||)$ is finitely representable in $(X,\|\cdot\|)$. Note that (1) yields (2). It suffices to prove (1). Density of B in D allows us to verify only that $(B,|||\cdot|||)$ is finitely representable in $(C,\|\cdot\|)$.

Given $\varepsilon>0$, let $S(B)=\mathrm{co}(y_0,\cdots,y_n)\subset B$ be an n-simplex, where $y_i=\sum_{j=1}^{n_i}\lambda_{ij}e_j$, $\lambda_{ij}\ge 0$ with $\sum_{j=1}^{n_i}\lambda_{ij}=1$ for $i=0,\cdots,n$; and let $F=\mathrm{span}(y_i-y_0)_{i=1}^n$. Since $(F,|||\cdot|||)$ is isomorphic to ℓ_1^n, there is $\beta>0$ such that

$$\beta\sum_{i=1}^n|a_i|\le|||\sum_{i=1}^n a_i(y_i-y_0)|||,\quad \text{for all } (a_1,\cdots,a_n)\in\ell_1^n. \tag{3.10}$$

Fix

$$0<\delta<\beta\varepsilon\Big/\Big(1+\mathrm{diam}_{\|\cdot\|}(C)+\mathrm{diam}_{|||\cdot|||}(D)\Big). \tag{3.11}$$

Let $\mathcal{F}=(a^k)_{k=1}^m\subset S_{\ell_1^n}$ be a finite δ-net of $S_{\ell_1^n}$. Then there exist a positive integer $n_\delta\in\mathbb{N}$ such that

$$\Big||||\sum_{i=1}^n a_i^k(y_i-y_0)|||-\|\sum_{i=1}^n a_i^k(y_{n_\delta+i}-y_{n_\delta})\|\Big|<\delta,\quad \text{for all } 1\le k\le m, \tag{3.12}$$

where $y_{n_\delta+i}=\sum_{j=1}^{n_i}\lambda_{ij}e_{n_\delta+j}$ and $0\le i\le n$. Since y_i-y_0 $(i=1,2,\cdots,n)$ are linearly independent, we can choose δ small enough so that $y_{n_\delta+i}-y_{n_\delta}$ $(i=1,2,\cdots,n)$ are linearly independent.

Set $S(C)=\mathrm{co}(y_{n_\delta},\cdots,y_{n_\delta+n})$. We want to show that the affine mapping $V:S(B)\to S(C)$ defined by $T(\sum_i a_iy_i)=\sum_i a_iy_{n_\delta+i}$ is a $(1+\varepsilon)$-affine embedding. It is clear that V is surjective.

Given

$$x=\sum_{j=1}^n\alpha_iy_i\ne y=\sum_{j=1}^n\beta_iy_i\in S(B)$$

for some $\alpha_i,\beta_i\ge 0$ with $\sum_{i=1}^n\alpha_i=1=\sum_{i=1}^n\beta_i$, let

$$t_i=(\alpha_i-\beta_i)/(\sum_{j=1}^n|\alpha_j-\beta_j|),\ \ s=\sum_{j=1}^n|\alpha_j-\beta_j|.$$

Then $(t_1,\cdots,t_n)\in S_{\ell_1^n}$, and consequently, there exist $a^k=(a_1^k,\cdots,a_n^k)\in\mathcal{F}$ so that $\sum_{i=1}^n|a_i^k-t_i|<\delta$. Hence, (3.10)–(3.12) together imply

$$\begin{aligned}
\Big|\|V(x)-V(y)\| - |||x-y|||\Big| &= \Big|\|\sum_{i=1}^{n}(\alpha_i-\beta_i)(y_{n_\delta+i}-y_{n_\delta})\| - |||\sum_{i=1}^{n}(\alpha_i-\beta_i)(y_i-y_0)|||\Big| \\
&= s\cdot\Big|\|\sum_{i=1}^{n}t_i(y_{n_\delta+i}-y_{n_\delta})\| - |||\sum_{i=1}^{n}t_i(y_i-y_0)|||\Big| \\
&\le s\cdot\Big(\Big|\|\sum_{i=1}^{n}t_i(y_{n_\delta+i}-y_{n_\delta})\| - \|\sum_{i=1}^{n}a_i^k(y_{n_\delta+i}-y_{n_\delta})\|\Big| \\
&\quad + \Big|\|\sum_{i=1}^{n}a_i^k(y_{n_\delta+i}-y_{n_\delta})\| - |||\sum_{i=1}^{n}a_i^k(y_i-y_0)|||\Big| \\
&\quad + \Big||||\sum_{i=1}^{n}a_i^k(y_i-y_0)||| - |||\sum_{i=1}^{n}t_i(y_i-y_0)|||\Big|\Big) \\
&\le s\cdot\Big(\sum_{i=1}^{n}|t_i-a_i^k|\mathrm{diam}_{\|\cdot\|}(C)+\delta+\sum_{i=1}^{n}|a_i^k-t_i|\mathrm{diam}_{|||\cdot|||}(B)\Big) \\
&\le \delta s\Big(1+\mathrm{diam}_{\|\cdot\|}(C)+\mathrm{diam}_{|||\cdot|||}(B)\Big) < \beta\varepsilon s \\
&= \varepsilon(\beta\cdot\sum_{i}|\alpha_j-\beta_i|) \le \varepsilon|||x-y|||.
\end{aligned}$$

Therefore,

$$(1-\varepsilon)|||x-y||| \le \|V(x)-V(y)\| \le (1+\varepsilon)|||x-y|||,$$

and which completes our proof. □

Theorem 3.6. *Suppose that C is a nonempty closed bounded convex subset of a Banach space X. Then the following statements are equivalent.*

i) C has the super fixed point property for linear isometries;
ii) C has the super fixed point property for affine isometries;
iii) C is super weakly compact.

Proof. i) $\Longrightarrow$ iii). Suppose, to the contrary, that C is not super weakly compact. Then there is a bounded, non-weakly compact, closed and convex subset B of a Banach space Y, which is finitely representable in C. By Lemma 3.5, there is a nonempty closed bounded convex set D of a Banach space Z such that (1) D is finitely representable in B; and (2) there is a self-linear isometry $V : D \to D$ which fails to have a fixed point. Since B is finitely representable in C and since D is finitely representable in B, D is necessarily finitely representable in C. Thus, C does not admit the super fixed point property for linear isometries.

iii) $\Longrightarrow$ ii). This is a consequence of Schauder's fixed point theorem [30]. Indeed, since C is super weakly compact, every bounded closed convex subset D of a Banach space is weakly compact if it is finitely representable in C. Since every affine isometry is necessarily weak-to-weak continuous, by Schauder's fixed point theorem, every such mapping has a fixed point.

ii) $\Longrightarrow$ i). It is clearly trivial. □

Corollary 3.7. *A Banach space is super reflexive if and only if it has the super fixed point property for affine isometries.*

Proof. By Theorem 3.6, it suffices to note that a Banach space is super reflexive if and only if its closed unit ball is super weakly compact [8, Corollary 2.15]. □

Remark 3.8. Maurey's fixed point theorem states that every closed bounded convex subset of a super reflexive Banach space has the super fixed point property for isometries. The super weak compactness of a closed bounded convex subset of a general Banach space cannot guarantee even the fixed point property for isometries though the theorem above says that it is true for affine isometries. Perhaps, this is one of the greatest difference between a bounded closed convex set in a super reflexive Banach space and a convex super weakly compact set in a general Banach space. We use the following Alspach's counterexample [1] to explain what happens.

Let $C = \{f \in L_1[0,1] : 0 \le f \le 2 \ a.e.\}$. Then C is weakly compact and there is an isometry $U : C \to C$, which does not have a fixed point [1]. On the other hand, since $L_1[0,1]$ is strongly super reflexive generated [16], i.e. there is a super reflexive space Z and a bounded linear mapping $T : Z \to L_1[0,1]$ such that for all $\varepsilon > 0$ and for every weakly compact set K, there exists $n \in \mathbb{N}$ so that $K \subset nT(B_Z) + \varepsilon B_{L_1[0,1]}$. Since Z is super reflexive, B_Z is super weakly compact [8, Corollary 2.15], and further $T(B_Z)$ is super weakly compact in $L_1[0,1]$ [8, Proposition 3.10]. Consequently, K (hence, C) is super weakly compact [7, Theorem 4.1].

Remark 3.9. Compared with a recent theorem of Lin [25] which asserts that ℓ_1 can be renormed to have the fixed point property (See, also [26].), Theorem 3.6 indicates a big difference between the super fixed point property and the fixed point properties: There exists a bounded closed convex set of a Banach space which can be renormed to admit the fixed point property, but it can never have the super fixed point property even if just for linearly isometric self-mappings.

4. Super fixed point property under renormings

In this section, we discuss the super fixed point property of bounded closed convex sets of Banach spaces in renorming sense. We shall show that for a bounded closed convex set C of a Banach space X there is an equivalent norm $|||\cdot|||$ on X such that C has the super fixed point property with respect to the new norm if and only if it is super weakly compact; and a strongly super weakly compact generated Banach space always admits an equivalent norm so that every weakly compact convex set has the super fixed point property.

A filter $\mathcal{F}$ is a collection of subsets of a set Ω satisfying i) $\emptyset \notin \mathcal{F}$; ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$; iii) $A \in \mathcal{F}$ and $A \subset B \subset \Omega$ entail $B \in \mathcal{F}$. A filter $\mathcal{F}$ is said to be free if $\cap\{F \in \mathcal{F}\} = \emptyset$. A filter $\mathcal{U}$ is called an ultrafilter if for any $A \subset \Omega$, either $A \in \mathcal{U}$, or, $\Omega \setminus A \in \mathcal{U}$. Let K be a topological space, and $f : \Omega \to K$ a function. We say f is convergent to some $k \in K$ with respect to a filter $\mathcal{F}$ if for every neighborhood U of k, we have $f^{-1}(U) \in \mathcal{F}$; in this case, we denote $\lim_{\mathcal{F}} f = k$.

We will recall the definition of an ultraproduct (ultrapower) of Banach spaces. For a nonempty set Ω, let $(X_\omega : \omega \in \Omega)$ be a collection of Banach spaces. Then their ultraproduct is defined by

$$\prod_{\mathcal{U}} X_\omega = (\bigoplus_{\omega \in \Omega} X_\omega)_{\ell_\infty} / \{(x_\omega) : \lim_{\mathcal{U}} \|x_\omega\| = 0\}. \tag{4.1}$$

$\lim_{\mathcal{U}} \|x_\omega\| = 0$ means for all $\varepsilon > 0$, $\{\omega \in \Omega : \|x_\omega\| < \varepsilon\} \in \mathcal{U}$. Please note that the ultraproduct is a quotient of the ℓ_∞-sum of (X_ω), so its elements are classes of the respective equivalences relation, not the generalized sequences itself. We will use in the sequel the notations $[(x_\omega)]$, or, $\boldsymbol{x} = [(\boldsymbol{x}(\omega))]$, to denote the equivalence class of (x_ω). Thus, for a collection $(A_\omega \subset X_\omega : \omega \in \Omega)$ of subsets, its ultraproduct is

$$\prod_{\mathcal{U}} A_\omega = \{[(x_\omega)] : (x_\omega) \in (\bigoplus_{\omega \in \Omega} X_\omega)_{\ell_\infty} : x_\omega \in A_\omega \text{ for } \omega \in \Omega\}. \tag{4.2}$$

In particular, if $X_\omega = X$, and $A_\omega = A \subset X$ for all $\omega \in \Omega$, then we denote by $A_{\mathcal{U}} = \prod_{\mathcal{U}} A$, the $\mathcal{U}$-ultrapower of A.

Proposition 4.1. *Suppose that X is a Banach space, $C \subset X$ is a nonempty bounded convex set and that $\mathcal{U}$ is a free ultrafilter.*

(1) If C is uniformly convex, then $C_{\mathcal{U}}$ is also uniformly convex in $X_{\mathcal{U}}$;

(2) If the norm $\|\cdot\|$ of X is C-uniformly Gâteaux differentiable, then the norm $\|\cdot\|_{\mathcal{U}}$ of $X_{\mathcal{U}}$ is also $C_{\mathcal{U}}$-uniformly Gâteaux differentiable.

Proof. (1). Given $\varepsilon > 0$, let $\boldsymbol{x} = [(\boldsymbol{x}(\omega))]$, $\boldsymbol{y} = [(\boldsymbol{y}(\omega))] \in C_{\mathcal{U}}$, with $\|\boldsymbol{x}-\boldsymbol{y}\|_{\mathcal{U}} > \varepsilon$. Then for all representative elements $(\boldsymbol{x}(\omega))$ of $\boldsymbol{x}$ and $(\boldsymbol{y}(\omega))$ of $\boldsymbol{y}$, we have

$$U \equiv \{\omega \in \Omega : \|\boldsymbol{x}(\omega) - \boldsymbol{y}(\omega)\| > \varepsilon\} \in \mathcal{U}. \tag{4.3}$$

Uniform convexity of C (Definition 2.8) entails that there exists $\delta > 0$ such that for all $(z(\omega)) \subset C$ we have

$$f_\omega(\boldsymbol{x}(\omega)) + f_\omega(\boldsymbol{y}(\omega)) - 2f_\omega(\frac{\boldsymbol{x}(\omega)+\boldsymbol{y}(\omega)}{2}) > \delta, \text{ whenever } \omega \in U, \tag{4.4}$$

where $f_\omega = \|\cdot - z(\omega)\|^2$ for all $\omega \in \Omega$. Therefore, $f_{\mathcal{U}} \equiv \|\cdot - \boldsymbol{z}\|^2_{\mathcal{U}}$ is uniformly convex on $C_{\mathcal{U}}$ for all $\boldsymbol{z} = [(z(\omega)] \in C_{\mathcal{U}}$. Consequently, $C_{\mathcal{U}}$ is uniformly convex.

(2). Recall that the smoothness modulus s_C of C is defined by

$$s_C(\tau) = \sup\{\frac{\|x+\tau y\| + \|x - \tau y\|}{2} - 1 : x \in S_X, y \in C\}, \tag{4.5}$$

for all $\tau > 0$, and the norm $\|\cdot\|$ is C-uniformly Gâteaux differentiable if and only if $s_C(\tau)/\tau \to 0$, as $\tau \to 0^+$. (See, for instance, [14].)

Note

$$s_{C_{\mathcal{U}}}(\tau) = \sup\{\frac{\|\boldsymbol{x}+\tau\boldsymbol{y}\|_{\mathcal{U}} + \|\boldsymbol{x}-\tau\boldsymbol{y}\|_{\mathcal{U}}}{2} - 1 : \boldsymbol{x} \in S_{X_{\mathcal{U}}}, \boldsymbol{y} \in C_{\mathcal{U}}\},$$

and note for each $\boldsymbol{x} \in S_{X_{\mathcal{U}}}$, and $\boldsymbol{y} \in C_{\mathcal{U}}$, we can choose representative elements $(x_\omega)_{\omega\in\Omega} \subset S_X$ of $\boldsymbol{x}$ and $(y_\omega)_{\omega\in\Omega} \subset C$ of $\boldsymbol{y}$ for all $\omega \in \Omega$ such that $\boldsymbol{x} = [(x_\omega)]$ and $\boldsymbol{y} = [(y_\omega)]$. It is not difficult to observe $s_{C_{\mathcal{U}}}(\tau) \le s_C(\tau)$. On the other hand, $C \subset C_{\mathcal{U}}$ and $S_X \subset S_{X_{\mathcal{U}}}$ imply $s_C(\tau) \le s_{C_{\mathcal{U}}}(\tau)$. Therefore, $s_C(\tau) = s_{C_{\mathcal{U}}}(\tau)$, and which further entails that the norm $\|\cdot\|_{\mathcal{U}}$ of $X_{\mathcal{U}}$ is also $C_{\mathcal{U}}$-uniformly Gâteaux differentiable. $\square$

Theorem 4.2. *A bounded closed convex subset C of a Banach space X can be renormed to have the super fixed point property if and only if it is super weakly compact.*

Proof. Sufficiency. Since C is super weakly compact, by Theorem 2.9, there exists an equivalent norm $|||\cdot|||$ on X such that $(C, |||\cdot|||)$ is uniformly convex. We first show that $(C, |||\cdot|||)$ has normal structure.

Without loss of generality we can assume that C is symmetric; otherwise, by [8, Corollary 3.11], we substitute $\overline{\mathrm{co}}(C \cup -C)$ for C. Let D be a convex subset of C with $d_D \equiv \mathrm{diam}(D) > 0$, and choose any $0 < \varepsilon < d_D$. Given any $x_0, x, y \in D$ with $\|x - y\| \ge \varepsilon$, let $\delta_D(x_0, d_D, \varepsilon)$ be defined as (2.10). Then

$$1 - \frac{1}{d_D}\|\frac{x+y}{2} - x_0\| \ge \delta_D(x_0, d_D, \varepsilon) \ge \delta_{2C}(0, d_D, \varepsilon) \equiv \alpha > 0.$$

Therefore,

$$\|\frac{x+y}{2} - x_0\| \le \alpha d_D. \tag{4.6}$$

The inequality above and arbitrariness of x_0 together imply that $z \equiv \frac{x+y}{2} \in D$ is a non-diameter point of D. Hence, C has normal structure.

To show that C has super normal structure, let G be a bounded convex set of a Banach space Y. Without loss of generality, we assume that G is not contained in a finite dimensional affine subspace of Y. Therefore, there is a dense subset H of G, which consists of linearly independent vectors. If G is finitely representable in C, then there exist a free ultrafilter $\mathcal{U}$ and an affine isometry T from H to $C_{\mathcal{U}}$ [7, Proposition 2.4]. Density of H in G and continuity of T imply T is an affine isometry from G to $C_{\mathcal{U}}$. By Proposition 4.1, G is also uniformly convex, and consequently, G has again normal structure. Thus, super weak compactness and super normal structure together guarantee that C has super fixed point property.

Necessity. This is a direct consequence of Theorem 3.6, since super weak compactness is invariant under equivalent renormings. □

Definition 4.3. A Banach space X is said to be (strongly, resp.) super weakly compact generated if there is a convex super weakly compact set $C \subset X$ such that $\mathrm{span}(C)$ is dense in X (for every weakly compact set $K \subset X$ and for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ so that $K \subset nC + \varepsilon B_X$, resp.). In particular, if there is a super reflexive Banach space Z and a bounded linear operator $T : Z \to X$ such that $C = T(B_Z)$, then we say that X is super reflexive generated (strongly super reflexive generated, resp.). For example, for any finite measure space (Ω, Σ, μ), the space $L_1(\mu) \equiv L_1(\Omega, \Sigma, \mu)$ is strongly super reflexive generated [16].

Proposition 4.4. *Every weakly compact subset K of a strongly super weakly compact generated Banach space X is super weakly compact.*

Proof. Let X be strongly generated by a super weakly compact convex set C, i.e. for every weakly compact set $K \subset X$, there is $n \in \mathbb{N}$ so that $K \subset nC + \varepsilon B_X$. It suffices to note the definition and that K satisfies that for every $\varepsilon > 0$ there exists a super weakly compact set G so that $K \subset G + \varepsilon B_X$ if and only K is relatively super weakly compact [7, Theorem 4.1]. □

The following theorem was motivated by Fabian et al. [16, Theorem 7], in which the authors have shown that a strongly super reflexive generated Banach space can be renormed so that it is M-uniformly Gâteaux differentiable with respect to every weakly compact set M of X.

Theorem 4.5. *A strongly super weakly compact generated Banach space X has an equivalent norm $\|\cdot\|$ such that it is M-uniformly Gâteaux differentiable with respect to every weakly compact set M of X.*

Proof. Suppose that C is a super weakly compact convex subset of X such that for every weakly compact set $M \subset X$ and for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ so that $M \subset mC + \varepsilon B_X$. Since C is super weakly compact if and only if it has finite-dual-index property ([7, Theorem 5.6]; See, also Theorem 2.5), and since X has an equivalent norm which is M-uniformly Gâteaux differentiable with respect to a set M if and only if M has finite-dual-index property [15, Theorem 5], X has an equivalent norm $\|\cdot\|$ such that it is C-uniformly Gâteaux differentiable, i.e. for any two sequences $(x_n^*), (y_n^*) \subset S_{X^*}$ with $\|x_n^* + y_n^*\| \to 2$, we have $|x_n^* - y_n^*|_C \equiv \sup_{x \in C} |\langle x_n^* - y_n^*, x\rangle| \to 0$, as $n \to \infty$.

In the following we show that for any weakly compact set $M \subset X$, the norm $\|\cdot\|$ is also M-uniformly Gâteaux differentiable. Given weakly compact set $M \subset X$ and $\varepsilon > 0$, let $m \in \mathbb{N}$ so that $M \subset mC + \varepsilon B_X$. Then,

$$\limsup_n |x_n^* - y_n^*|_M = \lim_n \sup_{x \in M} |\langle x_n^* - y_n^*, x\rangle| \le \limsup_n |x_n^* - y_n^*|_{mC + \varepsilon B_X} \le 2\varepsilon, \tag{4.7}$$

whenever $(x_n^*), (y_n^*) \subset S_{X^*}$ with $\|x_n^* + y_n^*\| \to 2$. Arbitrariness of ε implies $\limsup_n |x_n^* - y_n^*|_M = 0$, i.e. the norm $\|\cdot\|$ is M-uniformly Gâteaux differentiable. □

Theorem 4.6. *A strongly super weakly compact generated Banach space can be renormed so that every weakly compact convex set has super fixed point property.*

Proof. By Theorem 4.5, X has an equivalent norm $\|\cdot\|$ such that it is M-uniformly Gâteaux differentiable with respect to every weakly compact set M of X. Given a nonempty convex weakly compact $C \subset X$, we shall show that C has normal structure. We can assume that C is symmetric. Suppose, to the contrary, that there is nonempty closed convex set $D \subset C$ with $\operatorname{diam}(D) > 0$ such that

$$\varrho(x) \equiv \sup_{y \in D} \|x - y\| = \operatorname{diam}(D) \equiv d_D, \quad \text{for all } x \in D. \tag{4.8}$$

Then by Brodskii–Milman's characterization of normal structure property [4], there exists a diametral sequence $(x_n) \subset D$ such that

$$\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}(x_i; i \le n)) = d_D. \tag{4.9}$$

Without loss of generality we can assume that (x_n) is weakly convergent to some point $x_\infty \in D$. It is easy to observe that for each $x \in K \equiv \overline{\operatorname{co}}(x_n)$,

$$\lim_n \|x_n - x\| = d_D. \tag{4.10}$$

In particular,

$$\lim_n \|x_n - x_\infty\| = d_D. \tag{4.11}$$

Let $E = \overline{\operatorname{co}}(D \cup -D)(\subset C)$. Then convexity and symmetry of the function $\frac{\|x+\tau y\|+\|x-\tau y\|-2}{2}$ in y entail that the smoothness modules s_D of D (defined as (4.5)) equals the smoothness modulus s_E of E.

Given $\tau \in (0, 1)$, $m \ne n \in \mathbb{N}$ and $y \in E$, we have

$$\frac{1}{2}\Big[\|(x_n - x_m) + \tau y\| + \|(x_n - x_m) - \tau y\|\Big] - \|x_n - x_m\| \le \|x_n - x_m\| s_E(\frac{\tau}{\|x_n - x_m\|}).$$

Note $(x_\infty - x_m)/2 \in E$. Then

$$\begin{aligned} &\frac{1}{2}\Big[\|(x_n - x_m) + \tau(x_\infty - x_m)/2\| + \|(x_n - x_m) - \tau(x_\infty - x_m)/2\|\Big] \\ &\quad \le \|x_n - x_m\|(1 + s_E(\frac{\tau}{\|x_n - x_m\|})) \le d_D \cdot (1 + s_E(\frac{\tau}{\|x_n - x_m\|})). \end{aligned} \tag{4.12}$$

On the other hand, by (4.11) we obtain

$$\|x_n - x_m - \tau(x_\infty - x_m)/2\| = \|x_n - [(\tau/2)x_\infty + (1 - \tau/2)x_m]\| \to d_D, \tag{4.13}$$

as $n \to \infty$. This, weak lower semicontinuity of the norm, continuity of the smoothness modulus s_E, incorporating of (4.12) and (4.13) together entail

$$\begin{aligned} [(1 + \tau/2)]d_D &= \lim_{m\to\infty} \|(1 + \tau/2)x_\infty - (1 + \tau/2)x_m\| \\ &\le \liminf_{m\to\infty} \liminf_{n\to\infty} \|(x_n - x_m) + \tau(x_\infty - x_m)/2\| \\ &\le d_D \cdot \liminf_{m\to\infty} \liminf_{n\to\infty} \Big[1 + 2s_E(\frac{\tau}{\|x_n - x_m\|})\Big] \\ &= d_D\Big[1 + 2s_E(\frac{\tau}{d_D})\Big]. \end{aligned} \tag{4.14}$$

Therefore,

$$1/4 \le s_E(\frac{\tau}{d_D})/\tau, \quad \text{for all } \tau \in (0,1).$$

This contradicts to the assumption that the norm $\|\cdot\|$ is C-uniformly Gâteaux smooth.

We have shown that C has normal structure if the norm $\|\cdot\|$ is C-uniformly Gâteaux smooth. To show that C has super normal structure, it suffices to note every infinite dimensional convex set which is finitely representable in C can be regarded as a subset of $C_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ (by an argument similar to the proof of the sufficiency part of Theorem 4.2), and the norm of the ultraproduct space $X_{\mathcal{U}}$ is again $C_{\mathcal{U}}$-uniformly Gâteaux smooth (Proposition 4.1). □

Corollary 4.7. *For any finite measure space $(\Omega, \sum, \mu)$, $L_1(\mu) \equiv L_1(\Omega, \Sigma, \mu)$ can be renormed such that every weakly compact convex set of $L_1(\mu)$ has super fixed point property.*

Proof. By Theorem 4.6, it suffices to note that $L_1(\mu)$ is strongly super reflexive generated. □

Remark 4.8. The converse version of Theorem 4.6 is not true, i.e. a Banach space satisfying that every weakly compact convex set has the super fixed point property is not necessarily a strongly super weakly generated. For example, for any uncountable set Γ, the space $\ell_1(\Gamma)$ satisfies that every weakly compact convex set of it has the super fixed point property, since every weakly compact set in $\ell_1(\Gamma)$ is necessarily compact. However, $\ell_1(\Gamma)$ is weakly compact generated if and only if Γ is countable.

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