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Compactness of embeddings of function spaces on quasi-bounded domains and the distribution of eigenvalues of related elliptic operators. Part II

Hans-Gerd Leopold [a], Leszek Skrzypczak [b,*]

[a] *Mathematisches Institut, Friedrich-Schiller-Universität, Ernst-Abbe-Platz 1-2, 07740 Jena, Germany*
[b] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland*

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ABSTRACT

We prove the asymptotic behaviour of eigenvalues of elliptic self-adjoint differential operators defined on a wide class of quasi-bounded domains. The estimates are based on corresponding asymptotic behaviour of entropy numbers of Sobolev embeddings of Sobolev and Besov function spaces defined on the quasi-bounded domains. We consider also the inverse problem i.e. we identify the class of functions that can describe the asymptotic behaviour of eigenvalues of Dirichlet Laplacian of some quasi-bounded domain.

Let Ω be a domain in $\mathbb{R}^n$ and $\widetilde{W}^k_p(\Omega)$ a function space obtained by completing $C_0^\infty(\Omega)$ in the usual Sobolev norm. It is well known that spectral properties of the Dirichlet Laplacian on Ω, in particular the discreteness of the spectrum, are related to the properties of Sobolev embeddings of the function spaces $\widetilde{W}^k_p(\Omega)$ into $L_q(\Omega)$. It was noticed by Colin Clark in 1965 that the Sobolev embeddings can be compact not only if the domain Ω is bounded but also if the domain is unbounded but sufficiently narrow at infinity, cf. [4]. We refer to [1] and [9,10] for further discussion and references.

In [13] we considered the Besov and Triebel–Lizorkin spaces defined on a wide range of unbounded domains, so-called uniformly E-porous domains. Based on a wavelet characterisation of the spaces we obtain sufficient and necessary conditions for compactness of the Sobolev embeddings. The wavelet characterisation of the spaces is due to H. Triebel, cf. [14]. Moreover we prove the asymptotic behaviour of entropy numbers of the compact embeddings in some cases. In this paper we continue investigation of the degree of the compactness of the embeddings of the function spaces in terms of entropy numbers. The exact asymp-

* Corresponding author.
E-mail addresses: hans-gerd.leopold@uni-jena.de (H.-G. Leopold), lskrzyp@amu.edu.pl (L. Skrzypczak).
URLs: http://users.minet.uni-jena.de/~leopold/ (H.-G. Leopold), http://www.staff.amu.edu.pl/~lskrzyp/ (L. Skrzypczak).

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totic behaviour of the entropy numbers is calculated for the wide range of the quasi-bounded domains. In particular for any quasi-bounded uniformly E-porous domain $\Omega \subset \mathbb{R}^n$ we prove the following asymptotic behaviour of entropy number of compact Sobolev embedding with $p = q$

$$e_k\Big(\widetilde{W}_p^{k_1}(\Omega) \hookrightarrow \widetilde{W}_p^{k_2}(\Omega)\Big) \sim 2^{-(k_1-k_2)\mathbb{B}^{-1}(k)} \tag{1}$$

if $k_1 - k_2 > 0$. Here $\mathbb{B}$ denotes some function depending on the domain Ω and describing its quasi-boundedness cf. Section 1.2 for the definition. We also prove the asymptotic behaviour in the case $p \neq q$.

The asymptotic estimates of entropy numbers in (1) imply the following estimates of eigenvalues $\lambda_1(-\Delta) \le \lambda_2(-\Delta) \le \ldots$ of the Dirichlet Laplacian $-\Delta$ defined on Ω

$$\lambda_k(-\Delta) \sim 2^{2\mathbb{B}^{-1}(k)} . \tag{2}$$

Here $a_k \sim b_k$ stands for $c\,b_k \le a_k \le C b_k$, $c, C > 0$ being positive constants independent of k. Negative spectrum of the corresponding Schödinger type operator is studied as well.

The paper is organized as follows. In the first section we recall the definitions and some properties of the domains, the function spaces and their wavelet characterisation. In particular we recall the notion of box packing constant introduced in [13] and the sufficient criteria for the compactness of the Sobolev embeddings proved there. In Section 2 we prove the asymptotic behaviour of the entropy numbers in the case $p_1 = p_2$. Section 3 is devoted to the behaviour of entropy numbers in the case $p_1 \neq p_2$. In the last section we give applications to spectral theory of elliptic operators.

1. Preliminary

1.1. *Function spaces on arbitrary domains*

Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \neq \mathbb{R}^n$. Such a set will be called an arbitrary domain. We assume that the reader is familiar with definitions and basic facts concerning Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ defined on $\mathbb{R}^n$ as well as the Besov and Triebel–Lizorkin spaces, $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$, defined on Ω by restrictions. All we need can be found in the first chapter of [14]. We will use the common notations. In particular we put

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \qquad \text{and} \qquad \sigma_{p,q} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+, \qquad 0 < p, q \le \infty$$

and $A^s_{p,q}(\mathbb{R}^n)$, $A^s_{p,q}(\Omega)$ with $A = B$ or $A = F$.

Definition 1. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let

$$0 < p \le \infty, \qquad 0 < q \le \infty, \qquad s \in \mathbb{R},$$

with $p < \infty$ for the F-spaces.

(i) Let

$$\widetilde{A}^s_{p,q}(\Omega) = \Big\{ f \in D'(\Omega) : \quad f = g|_\Omega \text{ for some } g \in A^s_{p,q}(\mathbb{R}^n),\ \operatorname{supp} g \subset \overline{\Omega} \Big\},$$
$$\| f | \widetilde{A}^s_{p,q}(\Omega) \| = \inf \| g | A^s_{p,q}(\mathbb{R}^n) \|,$$

where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} g \subset \overline{\Omega}$ and $f = g|_\Omega$.

(ii) We define

$$\bar{F}^s_{p,q}(\Omega) = \begin{cases} \widetilde{F}^s_{p,q}(\Omega) & \text{if} \quad 0 < p < \infty,\ 0 < q \le \infty,\ s > \sigma_{p,q}\,, \\ F^0_{p,q}(\Omega) & \text{if} \quad 1 < p < \infty,\ 1 \le q \le \infty,\ s = 0\,, \\ F^s_{p,q}(\Omega) & \text{if} \quad 0 < p < \infty,\ 0 < q \le \infty,\ s < 0\,, \end{cases}$$

and

$$\bar{B}^s_{p,q}(\Omega) = \begin{cases} \widetilde{B}^s_{p,q}(\Omega) & \text{if} \quad 0 < p \le \infty,\ 0 < q \le \infty,\ s > \sigma_p\,, \\ B^0_{p,q}(\Omega) & \text{if} \quad 1 < p < \infty,\ 0 < q \le \infty,\ s = 0\,, \\ B^s_{p,q}(\Omega) & \text{if} \quad 0 < p < \infty,\ 0 < q \le \infty,\ s < 0\,. \end{cases}$$

Following H. Triebel we introduce E-thick (exterior thick) and E-porous domains, cf. [14, Chapter 3]. We start with the definition of porosity.

Definition 2. (i) A closed set $\Gamma \subset \mathbb{R}^n$ is said to be porous if there exists a number $0 < \eta < 1$ such that one finds for any ball $B(x,r) \subset \mathbb{R}^n$ centred at x and of radius r with $0 < r < 1$, a ball $B(y,\eta r)$ with

$$B(y,\eta r) \subset B(x,r) \qquad \text{and} \qquad B(y,\eta r) \cap \Gamma = \emptyset\,.$$

(ii) A closed set $\Gamma \subset \mathbb{R}^n$ is said to be uniformly porous if it is porous and there is a locally finite positive Radon measure μ on $\mathbb{R}^n$ such that $\Gamma = \operatorname{supp}\mu$ and

$$\mu(B(\gamma,r)) \sim h(r)\,, \qquad \text{with} \qquad \gamma \in \Gamma, \quad 0 < r < 1\,,$$

where $h : [0,1] \to \mathbb{R}$ is a continuous strictly increasing function with $h(0) = 0$ and $h(1) = 1$ (the equivalence constants are independent of γ and r).

Remark 1. The closed set Γ is called a d-set if there is a locally finite positive Radon measure μ on $\mathbb{R}^n$ such that $\Gamma = \operatorname{supp}\mu$ and

$$\mu(B(\gamma,r)) \sim r^d\,, \qquad \text{with} \qquad \gamma \in \Gamma, \quad 0 < r < 1\,.$$

Naturally $0 \le d \le n$. Any d-set with $d < n$ is uniformly porous.

Definition 3. Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \ne \mathbb{R}^n$ and $\Gamma = \partial\Omega$.

(i) The domain Ω is said to be E-thick if one can find for any interior cube $Q^i \subset \Omega$ with

$$\ell(Q^i) \sim 2^{-j}, \quad \text{and} \quad \operatorname{dist}(Q^i,\Gamma) \sim 2^{-j}\,, \quad j \ge j_0 \in \mathbb{N}$$

a complementing exterior cube $Q^e \subset \mathbb{R}^n \setminus \Omega$ with

$$\ell(Q^e) \sim 2^{-j}, \quad \text{and} \quad \operatorname{dist}(Q^e,\Gamma) \sim \operatorname{dist}(Q^e,Q^i) \sim 2^{-j}\,, \quad j \ge j_0 \in \mathbb{N}\,.$$

Here Q^i and Q^e denote cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates. Moreover $\ell(Q)$ denote a side-length of the cube Q.

(ii) The domain Ω is said to be E-porous if there is a number η with $0 < \eta < 1$ such that one finds for any ball $B(\gamma,r) \subset \mathbb{R}^n$ centred at $\gamma \in \Gamma$ and of radius r with $0 < r < 1$, a ball $B(y,\eta r)$ with

$$B(y,\eta r) \subset B(\gamma,r) \qquad \text{and} \qquad B(y,\eta r) \cap \overline{\Omega} = \emptyset\,.$$

(iii) The domain Ω is called uniformly E-porous if it is E-porous and Γ is uniformly porous.

Remark 2. If Ω is E-porous, then Ω is E-thick and $|\Gamma| = 0$. On the other hand, if Ω is E-thick and Γ is a d-set, then Ω is uniformly E-porous and $n-1 \le d < n$. If the domain is uniformly E-porous then one can characterize $\bar{A}^s_{p,q}(\Omega)$ spaces in terms of wavelet expansion of the distributions. cf. Section 1.3 below.

Remark 3. Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. Let $\widetilde{W}^k_p(\Omega)$, $1 < p < \infty$, $k = 1, 2, \ldots,$ be the completion of the space $C_0^\infty(\Omega)$ in the norm

$$\|f|\widetilde{W}^k_p(\Omega)\| = \sum_{|\alpha|\le k} \|\partial^\alpha f|L_p(\Omega)\|\,.$$

Then $\widetilde{W}^k_p(\Omega) = \tilde{F}^k_{p,2}(\Omega) = \bar{F}^k_{p,2}(\Omega)$ and the corresponding norms are equivalent, cf. [14, Theorem 4.30]. We put also $\widetilde{W}^0_p(\Omega) = L_p(\Omega)$.

1.2. Quasi-bounded domains

We recall that an unbounded domain Ω in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega, |x|\to\infty} \operatorname{dist}(x, \partial\Omega) = 0\,.$$

An unbounded domain is not quasi-bounded if, and only if, it contains infinitely many pairwise disjoint congruent balls, cf. [1], page 173. We will always assume in the paper that Ω in $\mathbb{R}^n$ is a quasi-bounded uniformly E-porous domain. There are quasi-bounded domains that are not E-porous or even not E-thick, cf. e.g. [1], page 176 for the example of quasi-bounded domain with empty exterior.

To formulate the properties of embedding of function spaces defined on quasi-bounded domains we need some quantities describing quasi-boundedness of the domain. For that reason we introduced in [13] *a box packing number* $b(\Omega)$ of an open set Ω. We recall the definition here.

Let $Q_{j,m}$ denote a dyadic cube in $\mathbb{R}^n$ with side-length 2^{-j}, $j \in \mathbb{N}_0$, whose vertices are adjacent points of the lattice $2^{-j}m$ and $m \in \mathbb{Z}^n$. More precisely we put

$$Q_{j,m} = [0, 2^{-j})^n + 2^{-j}m, \qquad m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0\,.$$

Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set $\Omega \ne \mathbb{R}^n$. Let

$$b_j(\Omega) = \sup\Big\{k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega\,, \quad Q_{j,m_\ell} \text{ being dyadic cubes of side-length } 2^{-j}\Big\}, \tag{3}$$

$j = 0, 1, \ldots$. If there is no dyadic cube of size 2^{-j} contained in Ω we put $b_j(\Omega) = 0$. The following properties of the sequence $\big(b_j(\Omega)\big)_{j=0,1,2,\ldots}$ are obvious:

- There exists a constant $j_0 = j_0(\Omega) \in \mathbb{N}_0$ such that for any $j \ge j_0$ we have

$$0 < 2^{n(j-j_0)} b_{j_0}(\Omega) \le b_j(\Omega). \tag{4}$$

- If $|\Omega| < \infty$, then

$$b_j(\Omega) 2^{-jn} \le |\Omega|\,. \tag{5}$$

It follows from (4) that if $0 < s < n$, then $\lim_{j\to\infty} b_j(\Omega)2^{-js} = \infty$. Moreover if $s_1 < s_2$ and the sequence $b_j(\Omega)2^{-js_1}$ is bounded, then $\lim_{j\to\infty} b_j(\Omega)2^{-js_2} = 0$. Thus there exists at most one number $b \in \mathbb{R}$ such that $\limsup_{j\to\infty} b_j(\Omega)2^{-js} = \infty$ if $s < b$ and $\lim_{j\to\infty} b_j(\Omega)2^{-js} = 0$ if $s > b$. We put

$$b(\Omega) = \sup\left\{t \in \mathbb{R}_+ : \limsup_{j\to\infty} b_j(\Omega)2^{-jt} = \infty\right\}. \tag{6}$$

Remark 4. For any non-empty open set $\Omega \subset \mathbb{R}^n$ we have $n \le b(\Omega) \le \infty$. If Ω is unbounded and not quasi-bounded, then $b(\Omega) = \infty$. But there are also quasi-bounded domains such that $b(\Omega) = \infty$, cf. Example 2 below. Moreover it follows from (4) and (5) that if the measure $|\Omega|$ is finite, then $b(\Omega) = n$.

Remark 5. Let again $j_0(\Omega) = \min\{j : b_j(\Omega) > 0\}$. We define the function $\mathbb{B} : [0,\infty) \to [0,\infty)$ by the following conditions:

$$\begin{aligned}
&\mathbb{B}(j) = b_j(\Omega) \qquad \text{for any} \quad j \ge \max(1, j_0),\\
&\mathbb{B}(t) \quad \text{is linear on any of the intervals} \quad [j, j+1], \quad j \ge \max(1, j_0),\\
&\mathbb{B}(t) \quad \text{is linear on the interval} \quad [0, \max(1, j_0)] \quad \text{and} \quad \mathbb{B}(0) = 0.
\end{aligned}$$

The function $\mathbb{B}$ is strictly increasing, cf. (4). Thus there exist the inverse function $\mathbb{B}^{-1}$ to the function $\mathbb{B}$ that is also strictly increasing and consequently $b_j \le k < b_{j+1}$ implies $j \le \mathbb{B}^{-1}(k) < j+1$ if $j \ge \max(1, j_0)$.

We give simple examples of quasi-bounded uniformly E-porous domains.

Example 1. Let $\alpha > 0$ and $\beta > 0$. We consider the open sets $\omega_\alpha, \omega_{1,\beta} \subset \mathbb{R}^2$ defined as follows

$$\begin{aligned}
\omega_\alpha &= \left\{(x,y) \in \mathbb{R}^2 : \ |y| < x^{-\alpha}, x > 1\right\},\\
\omega_{1,\beta} &= \left\{(x,y) \in \mathbb{R}^2 : \ |y| < x^{-1}(\log x)^{-\beta}, \ x > e\right\}.
\end{aligned}$$

One can easily calculate that

$$b_j(\omega_\alpha) \sim \begin{cases} 2^{j(\alpha^{-1}+1)} & \text{if} \quad 0 < \alpha < 1\,,\\ j2^{2j} & \text{if} \quad \alpha = 1\,,\\ 2^{2j} & \text{if} \quad \alpha > 1\,, \end{cases}$$

and

$$b_j(\omega_{1,\beta}) \sim \begin{cases} 2^{2j}\, j^{1-\beta} & \text{if} \quad \beta < 1\,,\\ 2^{2j}\, \log j & \text{if} \quad \beta = 1\,,\\ 2^{2j} & \text{if} \quad \beta > 1\,. \end{cases}$$

In consequence

$$b(\omega_\alpha) = \begin{cases} \alpha^{-1} + 1 & \text{if} \quad 0 < \alpha < 1\,,\\ 2 & \text{if} \quad \alpha \ge 1\,. \end{cases}$$

Moreover the limit $\lim_{j\to\infty} b_j(\omega_\alpha)2^{-jb(\omega_\alpha)}$ is a positive finite number if $\alpha \ne 1$. But if $\alpha = 1$, then the limit equals infinity.

In the similar way $b(\omega_{1,\beta}) = 2$ for all β and $\lim_{j\to\infty} b_j(\omega_{1,\beta})2^{-jb(\omega_{1,\beta})} = \infty$ if $0 < \beta \le 1$.

Example 2. Now we construct an example of a quasi-bounded domain in $\mathbb{R}^n$ with prescribed sequence $b_j(\Omega)$.

Let $Q_{j,m} = [0, 2^{-j})^n + 2^{-j}m$ be the usual dyadic cubes. Let $\{\widetilde{b}_j\}_j$, $j = 0, 1, 2, \ldots$, be a sequence of non-negative integers. We put $a_j = \sum_{\ell=0}^{j-1} 2^{-\ell}\widetilde{b}_\ell$ and

$$\mathcal{A}_j = \big\{(m_1, \ldots, m_n) \in \mathbb{Z}^n : \, a_j < 2^{-j}m_1 \le a_j + 2^{-j}\widetilde{b}_j, \, m_2 = \ldots = m_n = 0\big\}.$$

Please note that $\mathcal{A}_j$ is empty if $\widetilde{b}_j = 0$. At the end the domain Ω is defined in the following way

$$\Omega = \Big(\bigcup_{j\in\mathbb{N}_0} \widetilde{\Omega}_j\Big)^\circ \qquad \text{where} \qquad \widetilde{\Omega}_j = \bigcup_{m\in\mathcal{A}_j} \overline{Q}_{j,m}.$$

The set Ω is an open proper connected subset of $\mathbb{R}^n$. The cubes $Q_{j,m}$ and $Q_{k,n}$ are disjoint if $m \in \mathcal{A}_j$, $n \in \mathcal{A}_k$ and $j \neq k$. It follows by the construction that Ω is a uniformly E-porous domain. The cardinality of the set $\mathcal{A}_j$ equals $\widetilde{b}_j$, so $b_j = 2^n b_{j-1} + \widetilde{b}_j$ with $b_0 = \widetilde{b}_0$. Thus

$$b_j(\Omega) = \sum_{\ell=0}^{j} 2^{(j-\ell)n}\widetilde{b}_\ell \, .$$

This finishes the construction.

Let conversely $\{\sigma_j\}_j$ be an any sequence of non-negative integers such that

$$2^n\sigma_j \le \sigma_{j+1} \, .$$

Then taking $\widetilde{b}_0 = \sigma_0$ and $\widetilde{b}_j = \sigma_j - 2^n\sigma_{j-1}$ in case $j \ge 1$ we can construct a quasi-bounded domain such that $b_j(\Omega) = \sigma_j$. The case $b(\Omega) = \infty$ can be obtained for example in case $\sigma_j = 2^{j^\beta}$, $\beta > 1$.

For any domain $\Omega \neq \mathbb{R}^n$ and any $r > 0$ we put

$$\Omega_r = \{x \in \Omega : \, \operatorname{dist}(x, \partial\Omega) > r\}. \tag{7}$$

The domain Ω is the union of domains Ω_r, moreover if the domain Ω is quasi-bounded then $|\Omega_r| < \infty$ for any $r > 0$.

Proposition 1. *Let Ω be a quasi-bounded domain in $\mathbb{R}^n$. Then*

$$b(\Omega) = n + \limsup_{r\to 0} \left| \frac{\log_2 |\Omega_r|}{\log_2 r} \right| . \tag{8}$$

Proof. *Step 1.* If $|\Omega|$ is finite then $b(\Omega) = n$ and identity (8) holds trivially. Let $|\Omega|$ be infinite. For simplicity we assume that $|\Omega_r| > 1$ for any $0 < r < 1$. The rest is the case of rescaling.

First we prove that

$$b(\Omega) = \limsup_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} \tag{9}$$

Let $b = b(\Omega) \in \mathbb{R}$ and $s < b < t$. Then $\lim_{j\to\infty} 2^{-jt}b_j(\Omega) = 0$. In consequence for any $\varepsilon > 0$ there exists j_0 such that

$$\frac{\log_2 b_j(\Omega)}{j} \le t + \frac{\log_2 \varepsilon}{j} \qquad \text{if} \qquad j \ge j_0. \tag{10}$$

In consequence

$$\limsup_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} \le t\,. \tag{11}$$

In the similar way one can find subsequence j_k such that

$$s \le \lim_{k\to\infty} \frac{\log_2 b_{j_k}(\Omega)}{j_k}\,. \tag{12}$$

Taking the sequence $s_\ell \to b$, $s_\ell < b$, we find sequences $(j_k^{(\ell)})_k$ that satisfy (12) with s_ℓ instead of s. We may assume also that $\log_2 b_{j_k^{(\ell)}}(\Omega)/j_k^{(\ell)} > s_\ell - 2^{-\ell}$ for any k. Then using the diagonal argument we find that

$$b(\Omega) \le \limsup_{\ell\to\infty} \frac{\log_2 b_{j_\ell^{(\ell)}}(\Omega)}{j_\ell^{(\ell)}}\,. \tag{13}$$

Now (11) and (13) imply

$$\limsup_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} = b(\Omega)\,. \tag{14}$$

This proves (9) for $b(\Omega) \in \mathbb{R}$. If $b(\Omega) = \infty$ then the proof is similar but easier.

Step 2. Let

$$\mathcal{M}_j = \{m \in \mathbb{Z}^n : Q_{j,m} \subset \Omega\}, \qquad \text{and} \quad \widetilde{\Omega}_j = \bigcup_{m\in\mathcal{M}_j} \overline{Q}_{j,m}, \qquad j = 0,1,2,\ldots \tag{15}$$

Then $|\widetilde{\Omega}_j| = 2^{-jn} b_j(\Omega)$ and we have immediately from (9) that

$$b(\Omega) = \limsup_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} = n + \limsup_{j\to\infty} \frac{\log_2 |\widetilde{\Omega}_j|}{j}. \tag{16}$$

One can easily see that

$$0 \le \operatorname{dist}(\widetilde{\Omega}_j, \partial\Omega) \le 2^{-j}\sqrt{n}$$

and consequently

$$\Omega_{r_j} \subset \widetilde{\Omega}_j \qquad \text{with} \qquad r_j = 2^{-j}\sqrt{n}\,. \tag{17}$$

Let $\kappa \in \mathbb{N}$ be fixed with $1 > 2^{-\kappa+1}\sqrt{n}$. For each $Q_{j,m} \subset \Omega$ there exist always an open cube $\widetilde{Q}_m$ of side length $2^{-j-\kappa}$ with $\widetilde{Q}_m \subset Q_{j,m}$ such that $\operatorname{dist}(\widetilde{Q}_m, \partial\Omega) \ge 2^{-j-1} \ge 2^{-j-\kappa}\sqrt{n}$. In consequence

$$|\Omega_{r_j}| \le |\widetilde{\Omega}_j| \le \sum_{m\in\mathcal{M}_j} 2^{\kappa n} |\widetilde{Q}_m| \le 2^{\kappa n} |\Omega_{r_{j+\kappa}}|\,, \tag{18}$$

and

$$\frac{|\log_2 r_j|}{j} \cdot \frac{\log_2 |\Omega_{r_j}|}{|\log_2 r_j|} \le \frac{\log_2 |\widetilde{\Omega}_j|}{j} \le \frac{\kappa n}{j} + \frac{|\log_2 r_{j+\kappa}|}{j} \cdot \frac{\log_2 |\Omega_{r_{j+\kappa}}|}{|\log_2 r_{j+\kappa}|}\,. \tag{19}$$

Now (8) follows from the last inequalities and (16) at first for the special sequence $(r_j)_j$ and then for general r. □

For a real number a we put $a_+ := \max(a, 0)$. We will use also the following abbreviations

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \qquad \text{and} \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ .$$

In the next theorem we recall the sufficient and necessary conditions for boundedness and compactness of embeddings of the function spaces defined on a uniformly E-porous quasi-bounded domain. We refer to [13] for the proof.

Theorem 1. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$ and let $s_1 > s_2$.*

(i) *If $b(\Omega) = \infty$, then the embedding*

$$\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega) \tag{20}$$

is compact if, and only if, $p_1 \le p_2$ and

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0\,. \tag{21}$$

(ii) *Let $b(\Omega) < \infty$. The embedding*

$$\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega) \tag{22}$$

is compact if

$$s_1 - s_2 - n\big(\frac{1}{p_1} - \frac{1}{p_2}\big) > \frac{b(\Omega)}{p^*}\,. \tag{23}$$

If the embedding (22) is compact and $\frac{1}{p^*} = 0$, then $s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) > 0$.
If the embedding (22) is compact and $\frac{1}{p^*} > 0$, then $s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) \ge \frac{b(\Omega)}{p^*}$.

Remark 6.

1. Using the same method one can prove that (23) is a sufficient and necessary condition for compactness of the embeddings, except of the case $\limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} = 0$ and $\frac{1}{p^*} > 0$.
2. If the domain Ω is not quasi-bounded, then the embedding (22) is never compact, cf. [13].
3. If Ω is a domain in $\mathbb{R}^n$ with finite Lebesgue measure, then the embedding (22) is compact if, and only if,
$$s_1 - s_2 - \big(\frac{n}{p_1} - \frac{n}{p_2}\big)_+ > 0\,,$$
cf. [13].
4. Thus for a set of finite Lebesgue measure we get the same conditions for compactness as for bounded smooth domains. On the other hand if the domain is not quasi-bounded then the Sobolev embeddings are never compact. So the most interesting case are the quasi-bounded domains with infinite measure. If Ω is such a domain, then all numbers $b_j(\Omega)$ are finite. But in contrast to the domain with finite measure the numbers $b_j(\Omega)$ are not asymptotically equivalent to 2^{jn}.

1.3. Admissible sequences

To estimate the entropy numbers of embeddings of the Besov spaces on quasi-bounded domains we used in [13] the following rather strong condition

$$0 < \liminf_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty\,. \tag{24}$$

Now we weaken this condition. To do it we use the notion of admissible sequences.

Definition 4. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$, $\sigma_j > 0$, is called an *admissible sequence*, if there are two constants $0 < d_0 = d_0(\sigma) \le d_1 = d_1(\sigma) < \infty$ such that

$$d_0\,\sigma_j \le \sigma_{j+1} \le d_1\sigma_j \quad \text{for any} \quad j \in \mathbb{N}_0. \tag{25}$$

Remark 7. M. Bricchi and S. Moura introduced in [2] the upper, respectively the lower, Boyd (Matuszewska–Orlicz) index of a given admissible sequence as follows. First let

$$\bar{\sigma}_j := \sup_{k\ge 0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \underline{\sigma}_j := \inf_{k\ge 0} \frac{\sigma_{j+k}}{\sigma_k} \qquad j \in \mathbb{N}_0.$$

Then the expressions

$$\alpha_\sigma := \inf_{j\in\mathbb{N}} \frac{\log_2 \bar{\sigma}_j}{j} = \lim_{j\to\infty} \frac{\log_2 \bar{\sigma}_j}{j} \quad \text{and} \quad \beta_\sigma := \sup_{j\in\mathbb{N}} \frac{\log_2 \underline{\sigma}_j}{j} = \lim_{j\to\infty} \frac{\log_2 \underline{\sigma}_j}{j}$$

are called the (*upper* and respectively *lower*) *Boyd indices* of the sequence σ.

Remark 8. 1. Given an admissible sequence σ with Boyd indices α_σ and β_σ then it is possible to find for any $\varepsilon > 0$ a constant c_ε such that

$$c_\varepsilon^{-1} 2^{j(\beta_\sigma-\varepsilon)} \le \sigma_j \le c_\varepsilon 2^{j(\alpha_\sigma+\varepsilon)}\,. \tag{26}$$

2. If Ω is a quasi-bounded domain in $\mathbb{R}^n$ and the sequence $b = (b_j(\Omega))_{j\in\mathbb{N}_0}$ defined by (3) is admissible then it follows easily from (4), (9) and (26) that

$$n \le \beta_b \le \liminf_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} \le \limsup_{j\to\infty} \frac{\log_2 b_j(\Omega)}{j} = b(\Omega) \le \alpha_b. \tag{27}$$

If the sequence $(b_j(\Omega))$ satisfies the condition (24) then it is admissible and $b(\Omega) = \alpha_b = \beta_b$. On the other hand there are quasi-bounded domains such that the sequence $(b_j(\Omega))$ is admissible but $b(\Omega) < \alpha_b$. We postpone the detail discussion to the end of Section 3.

3. Let $n < b(\Omega) < \infty$. The assumption $b(\Omega) < \infty$ implies that for any $\varepsilon > 0$ we have

$$b_k(\Omega) \le 2^{k(b(\Omega)+\varepsilon)}$$

at least for sufficiently large k. On the other hand (9) guarantees that there is a subsequence k_ℓ such that

$$2^{k_\ell(b(\Omega)-\varepsilon)} \le b_{k_\ell}(\Omega).$$

Thus $b(\Omega)$ says us that there is an increasing sequence of positive integers k_ℓ such that the domain Ω contains approximately $2^{k_\ell b(\Omega)}$ dyadic cubes of size 2^{-k_ℓ}.

Let $j,k \in \mathbb{N}_0$. Then $b_{j+k}(\Omega) = 2^{nj} b_k(\Omega) + \widetilde{b_{k,j}}(\Omega)$, where $\widetilde{b_{k,j}}(\Omega)$ is a number of dyadic cubes of size $2^{-(j+k)}$ contained in Ω but not in $\widetilde{\Omega}_k$, cf. (15). The sequence $(b_k(\Omega))_k$ is admissible if and only if there exists a positive constant $\tilde{d}_1$ such that $\widetilde{b_{k,1}}(\Omega) \le \tilde{d}_1\, b_k(\Omega)$. More generally, if $(b_k(\Omega))_k$ is admissible then for any k we have

$$\widetilde{b_{k,j}}(\Omega) \le \tilde{d}_j\, b_k(\Omega)\,, \qquad \text{with} \qquad \tilde{d}_j = d_1^j - 2^{jn}\,.$$

On the other hand the definition of the upper Boyd index implies that for any step j there is an increasing subsequence k_ℓ, $\ell \in \mathbb{N}$, such that

$$b_{k_\ell}(\Omega)\big(2^{j(\alpha_b - \frac{1}{\ell})} - 2^{jn} - \frac{1}{\ell}\big) \le \widetilde{b_{k_\ell,j}}(\Omega)\,.$$

Let the sequence $(b_k(\Omega))_k$ be admissible and let the step j be fixed. The meaning of the last formula is the following. There is an increasing sequence $(k_\ell)_\ell$, $k_\ell \in N_0$, such that the number of dyadic cubes of size $2^{-(j+k_\ell)}$ belonging to Ω but not to $\widetilde{\Omega}_{k_\ell}$ is proportional to the number of cubes of the same size contained in $\widetilde{\Omega}_{k_\ell}$. Moreover the ratio of $\widetilde{b_{k_\ell,j}}(\Omega)$ and $b_{k_\ell}(\Omega)$ is approximately not smaller than $2^{j\alpha_b} - 2^{jn}$. So, roughly speaking, the Boyd index α_b describes somehow the regularity of the process how the domain Ω becomes thin.

We will need also the following definition, cf. [12].

Definition 5. A sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is called *almost strongly increasing* if there is a natural number κ_0 such that

$$2\gamma_j \le \gamma_k \quad \text{for all} \quad j \quad \text{and} \quad k \quad \text{with} \quad j+\kappa_0 \le k.$$

1.4. Wavelet characterisation of function spaces on uniformly E-porous domains

Let $\beta = \{\beta_j\}_{j=0}^\infty$ be a sequence of positive numbers and let $N_j \in \overline{\mathbb{N}}$, $j \in \mathbb{N}_0$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We will work with the following sequence spaces

$$\ell_q\Big(\beta_j \ell_p^{N_j}\Big) := \Bigg\{\lambda = \{\lambda_{j,k}\}_{j,k} : \quad \lambda_{j,k} \in \mathbb{C}\,, \\ \Big\|\lambda\,\Big|\ell_q\Big(\beta_j \ell_p^{N_j}\Big)\Big\| = \left(\sum_{j=0}^{\infty} \beta_j^q \left(\sum_{k=0}^{N_j} |\lambda_{j,k}|^p\right)^{q/p}\right)^{1/q} < \infty\Bigg\}$$

(usual modifications if $p=\infty$ and/or $q=\infty$). If $N_j = \infty$ then $\ell_p^{N_j} = \ell_p$.

Theorem 2. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$. Let $u \in \mathbb{N}_0$. Then there exists an orthonormal basis*

$$\big\{\Phi_r^j :\ j \in \mathbb{N}_0;\ r = 1,\dots,N_j\big\}\,, \qquad \Phi_r^j \in C^u(\Omega), \quad j \in \mathbb{N}_0,\ r = 1,\dots,N_j$$

in $L_2(\Omega)$, such that if $u > \max(s, \sigma_p - s)$ then $f \in D'(\Omega)$ is an element of $\bar{B}^s_{p,q}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty}\sum_{r=1}^{N_j} \lambda_r^j 2^{-jn/2}\Phi_r^j, \qquad \lambda \in \ell_q\Big(2^{j(s-\frac{n}{p})}\ell_p^{N_j}\Big), \tag{28}$$

and the series is unconditionally convergent in $D'(\Omega)$.

Furthermore, if $f \in \bar{B}^s_{p,q}(\Omega)\beta$, then the representation (28) is unique with $\lambda = \lambda(f)$

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2}(f, \Phi^j_r),$$

where (,) is a dual pairing and

$$I : \bar{B}^s_{p,q}(\Omega) \ni f \mapsto \lambda(f) \in \ell_q\left(2^{j(s-\frac{n}{p})}\ell_p^{N_j}\right)$$

is an isomorphism. If in addition $\max(p,q) < \infty$, then $\left\{\Phi^j_r\right\}$ is an unconditional basis in $\bar{B}^s_{p,q}(\Omega)$.

There is a strict relation between the numbers N_j used in the last theorems and the sequence $b_j(\Omega)$. Namely it was proved in [13] that

$$b_{j-2}(\Omega) \le N_j \le b_j(\Omega)\,. \tag{29}$$

Remark 9. The above theorem was proved by Hans Triebel, cf. [14, Theorem 3.23]. He used so-called u-wavelet systems, cf. Chapter 2 of the above book for the construction of this wavelet system. The sketch of the construction can be found in [13]. If we assume that the domain Ω is E-thick, then the theorem holds for $\bar{B}^s_{p,q}(\Omega)$ with $s \neq 0$, cf. Theorem 3.13 ibidem. Similar results hold for $\bar{F}^s_{p,q}(\Omega)$ spaces.

2. Entropy numbers of the compact embeddings: the case $p_1 = p_2$

First we recall the definition of entropy numbers.

Definition 6. Let $T : X \to Y$ be a bounded linear operator between complex quasi-Banach spaces, and let $k \in \mathbb{N}$. Then the k-th entropy number of T is defined as

$$e_k(T : X \to Y) := \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ of radius } \varepsilon\}\,,$$

where $B_X := \{x \in X : \|x\|_X \le 1\}$ denotes the closed unit ball of X.

In particular, $T : X \to Y$ is compact if, and only if, $\lim_{k\to\infty} e_k(T) = 0$. For details and basic properties like multiplicativity, additivity, behaviour under interpolation and the relation to eigenvalues of the compact operator we refer to the monographs [3,6,7].

Remark 10. Now our aim is to describe the asymptotic behaviour of entropy numbers of compact embeddings (20) and (22). It follows from Theorem 2 that this can be reduced to estimates of entropy numbers of the corresponding embeddings of sequences spaces $\ell_q(2^{j(s-\frac{n}{p})}\ell_p^{N_j})$. Furthermore it follows from the inequalities (29) that the asymptotic behaviour of the entropy numbers of embeddings between the spaces $\ell_q(2^{j(s-\frac{n}{p})}\ell_p^{N_j})$ is the same as the asymptotic behaviour of the entropy numbers of embeddings between the spaces $\ell_q(2^{j(s-\frac{n}{p})}\ell_p^{b_j(\Omega)})$. Thus we can use $\ell_q(2^{j(s-\frac{n}{p})}\ell_p^{b_j(\Omega)})$ instead of $\ell_q(2^{j(s-\frac{n}{p})}\ell_p^{N_j})$.

If $p_1 = p_2$ then in both cases, $b(\Omega) = \infty$ or $b(\Omega) < \infty$, the embedding (20) is compact if and only if $s_1 - s_2 > 0$. This simplified the estimation of the asymptotic behaviour of entropy numbers of the embeddings. For convenience we use the function $\mathbb{B}$ related to the sequence $b_j(\Omega)$ introduced in Remark 5.

The result in the special case $p_1 = p_2$ holds for arbitrary sequences $b_j(\Omega)$ without any additional properties. The estimation is based on Lemma 1 and Corollary 1 from [12]. For convenience of the reader we recall them bellow.

Lemma 1. *Let $0 < p_1, p_2, q_1, q_2 \le \infty$ and $\delta > 0$. Let $\{b_j(\Omega)\}_{j=0}^\infty$ be an arbitrary sequence defined by (3) and let $\{2^{-j\delta} b_j(\Omega)^{(\frac{1}{p_2}-\frac{1}{p_1})_+}\}_{j=0}^\infty \in \ell_{q*}$.*

(i) Then

$$c\,2^{-L\delta}(2b_L(\Omega))^{-(\frac{1}{p_1}-\frac{1}{p_2})} \le e_{2b_L(\Omega)}(id : \ell_{q_1}(2^{j\delta}\ell_{p_1}^{b_j(\Omega)}) \to \ell_{q_2}(\ell_{p_2}^{b_j(\Omega)})) \tag{30}$$

for all natural numbers L, where c is positive constant independent of L.

(ii) Denote for $x \in \ell_{q_2}(\ell_{p_2}^{b_j(\Omega)})$ the j-part $id_j\, x$ of $id\, x$ by

$$id_j\, x = (\delta_{j,k} x_{k,l})_{k\in\mathbb{N}_0, l=1,\dots,b_j(\Omega)} = (0,\dots,0,x_{j,1},\dots,x_{j,b_j(\Omega)},0,\dots)\,.$$

Then

$$\begin{aligned} &\|\,(id - \sum_{j=0}^{N} id_j)\, x \mid \ell_{q_2}(\ell_{p_2}^{b_j(\Omega)})\| \\ &\le \|\{2^{-j\delta} b_j(\Omega)^{(\frac{1}{p_2}-\frac{1}{p_1})_+}\}_{j=N+1}^\infty | \ell_{q*}\|\ \|\, x \mid \ell_{q_1}(2^{j\delta}\ell_{p_1}^{b_j(\Omega)})\|\,. \end{aligned} \tag{31}$$

Theorem 3. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. Let $\bar{A}^{s_1}_{p,q_1}(\Omega)$ and $\bar{A}^{s_2}_{p,q_2}(\Omega)$ be the function spaces defined in Definition 1. Let $\delta = s_1 - s_2 > 0$. Then for $k \in \mathbb{N}$*

$$e_k\Big(\bar{A}^{s_1}_{p,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p,q_2}(\Omega)\Big) \sim 2^{-(s_1-s_2)\mathbb{B}^{-1}(k)}. \tag{32}$$

Proof. *Step 1.* The estimation from below follows from the wavelet characterisation of the function spaces given in Theorem 2, Remark 10 and Lemma 1.

Let $b_N(\Omega) \le k \le b_{N+1}(\Omega) \le 2b_{N+1}(\Omega)$, then $N \le \mathbb{B}^{-1}(k) < N+1$ and by (30) we have

$$c\,2^{-(N+1)\delta} \le e_{2b_{N+1}}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)}))$$

respectively

$$c\,2^{-\delta}\,2^{-\mathbb{B}^{-1}(k)\delta} \le e_k(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)}))\,.$$

Step 2. For the estimation in the opposite direction we can use some ideas and notations from the proof of Theorem 3 in [12]. Without loss of generality we will assume $b_0(\Omega) > 0$.

Substep 2.1. The assumptions $p = p_1 = p_2$ and $\delta > 0$ imply $\{2^{-j\delta}\}_{j=0}^\infty \in \ell_{q*}$ and we obtain by Lemma 1

$$\|(id - \sum_{j=0}^{N} id_j)\ x \mid \ell_{q_2}(\ell_p^{b_j(\Omega)})\| \le c\,2^{-N\delta}\|\, x \mid \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)})\|$$

and

$$\|(id - \sum_{j=0}^{N} id_j) \mid \mathcal{L}(\ell_{q_2}(\ell_p^{b_j(\Omega)}), \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}))\| \le c\,2^{-N\delta}\,.$$

Substep 2.2. Let $\rho = \min(1, p, q_2)$, then $\ell_{q_2}(\ell_p^{b_j(\Omega)})$ is a ρ-Banach space and we get

$$e_k^\rho(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \le c2^{-N\delta\rho} + \sum_{j=0}^{N} e_{k_j}^\rho(id_j) \tag{33}$$

where

$$k = \sum_{j=0}^{N} k_j - (N+2)\,.$$

We consider the diagram

$$\begin{array}{ccc} \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) & \xrightarrow{T_j} & \ell_p^{b_j(\Omega)}) \\ \mathrm{id}_j \big\downarrow & & \big\downarrow \mathrm{id}^{(j)} \\ \ell_{q_2}(\ell_p^{b_j(\Omega)}) & \xleftarrow{\tilde{id}^{(j)}} & \ell_p^{b_j(\Omega)}, \end{array}$$

and obtain

$$e_{k_j}(id_j) \le 2^{-j\delta}\, e_{k_j}(id^{(j)} : \ell_p^{b_j(\Omega)} \to \ell_p^{b_j(\Omega)})\,. \tag{34}$$

But in this case the estimation of the entropy numbers in finite dimensional ℓ_p-spaces, see for example [7, Section 3.2.2] or [12], leads to

$$e_{k_j}(id_j) \le c\, 2^{-j\delta}\, 2^{-\frac{k_j}{2b_j(\Omega)}} \quad \text{if} \quad k_j \ge 2b_j(\Omega)\,. \tag{35}$$

Substep 2.3. For arbitrary fixed ε with $0 < \varepsilon < n$ we put

$$k_j := 2b_j(\Omega) 2^{(N-j)\varepsilon}\,.$$

Then obviously $k_j \ge 2b_j(\Omega)$ and

$$\begin{aligned} & e_k^\rho(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \\ & \quad \le c\, 2^{-N\delta\rho} + c\, 2^{-N\delta\rho} \sum_{j=0}^{N} 2^{(N-j)\delta\rho}\, 2^{-2^{(N-j)\varepsilon}\rho}\,. \end{aligned} \tag{36}$$

The sum in (36) converges and can be estimated independently of N. So we obtain

$$e_k(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \le c'\, 2^{-N\delta}\,. \tag{37}$$

Moreover the inequality $2^n b_j(\Omega) \le b_{j+1}(\Omega)$ and the choice of ε imply

$$k = \sum_{j=0}^{N} (2b_j(\Omega)) 2^{(N-j)\varepsilon} \le 2b_N(\Omega) \sum_{j=0}^{N} 2^{-(N-j)n+(N-j)\varepsilon} \le c'' b_N(\Omega)\,.$$

This together with (37) gives

$$e_{c'' b_N(\Omega)}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \le c'\, 2^{-N\delta}\,.$$

Substep 2.4. Let $\kappa \in \mathbb{N}$ be chosen in such a way that $c'' 2^{-\kappa n} \le 1$. Then we get

$$c'' b_{N-\kappa} \le c'' 2^{-\kappa n} b_N \le b_N\,.$$

For all k with $b_N \le k < b_{N+1}$ it follows that $N \le \mathbb{B}^{-1}(k) < N+1$ and

$$\begin{aligned} e_k(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) &\le e_{c'' b_{N-\kappa}}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \\ &\le c'\, 2^{-(N-\kappa)\delta} \le c'\, 2^{(\kappa+1)\delta}\, 2^{-\mathbb{B}^{-1}(k)\delta} . \end{aligned}$$

Now the theorem follows from Remark 10 and the wavelet-characterisation, cf. Theorem 2. □

3. Entropy numbers of the compact embeddings: the case $p_1 \neq p_2$

In [13] we proved

$$e_k\Big(\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)\Big) \sim k^{-\gamma} \tag{38}$$

under the condition (24) where

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) .$$

But already Example 4.6 in [13] showed that this is a very restrictive one. Now we want to avoid this strong condition (24). A general approach to estimation of entropy numbers of embeddings of the sequence spaces $\ell_q(\beta_j \ell_p^{b_j})$ with the admissible almost strongly increasing indices β_j and b_j was developed by one of the authors in [12]. We follow this approach.

Lemma 2. *If $\{b_j(\Omega)\}_j$ is an admissible sequence with upper Boyd index α_b and $\delta > \frac{\alpha_b}{p*}$ then the sequence $\{2^{j\delta} b_j(\Omega)^{-1/p*}\}_j$ is almost strongly increasing.*

Proof. The assumption $\delta > \frac{\alpha_b}{p*}$ implies that $\delta > \frac{\alpha_b+\varepsilon}{p*}$ for some suitable $\varepsilon > 0$ that will be fixed. There exists an index κ_1, depending on ε, such that

$$2^{-\kappa_1(\delta - \frac{\alpha_b+\varepsilon}{p*})} \le \frac{1}{2}, \quad \frac{\log_2 \bar{b}_{\kappa_1}}{\kappa_1} < \alpha_b + \varepsilon \quad \text{and consequently} \quad \bar{b}_{\kappa_1} < 2^{\kappa_1(\alpha_b+\varepsilon)}.$$

Let $\gamma_j := 2^{j\delta} b_j^{-1/p*}$. Then

$$\begin{aligned} \gamma_j = 2^{j\delta} b_j^{-1/p*} &\le 2^{-\kappa_1\delta}\, (\bar{b}_{\kappa_1})^{1/p^*} 2^{(j+\kappa_1)\delta} b_{j+\kappa_1}^{-1/p*} \le 2^{-\kappa_1\delta} 2^{(\kappa_1(\alpha_b+\varepsilon))/p^*} \gamma_{j+\kappa_1} \\ &\le 2^{-\kappa_1(\delta - \frac{\alpha_b+\varepsilon}{p*})} \gamma_{j+\kappa_1} \le \frac{1}{2}\gamma_{j+\kappa_1} \qquad \text{since} \qquad \bar{b}_{\kappa_1} \ge \frac{b_{j+\kappa_1}}{b_j} . \end{aligned}$$

Taking $1 \le l \le \kappa_1 - 1$ we get

$$\begin{aligned} 2\gamma_j \le \gamma_{j+\kappa_1} = 2^{(j+\kappa_1)\delta} b_{j+\kappa_1}^{-1/p*} &\le 2^{(\kappa_1 - l)\delta} 2^{n(\kappa_1 - l)/p^*} 2^{(j+l)\delta} b_{j+l}^{-1/p*} \\ &\le 2^{(\kappa_1 - l)(\delta + n/p^*)} \gamma_{j+l} \le c_{\kappa_1} \gamma_{j+l} \end{aligned}$$

where $c_{\kappa_1} = 2^{\kappa_1\delta} 2^{n\kappa_1/p^*}$. Here we used $2^n b_j \le b_{j+1}$.

Now for $\tilde{l} = m\kappa_1 + l$, $1 \le l \le \kappa_1 - 1$, $m \in \mathbb{N}$, we have

$$\gamma_j \le 2^{-1}\gamma_{j+\kappa_1} \le 2^{-m}\gamma_{j+m\kappa_1} \le 2^{-m} c_{\kappa_1} \gamma_{j+\tilde{l}} .$$

At the end we choose m such that $2^{-m}c_{\kappa_1} \le 1/2$ and $\kappa_0 := m\kappa_1$. Then

$$2\gamma_j \le \gamma_{j+\tilde{l}} = \gamma_k \quad \text{if} \quad k = j + \tilde{l} \ge j + \kappa_0.$$

So the sequence is almost strongly increasing. $\square$

First we consider domains with $b_j(\Omega) := 2^{jb(\Omega)}\psi(2^j)$ where ψ is a weak perturbation (slowly varying of order zero) of the polynomial order. In particular

$$c_{\varepsilon,1}2^{-\varepsilon|j-k|} \le \frac{\psi(2^j)}{\psi(2^k)} \le c_{\varepsilon,2}2^{\varepsilon|j-k|} \qquad \text{for all} \quad \varepsilon > 0 \text{ and } k, j\,. \tag{39}$$

See also Example 1. In this case the identity $\alpha_b = b(\Omega)$ holds.

Theorem 4. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$ with $b_j(\Omega) := 2^{jb(\Omega)}\psi(2^j)$ where ψ is a weak perturbation which fulfils (39). Let $\bar{A}^{s_1}_{p,q_1}(\Omega)$ and $\bar{A}^{s_2}_{p,q_2}(\Omega)$ be the function spaces defined in Definition 1. Let $\delta = s_1 - s_2 - n\big(\frac{1}{p_1} - \frac{1}{p_2}\big) > \frac{b(\Omega)}{p*}$. Then for $k \in \mathbb{N}$*

$$e_k\Big(\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)\Big) \sim k^{-\frac{\delta}{b(\Omega)}+(\frac{1}{p_2}-\frac{1}{p_1})}\psi(k^{\frac{1}{b(\Omega)}})^{\frac{\delta}{b(\Omega)}}\,. \tag{40}$$

Proof. Because of $\alpha_b = b(\Omega)$ the sequence $\{2^{j\delta}b_j^{-1/p*}\}_j$ is admissible and almost strongly increasing by Lemma 2. Of course the sequence $b_j(\Omega)$ is admissible too and so we can apply [12, Theorem 3 and Theorem 4]. This gives

$$e_{2b_L}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \sim 2^{-L\delta}b_L^{(\frac{1}{p_2}-\frac{1}{p_1})}. \tag{41}$$

Because the sequence $\{b_j(\Omega)\}_j$ is admissible we have $c_1b_j \le b_{j+1} \le c_2b_j$. Now we consider k with $2b_L \le k < 2b_{L+1}$. Then

$$\begin{aligned} e_k(id :\ \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) &\le e_{2b_L}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \\ &\le b_L^{-\frac{\delta}{b(\Omega)}+(\frac{1}{p_2}-\frac{1}{p_1})}\psi^{\frac{\delta}{b(\Omega)}}(2^L) \le c\,k^{-\frac{\delta}{b(\Omega)}+(\frac{1}{p_2}-\frac{1}{p_1})}\psi^{\frac{\delta}{b(\Omega)}}(k^{\frac{1}{b(\Omega)}}). \end{aligned}$$

The estimation from below is obtained in a similar way.

Now Remark 10 and the wavelet characterisation from Theorem 2 prove the theorem. $\square$

Now we consider more general quasi-bounded domains Ω where $\{b_j(\Omega)\}_j$ is an admissible sequence with upper Boyd index α_b – see also Example 2.

Theorem 5. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$ and $\{b_j(\Omega)\}_j$ is an admissible sequence with upper Boyd index α_b. Let $\bar{A}^{s_1}_{p,q_1}(\Omega)$ and $\bar{A}^{s_2}_{p,q_2}(\Omega)$ be the function spaces defined in Definition 1. Let $\delta = s_1 - s_2 - n\big(\frac{1}{p_1} - \frac{1}{p_2}\big) > \frac{\alpha_b}{p*}$. Then for $k \in \mathbb{N}$*

$$e_k\Big(\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)\Big) \sim 2^{-\mathbb{B}^{-1}(k)\delta}k^{(\frac{1}{p_2}-\frac{1}{p_1})}. \tag{42}$$

Here $\mathbb{B}^{-1}$ is the inverse function to the strongly increasing sequence $\{b_j(\Omega)\}_j$ – see Remark 5.

Proof. Let $\beta_j := 2^{j\delta}$ with $\delta > \frac{\alpha_b}{p*}$. Then the sequence $\{2^{j\delta} b_j^{1/p*}\}_j$ is almost strongly increasing by Lemma 2 and we can apply again as above [12, Theorem 3 and Theorem 4]. This gives again

$$e_{2b_L}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \sim 2^{-L\delta} b_L^{(\frac{1}{p_2}-\frac{1}{p_1})}.$$

The sequence $\{2^{j\delta} b_j^{1/p*}\}_j$ is admissible and we have

$$c_1\, 2^{j\delta} b_j^{1/p*} \le 2^{(j+1)\delta} b_{j+1}^{1/p*} \le c_2\, 2^{j\delta} b_j^{1/p*}\,. \tag{43}$$

Let k with $2b_L \le k < 2b_{L+1}$. As in the previous proof we obtain

$$\begin{aligned} e_k(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) &\sim e_{2b_L}(id : \ell_{q_1}(2^{j\delta}\ell_p^{b_j(\Omega)}) \to \ell_{q_2}(\ell_p^{b_j(\Omega)})) \\ &\sim 2^{-L\delta} b_L^{(\frac{1}{p_2}-\frac{1}{p_1})} \sim 2^{-\mathbb{B}^{-1}(k)\delta} k^{(\frac{1}{p_2}-\frac{1}{p_1})}\,. \end{aligned}$$

Here we used substantial the property (43).

Again the theorem follows from Remark 10 and Theorem 2. □

In the case $b(\Omega) < \alpha_b$ the assumption $\delta > \frac{\alpha_b}{p^*}$ is stronger than $\delta > \frac{b(\Omega)}{p^*}$. In both cases the embedding is compact but only in the first one we can describe asymptotically the behaviour of the entropy numbers. One can ask if there exist uniformly E-porous quasi-bounded domains with $b(\Omega) < \alpha_b$. Due to the construction in Example 2 we can reduce this question to the construction of the sequences $\{\sigma_j\}_j$.

Example 3. Let $n \le s_0 < s_1$. Consider a sequence $j_l = 2^l$ and a sequence $\{\sigma_j\}_j$ defined by

$$\begin{aligned} \sigma_{j_{2l}} &:= 2^{j_{2l}\frac{2s_1+s_0}{3}} \\ \sigma_j &:= \sigma_{j_{2l}}\, 2^{(j-j_{2l})s_0} && j_{2l} \le j < j_{2l+1} \\ \sigma_{j_{2l+1}} &:= 2^{j_{2l+1}\frac{s_1+2s_0}{3}} \\ \sigma_j &:= \sigma_{j_{2l+1}}\, 2^{(j-j_{2l+1})s_1} && j_{2l+1} \le j < j_{2l+2}\,. \end{aligned}$$

Then $\bar{\sigma}_j = \sup_{k\ge 0} \frac{\sigma_{j+k}}{\sigma_k} = 2^{js_1}$ and $\underline{\sigma}_j = \inf_{k\ge 0} \frac{\sigma_{j+k}}{\sigma_k} = 2^{js_0}$, so $\alpha_\sigma = s_1$ and $\beta_\sigma = s_0$, see also Remark 7. But any σ_j can be estimated in the following way

$$2^{j\frac{s_1+2s_0}{3}} \le \sigma_j \le 2^{j\frac{2s_1+s_0}{3}},$$

so $b(\Omega) = \frac{2s_1+s_0}{3}$.

Example 4. Let $n \le s$. Consider now a sequence $j_l = l!$ and a sequence $\{\tau_j\}_j$ defined by

$$\tau_{j_l} := l!\, 2^{j_l s} \quad \text{and} \quad \tau_j := \tau_{j_l}\, 2^{(j-j_l)s} \qquad j_l \le j < j_{l+1}\,.$$

Then the sequence $\{\tau_j\}_j$ is not admissible. Moreover, $\bar{\tau}_j = \sup_{k\ge 0} \frac{\tau_{j+k}}{\tau_k} = \infty$ and α_τ does not exist. But all τ_j can be estimated by

$$2^{js} \le \tau_j \le j\, 2^{js}\,,$$

consequently $b(\Omega) = s$ and $1 = \liminf_{j\to\infty} \tau_j 2^{-js} \le \limsup_{j\to\infty} \tau_j 2^{-js} = \infty$.

Example 5. Let a sequence be defined by

$$v_j := j! \qquad \text{or by} \qquad v_j^\beta := e^{j^\beta} \quad \beta > 1\,.$$

These sequences are not admissible and in both cases $b(\Omega) = \infty$, however the domain Ω defined by Example 2 is again uniformly E-porous and quasi-bounded.

Example 3 corresponds to Example 4.12 in [11] and shows that there exist uniformly E-porous quasi-bounded domains with $b(\Omega) < \alpha_b$. For another example we refer to Example 3 in [12] where $\alpha_\sigma = s+2$ but $\sigma(\Omega) = s+1$

Examples 4 and 5 lead to uniformly E-porous quasi-bounded domains for which we have the criterion for compactness holds, cf. Theorem 1, but the asymptotic behaviour of the entropy numbers can be described only in the case $p_1 = p_2$, cf. Theorem 3.

4. Applications to spectral theory on unbounded domains

Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. Then $\bar{B}^{2m}_{2,2}(\Omega) = \bar{F}^{2m}_{2,2}(\Omega) = \widetilde{W}^{2m}_2(\Omega)$, $m \in \mathbb{N}$, cf. Remark 3.

Let

$$A(x,D) = \sum_{|\alpha| \le 2m} a_\alpha(x)\partial^\alpha$$

be a formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \le 2m$. Then the operator $A = A(x,D)$ with domain

$$\mathcal{D}(A) = \bar{B}^{2m}_{2,2}(\Omega)$$

is a closed linear operator with discrete spectrum $\sigma(A)$ of eigenvalues having no finite accumulation point, cf. [4,9].

We assume that A is a positive self-adjoint operator in $L_2(\Omega)$.

Theorem 6. *Let Ω be a quasi-bounded uniformly E-porous domain in $\mathbb{R}^n$. Let A be the operator defined above and let $\lambda_1, \lambda_2, \ldots$ be eigenvalues of A ordered by their magnitude and counted according to their multiplicities. Then*

$$\lambda_k \sim 2^{\mathbb{B}^{-1}(k)2m}\,, \qquad k \in \mathbb{N}\,.$$

Proof. The proof is standard. Since A is a positive self-adjoint operator with compact resolvent, λ_k is an eigenvalue of A if, and only if, $\mu_k = \lambda_k^{-1}$ is an eigenvalue of A^{-1}. But A is a bounded operator mapping $\bar{B}^{2m}_{2,2}(\Omega)$ onto $L_2(\Omega)$. So we can factorize A^{-1} through the compact embedding $\bar{B}^{2m}_{2,2}(\Omega) \hookrightarrow L_2(\Omega)$. Using Carl's inequality and Theorem 3 we get

$$\mu_k \le C2^{-2m\mathbb{B}^{-1}(k)}\,.$$

This proves the estimates of λ_k from below.

On the other hand let ψ be a smooth function such that $\operatorname{supp}\psi \subset (0,1)^n$ and $\|\psi|L_2(\Omega)\| = 1$. Let $\psi_j(x) = 2^{jn/2}\psi(2^j x)$, $j = 1,2,\ldots$, and $\psi_{j,h}(x) = \psi_j(x-h)$, $h \in \mathbb{R}^n$. If $\operatorname{supp}\psi_{j,h} \subset \Omega$, then

$$C \le \|\psi_{j,h}|L_2(\Omega)\|^2 \le \|A\psi_{j,h}|L_2(\Omega)\|\,\|A^{-1}\psi_{j,h}|L_2\| \le$$
$$\le C\,\|\psi_{j,h}|\widetilde{W}_2^{2m}(\Omega)\|\,\|A^{-1}\psi_{j,h}|L_2(\Omega)\| \le C\,2^{2mj}\,\|A^{-1}\psi_{j,h}|L_2(\Omega)\|.$$

By translations one can find $b_j(\Omega)$ functions of the form $\psi_{j,h}$ with pairwise disjoint supports contained in Ω. So for any linear operator T of rank smaller than $b_j(\Omega)$ we can find $\psi_{j,h}$ such that $T(\psi_{j,h}) = 0$. But this implies

$$a_{b_j(\Omega)}(A^{-1}) = \inf\{||A^{-1} - T|\mathcal{L}(L_2(\Omega)||\,,\,\mathrm{rank}\,T < b_j(\Omega)\} \ge$$
$$\ge \|A^{-1}\psi_{j,h}|L_2(\Omega)\| \ge c2^{-2mj}\,. \tag{44}$$

Now using the properties of approximation numbers $a_k(A^{-1})$ in Hilbert spaces and formula (44) we get

$$\lambda_{b_j(\Omega)} = \Big(\mu_{b_j(\Omega)}\Big)^{-1} = \Big(a_{b_j(\Omega)}(A^{-1})\Big)^{-1} \le c2^{2m\mathbb{B}^{-1}(b_j(\Omega))}\,.$$

If $b_j(\Omega) < k < b_{j+1}(\Omega)$ then one get easily

$$\lambda_k \le \lambda_{b_{j+1}(\Omega)} \le c\,2^{2m\mathbb{B}^{-1}(b_{j+1}(\Omega))} \le c\,2^{2m}\,2^{2m\mathbb{B}^{-1}(k)},$$

since $\mathbb{B}^{-1}(b_j(\Omega)) = j$ and the function $\mathbb{B}^{-1}$ is increasing. This finishes the proof. □

Remark 11. If $|\Omega| < \infty$, then we have

$$\lambda_k \sim k^{\frac{2m}{n}}\,.$$

This formula is well known for bounded regular domains and goes back to Hermann Weyl, cf. [15]. On the other hand Hermann König considered a similar problem for so-called quasi-bounded full C_1^ℓ-domains, $\ell > 0$. He proved that

$$\lambda_k \sim k^{\frac{\ell}{\ell+1}\frac{2m}{n}}\,,$$

cf. [9,10]. Theorem 6 is more general as it is shown in the proof of the next theorem, cf. also Example 6 below.

Theorem 7. *Let $\mathbb{S} : [0,\infty) \to [1,\infty)$ be a non-negative strictly increasing and continuously differentiable function with*

$$0 < \frac{d\mathbb{S}}{ds}(s) \le \frac{2}{n}\,\frac{\mathbb{S}(s)}{s} \quad \textit{for all} \quad s \ge s_0 \ge 0\,. \tag{45}$$

Then there exists a quasi-bounded uniformly E-porous domain Ω in $\mathbb{R}^n$ such that, if $\lambda_1(-\Delta), \lambda_2(-\Delta), \ldots$ are eigenvalues of positive Dirichlet Laplacian $-\Delta$ defined on Ω, ordered by their magnitude and counted according to their multiplicities, then

$$\lambda_k(-\Delta) \sim \mathbb{S}(k)\,, \quad k \in \mathbb{N}\,. \tag{46}$$

Proof. Let $D(s) = \frac{1}{2}\log_2 \mathbb{S}(s)$. Then D is a non-negative strictly increasing and continuously differentiable function and

$$0 < \frac{dD}{ds}(s) \le \frac{\log_2 e}{n}\,\frac{1}{s} \quad \text{for all} \quad s \ge \tilde{s}_0 \ge 0\,. \tag{47}$$

Let us denote by $B(t) := D^{-1}(t)$ the inverse function to the strictly increasing function D. Then by (47) we have

$$\frac{dB}{dt}(t) \geq \frac{n}{\log_2 e}\, B(t)$$

and this leads to

$$\frac{d(\log_2 B)}{dt}(t) \geq n \quad \text{for all} \quad t \geq t_0 := D^{-1}(\tilde{s}_0)\,.$$

Consider the function $\log_2 B(t)$ and $j \geq t_0$ then we obtain

$$\log_2 B(j+1) - \log_2 B(j) = \frac{d(\log_2 B)}{dt}(\tau) \geq n\,.$$

Define

$$\sigma_j := B(j)$$

then the last inequality implies

$$\sigma_{j+1} \geq 2^n\, \sigma_j$$

and in the same way as in Example 2 we can construct a quasi-bounded domain Ω in $\mathbb{R}^n$ such that $b_j(\Omega) = \sigma_j$.

Now Theorem 6 implies for $A = -\Delta$ that

$$\lambda_k \sim 2^{2\mathbb{B}^{-1}(k)}\,, \quad k \in \mathbb{N}\,.$$

But by construction we have $D(\sigma_j) = \mathbb{B}^{-1}(\sigma_j) = j$ and consequently follows $|\mathbb{B}^{-1}(k) - D(k)| \leq 1$. $\square$

Remark 12. 1. The equality in the right-hand inequality in (45) holds for $\mathbb{S}(s) = s^{2/n}$. This is the function describing the asymptotic behaviour of eigenvalues of Dirichlet Laplace operator on bounded domains. Boundedness of a domain is the most restrictive condition for the domains we take into account. Thus the inequalities in (45) are natural bounds for the behaviour of derivative of $\mathbb{S}$.

2. The last theorem holds also for a differential operator A of higher order than 2 if it satisfies the following conditions:

- A is a uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$ which are uniformly bounded and uniformly continuous for $|\alpha| \leq 2m$,
- is a positive self-adjoint operator in $L_2(\Omega)$ with domain of definition $\widetilde{W}_2^{2m}(\Omega)$,
- $\mathbb{S} : [0, \infty) \to [1, \infty)$ is again strictly increasing and continuously differentiable function with property the following property:

$$0 < \frac{d\mathbb{S}}{ds}(s) \leq \frac{2m}{n}\,\frac{\mathbb{S}(s)}{s} \quad \text{for all} \quad s \geq s_0 \geq 0\,.$$

Then eigenvalues $\lambda_1, \lambda_2, \ldots$ of A, ordered by their magnitude and counted according to their multiplicities, satisfied the following asymptotic estimates (46).

Example 6. Below we give examples of functions satisfying the assumptions of the last theorem and the corresponding asymptotic behaviours of eigenvalues of $-\Delta$:

$$
\begin{aligned}
&\text{(i)} \quad \mathbb{S}(s) = (s+1)^{2/b}, \\
&\text{(ii)} \quad \mathbb{S}(s) = (\log_2(s+1)+1)^{2\gamma}, \quad \gamma > 0, \\
&\text{(iii)} \quad \mathbb{S}(s) = (s+1)^{2/b}(\log_2(s+1)+1)^{2\gamma},\ b > n,\ \gamma \in \mathbb{R}, \\
&\text{(iv)} \quad \mathbb{S}(s) = (s+1)^{2/n}(\log_2(s+1)+1)^{2\gamma}, \quad \gamma \le 0, \\
&\text{(v)} \quad \mathbb{S}(s) = \big(\log_2(\log_2(s+1)+1)+1\big)^{2\gamma}, \quad \gamma > 0.
\end{aligned}
$$

We can also perturb A by a multiplication operator $f \mapsto V^2 f$, giving

$$H_\alpha f = Af - \alpha V^2 f, \qquad V(x) \text{ real valued on } \Omega,\ \alpha > 0\,,$$

and ask for the behaviour of the cardinality of the negative spectrum $\#\{\sigma(H_\alpha) \cap (-\infty, 0]\}$ as $\alpha \to \infty$. This is the usual question in the study of spectral properties of elliptic operators, cf. [6,7]. For the more results on unbounded domains we refer to [8].

For $\delta > 0$ and $2 < r < \infty$ we define a function $\eta_{\delta,r}$ in the following way:

$$\eta_{\delta,r}(s) = \delta \mathbb{B}^{-1}(s) + \frac{2}{r}\log_2 s \qquad \text{if} \quad s > 0.$$

Please confer the definition of $\mathbb{B}$ and $\mathbb{B}^{-1}$ in Remark 5. Thus $\eta_{\delta,r}$ is strictly increasing with co-domain $\mathbb{R}$.

Using Theorem 5 and the entropy version of the Birman–Schwinger principle, cf. [7], one can easily prove the following theorem.

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a quasi-bounded uniformly E-porous domain such that the sequence $b_j(\Omega)$ is admissible. Suppose that $2 < r < \infty$, $mr > n$ and $V \in L_r(\Omega)$. We assume that $\|V|L_r(\Omega)\| = 1$.*

Let $\delta = 2m - \frac{2n}{r}$. Then there exists $c > 0$, which is independent of α and V, such that

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le c\,\eta^{-1}_{\delta,r}(\log_2 \alpha), \qquad \alpha > 0\,. \tag{48}$$

Remark 13. The estimate (48) is optimal in the following sense. There are positive constants $c_1, c_2 > 0$ and a sequence of operators $H_{\alpha,j} = A - \alpha V_j^2$ with $\|V_j|L_r(\Omega)\| = 1$, $j \ge j_0$ such that there are α_j with

$$c_1\, b_j(\Omega) \le \eta^{-1}_{\delta,r}(\log_2 \alpha_j) \le c_2\, b_j(\Omega) \tag{49}$$

and

$$c_1\, \eta^{-1}_{\delta,r}(\log_2 \alpha_j) \le \#\{\sigma(H_{\alpha_j,j}) \cap (-\infty, 0]\} \le c_2\, \eta^{-1}_{\delta,r}(\log_2 \alpha_j)\,. \tag{50}$$

Please note that the constants c_1, c_2 are independent of j.

We sketch the proof. For any $\alpha \ge 0$ the operator H_α is self-adjoint, semibounded operator with compact resolvent, therefore it has purely discrete spectrum and its eigenvectors form a complete orthonormal basis in $L_2(\Omega)$. Moreover if $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of H_α written in increasing order and repeated according to their multiplicity then

$$\lambda_k = \inf\{\lambda(L) \,:\, L \subset \mathcal{D}(H_\alpha) \text{ and } \dim L = k\}\,,$$

where

$$\lambda(L) = \sup\{(H_\alpha f, f) \,:\, f \in L \text{ and } \|f|L_2(\Omega)\| = 1\}$$

and L is any finite-dimensional subspaces of the domain of H_α, cf. [5, Theorem 4.5.1].

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be supported in the cube $Q_{0,0}$ and let $\|\psi|L_2(\Omega)\| = 1$. We put $\psi_j(x) = 2^{jn/2}\psi(2^j x)$. We consider an operator $H_{\alpha,j}$ with the potential $V_j(x) = 2^{j\frac{n}{r}} b_j(\Omega)^{-\frac{1}{r}} \chi_{\widetilde{\Omega}_j}(x)$, where $\widetilde{\Omega}_j$ is the sum of dyadic cubes of size 2^{-j} contained in Ω, cf. (15). Moreover we take a $b_j(\Omega)$-dimensional subspace L_j spanned by the translates ψ_{j,m_ℓ} of the function ψ_j supported in the cubes Q_{j,m_ℓ}, cf. (3). Now if $f \in L_j$ then

$$(H_{\alpha,j} f, f) \le 0 \qquad \text{if} \qquad \alpha \ge c_A 2^{2j(m-\frac{n}{r})} b_j(\Omega)^{\frac{2}{r}}\,, \tag{51}$$

where c_A is the constant depending of operator A. In consequence the negative spectrum of $H_{\alpha,j}$ contains for these α at least $b_j(\Omega)$ eigenvalues. The condition for α in (51) is equivalent to $\log_2 \alpha \ge \log_2 c_A + \eta_{\delta,r}(b_j(\Omega))$. So we choose α_j satisfying the identity $\log_2 \alpha_j = \log_2 c_A + \eta_{\delta,r}(b_j(\Omega))$ we get (49) and (50) since the sequence $b_j(\Omega)$ is admissible.

Remark 14. If $r = \infty$ then we do not need the assumption that the sequence $b_j(\Omega)$ is admissible since we can use Theorem 3 instead of Theorem 5. So in that case for any quasi-bounded uniformly E-porous domain we get the estimates

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le \mathbb{B}\Big(\frac{\log_2 \alpha}{2m} + c_0\Big), \qquad \alpha \ge 1$$

for some $c_0 > 0$ which is independent of α.

If the sequence $b_j(\Omega)$ is in addition admissible, we have for large α

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le c\,\mathbb{B}\Big(\frac{\log_2 \alpha}{2m}\Big)$$

where the constant $c > 0$ is independent of α.

Moreover if we use Theorem 6, the example $V(x) \equiv 1$ on Ω shows, that we cannot expect a better estimation for general potentials $V \in L_\infty(\Omega)$.

Remark 15. For some domains we can prove the more explicit formula. In [13] we prove that

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le c\alpha^\rho, \qquad \text{where} \qquad \rho = \frac{b(\Omega)r}{2mr + 2(b(\Omega) - n)}\,,$$

if

$$0 < \liminf_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} \le \limsup_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty\,.$$

The last theorem generalizes this result. For example, if we take $b_j(\Omega) = [2^{jb} j^\beta]$, $b \ge n$, $\beta \in \mathbb{R}$ ($\beta \ge 0$ if $b = n$) then the sequence $b_j(\Omega)$ is admissible and $\mathbb{B}^{-1}(s) \sim \frac{1}{b}\log_2(s+1) - \frac{\beta}{b}\log_2(\log_2(s+1)+1)$. In consequence $\eta^{-1}_{\delta,r}(t) \sim 2^{t(\frac{\delta}{b}+\frac{2}{r})^{-1}} t^{\frac{\beta\delta}{b}(\frac{\delta}{b}+\frac{2}{r})^{-1}}$ and

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le c\alpha^\rho (\log_2 \alpha)^\sigma$$

where

$$\rho = \frac{b(\Omega)r}{2mr + 2(b(\Omega) - n)} \qquad \text{and} \qquad \sigma = \frac{\beta(2mr - 2n)}{2mr + 2(b(\Omega) - n)}.$$

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