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Nonlocal elliptic equations involving measures

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ABSTRACT

In this article, we study the existence of solution for the problem $(-\Delta)^\alpha u = \lambda f(u) + \nu$ in Ω, $u \equiv 0$ in $\mathbb{R}^N \backslash \Omega$, where $\lambda > 0$ is a parameter, $\alpha \in (0,1)$ and ν is a Radon measure. A weak solution is obtained by using Schauder's fixed point theorem. In the case where ν is Dirac measure, the symmetry of the solution is obtained by using the moving plane method.

1. Introduction

In recent years, fractional (nonlocal) Laplacian operator has been extensively studied by many authors [7,8,6,2,10,16,17,15]. It is well known that the corresponding Fokker–Planck equation to a stochastic differential equation with Brownian motion is the traditional diffusion equation. When the Brownian motion is replaced by an α-stable Lévy motion (a non-Gaussian process) L_t^α, $\alpha \in (0,2)$, the Fokker–Planck equation becomes a nonlocal partial differential equation [1] with a fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$. There are many physical motivations to consider the fractional Laplacian operator, which appears in many models in non-Newtonian fluids, in models of viscoelasticity such as Kelvin–Voigt models, various heat transfer processes in fractal and disordered media and models of fluid flow and acoustic propagation in porous media [3,13,14]. Interestingly, it has also been applied to pricing derivative securities in financial market, see [3] for details.

Recently, Chen and Véron [8] considered the following problem:

$$\begin{cases} (-\Delta)^\alpha u + g(u) = \nu, & t > 0, x \in \Omega, \\ u|_{\Omega^c} = 0, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is an open bounded C^2 domain, $\alpha \in (0,1)$, ν is a Radon measure such that $\int_\Omega \delta^\beta d|\nu| < \infty$ for some $\beta \in [0,\alpha]$ and $\delta(x) = \mathrm{dist}(x, \Omega^c)$. The nonlocal Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \downarrow 0} (-\Delta)^\alpha_\varepsilon u(x),$$

where for $\varepsilon > 0$

$$(-\Delta)^\alpha_\varepsilon u(x) = c_{n,\alpha} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}} \chi_\varepsilon(|x-y|) dy,$$

and

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } \ x \in [0,\varepsilon], \\ 1, & \text{if } \ x > \varepsilon. \end{cases}$$

They proved that (1.1) admits a unique weak solution u under the condition that $g : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function, satisfying

$$g(r)r \geq 0, \quad \forall r \in \mathbb{R} \ \text{ and } \ \int_1^\infty (g(s) - g(-s)) s^{-1-k_{\alpha,\beta}} ds < \infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \dfrac{N}{N-2\alpha}, & \text{if } \beta \in [0, \dfrac{N-2\alpha}{N}\alpha], \\ \dfrac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\dfrac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \tag{1.2}$$

In their another paper [7], they obtained the existence of weak solution to (1.1), where $g(u)$ was replaced by $\varepsilon g(|\nabla u|)$, $\varepsilon = \pm 1$. When the measure ν is just a bounded function $g(x)$, the existence of solutions to (1.1) has been studied in [18] via variational methods.

Ros-Oton and Serra [16] studied the extremal solution for the following problem:

$$\begin{cases} (-\Delta)^\alpha u = \lambda f(u), & \text{in } \Omega, \\ u|_{\Omega^c} = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter, $\alpha \in (0,1)$ and $f : [0,\infty) \to \mathbb{R}$ satisfies

$$f \in C^1, \quad \text{non-decreasing}, \quad f(0) > 0, \text{ and } \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.4}$$

Under the above assumptions, they proved that there exists $\lambda^* \in (0,\infty)$ such that

(i) If $0 < \lambda < \lambda^*$, problem (1.3) admits a minimal classical solution u_λ;
(ii) The family of functions $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*}$ is a weak solution of (1.3) with $\lambda = \lambda^*$;
(iii) For $\lambda > \lambda^*$, problem (1.3) admits no classical solution.

It follows from (1.4) that f must satisfy $f(u) \geq f(0) > 0$, $\forall u > 0$. Furthermore, they considered the solutions u_λ and u_{λ^*} must be positive. Besides the above results, they also obtained the bounded domain of the extremal solution u_{λ^*}.

Motivated by papers [7,8,16], in this paper, we consider the following nonlocal elliptic equations involving measure:

$$\begin{cases} (-\Delta)^\alpha u = \lambda f(u) + \nu, & \text{in } \Omega, \\ u|_{\Omega^c} = 0, \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a parameter, $\nu \in \mathfrak{M}_+(\Omega, \delta^\beta)$. Here $\delta(x) = \mathrm{dist}(x, \Omega^c)$ and $\mathfrak{M}(\Omega, \delta^\beta)$ is the space of Radon measures in Ω satisfying

$$\int_\Omega \delta^\beta d\nu < \infty.$$

The associated positive cones are denoted by $\mathfrak{M}_+(\Omega, \delta^\beta)$.

Our interest in this article is to investigate the existence of weak solutions to (1.5). Firstly, we remark that if $f(u) = u^p$, $p > 0$, then $-f(r)r \geq 0$, $\forall r \in \mathbb{R}$ will not hold, that is, the result in [7] does not contain the case. Moreover, in paper [16], the authors assume that $f(0) > 0$ and $\lim\limits_{t\to\infty} \frac{f(t)}{t} = \infty$. In this paper, we only assume that

(H) $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is a C^1 continuous non-negative function which satisfies

$$f(u) \leq au^p + b, \qquad \forall u \geq 0$$

for some $p \in (0, p_{\alpha,\beta})$, where $a > 0$, $b \geq 0$ and $p_{\alpha,\beta}$ is defined in (1.6).

Secondly, it follows from [11] that the eigenvalues of $(-\Delta)^\alpha$ are different from those of $(-\Delta)$ in the bounded domain. For $n = 1$, Kwaśnicki [11] obtained the existence of eigenvalues of $(-\Delta)^\alpha$. When $n \geq 2$, the problem was solved in papers [19,20]. However, in this paper, we will use a different method to prove the existence of weak solutions to (1.5). Before stating our main theorem we make precise the notion of weak solution used in this article. The following definition is used in [8].

Definition 1.1. We say that u is a weak solution of (1.5), if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \delta^\alpha dx)$ and

$$\int_\Omega u(-\Delta)^\alpha v dx = \int_\Omega \lambda f(u) v dx + \int_\Omega v d\nu, \quad \forall v \in \mathbb{X}_\alpha,$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions v satisfying:

(i) $\mathrm{supp}(v) \subset \bar{\Omega}$;
(ii) $(-\Delta)^\alpha v(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha v(x)| \leq C$ for some constant $C > 0$;
(iii) there exist $\phi \in L^1(\Omega, \delta^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha v(x)| \leq \phi$ a.e. in Ω for all $\epsilon \in (0, \epsilon_0]$.

Our main result is the following.

Theorem 1.1. *Assume that $\nu \in \mathfrak{M}_+(\Omega, \delta^\beta)$ with $\beta \in [0, \alpha]$, and f satisfies* (H).

(i) *If $p \in (0, 1)$, then problem* (1.5) *admits a weak solution u;*
(ii) *If $p = 1$ and λ or a is small enough, then problem* (1.5) *admits a weak solution u;*

(iii) *If $p \in (1, p_{\alpha,\beta})$, a and $b + \|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} > 0$ fixed, then there exists a positive number λ^* such that when $\lambda \in (0, \lambda^*]$, the problem (1.5) admits a non-negative weak solution u, when $\lambda > \lambda^*$, the problem (1.5) will admit no non-negative weak solution if $f(u) = au^p + b$. On the other hand, if λ and a fixed, and $b + \|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} > 0$ is small enough, then the problem (1.5) admits a non-negative weak solution u. Here*

$$p_{\alpha,\beta} = \frac{N}{N - 2\alpha + \beta}. \tag{1.6}$$

If $f(u) \geq u$, then there exists $\lambda_* > 0$ such that when $\lambda > \lambda_*$, problem (1.5) admits no non-negative solution.

Remark 1.1. 1. From Theorem 1.1, we say that the value of p can be less than 1, which does not satisfy (1.4). On the other hand, we can let $\beta = 0$, then $\nu \in \mathfrak{M}_+(\Omega, dx)$, that is, $0 < \int_\Omega \nu(x)dx < \infty$. In particular, we can assume that $\nu = 1$ or $\nu(x) = g(x) > 0$.

2. Although Theorem 1.1 is similar to Theorem 1.2 in [7], the authors in [7] considered the case $\lambda f(u)$ replaced by $g(|\nabla u|)$. Moreover, the value of p in Theorem 1.1 is different from that of Theorem 1.2 in [7]. What's more, we take different work space from that in [7].

The rest of this paper is organized as follows. In Section 2, we will recall some known results and interpret the difference between our work space and the ordinary fractional Sobolev space. Section 3 is concerned with the proof of Theorem 1.1. In the last section, we consider a special case where ν is Dirac measure and obtain that the solution is symmetric by using the moving plane method.

2. Preliminaries

In this section, we first introduce our work space. Then we recall the estimate of Green function and lastly we prove some properties of the work space.

We say $(-\Delta)^\alpha$ is a nonlocal operator because of its definition. Sometimes many authors call it fractional Laplacian operator. Now we will interpret the difference between nonlocal and fractional operators. Firstly, for a positive operator A on bounded domain Ω. We can define the fractional operator of A in the following way. Suppose $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ are the eigenvalues of A and $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ are the corresponding eigenfunctions of A. Then we define

$$A^\beta u = \sum_{i=1}^{\infty} \lambda_i^\beta (\phi_i, u)\phi_i, \quad \forall u \in L^2(\Omega),$$

where $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ is an orthonormal basis of $L^2(\Omega)$. That is, λ_i^β $(i = 1, 2, \cdots)$ are the eigenvalues of A^β. However, it follows from Theorem 1 of [11] that

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^\alpha + O(\frac{1}{n}),$$

where $n = 1, 2, \cdots$, $\Omega = (0, 1)$ and

$$(-\Delta)^\alpha u = C_\alpha P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+\alpha}} dy, \quad x \in \Omega, \text{ and } u \equiv 0 \text{ in } \mathbb{R} \setminus \Omega.$$

It is well known that the eigenvalues of $-\Delta$ in $(0, 1)$ is $\lambda_n = \frac{n\pi}{2}$. Thus the two operators are different, see [12] for details.

On the other hand, the work space of the two operators is different. For fractional operator, we can take the classical fractional Sobolev space as its work space. But for nonlocal operator, we must take nonlocal Sobolev space as its work space, see Remark 2.1 in [12]. Here nonlocal Sobolev space is the weight fractional Sobolev space. More precisely, we define for $0 < s < 1$, $p \geq 1$,

$$W^{s,p}_{\rho}(\Omega) = \{u \in W^{s,p}(\Omega) : \ \rho(x)u(x) \in L^p(\Omega)\}$$

with the norm

$$\|u\|^p_{W^{s,p}_{\rho}(\Omega)} = \int\limits_{\Omega} |\rho(x)u(x)|^p dx + \int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy,$$

where $\rho(x) \sim \frac{1}{\delta^{\alpha}(x)}$, $\delta(x) = \mathrm{dist}(x, \Omega^c)$. Actually,

$$W^{s,p}_{\rho}(\Omega) = \left\{u \in W^{s,p}(\mathbb{R}^N) : \ u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}$$

and

$$\|u\|^p_{W^{s,p}_{\rho}(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy,$$

which coincides with that in [9,10]. It is easy to see that $W^{s,p}_{\rho}(\Omega) \subset W^{s,p}(\Omega)$. There is another reason why we introduce the nonlocal Sobolev space. It is well known that $\|\Delta u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s+2,p}(\Omega)}$. But for nonlocal operator $(-\Delta)^{\alpha}$, $\|(-\Delta)^{\alpha}u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s+2\alpha,p}(\Omega)}$ will not hold. By using Fourier transform, we have $\|(-\Delta)^{\alpha}u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{W^{s+2\alpha,p}(\mathbb{R}^N)}$, that is, $\|(-\Delta)^{\alpha}u\|_{W^{s,p}_{\rho}(\Omega)} = \|u\|_{W^{s+2\alpha,p}_{\rho}(\Omega)}$.

Next, we denote by G_{α} the Green kernel of $(-\Delta)^{\alpha}$ in Ω and by $\mathbb{G}$ the Green operator defined by

$$\mathbb{G}[f](x) = \int\limits_{\Omega} G_{\alpha}(x,y)f(y)dy.$$

It follows from [5] that there exists a constant $C > 0$ such that for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$

$$\begin{aligned} G_{\alpha}(x,y) &\leq C \min\left\{\frac{1}{|x-y|^{N-2\alpha}},\ \frac{\delta^{\alpha}(x)}{|x-y|^{N-\alpha}},\ \frac{\delta^{\alpha}(y)}{|x-y|^{N-\alpha}}\right\}, \\ G_{\alpha}(x,y) &\leq C\frac{\delta^{\alpha}(y)}{\delta^{\alpha}(x)|x-y|^{N-\alpha}}. \end{aligned} \tag{2.1}$$

From [4, Corollary 3.3], we have

$$\left|\nabla_x G_{\alpha}(x,y)\right| \leq N G_{\alpha}(x,y) \max\left\{\frac{1}{|x-y|},\ \frac{1}{\delta}\right\}. \tag{2.2}$$

Combining with (2.1) and (2.2), we get

$$\left|\nabla_x G_{\alpha}(x,y)\right| \leq C \max\left\{\frac{\delta^{\alpha}(y)}{\delta^{\alpha}(x)|x-y|^{N-\alpha}},\ \frac{\delta^{\frac{(2\alpha-1)(N-\alpha)}{N-2\alpha+1}}(y)\delta^{\frac{2\alpha-1-N\alpha}{N-2\alpha+1}}(x)}{|x-y|^{N-\alpha}}\right\},$$

$$\left|\nabla_x G_{\alpha}(x,y)\right|\delta^{\alpha}(x) \leq C \max\left\{\frac{\delta^{\alpha}(y)}{|x-y|^{N-\alpha}},\ \frac{\delta^{\frac{(2\alpha-1)(N-\alpha)}{N-2\alpha+1}}(y)\delta^{\frac{(2\alpha-1)(1-\alpha)}{N-2\alpha+1}}(x)}{|x-y|^{N-\alpha}}\right\}.$$

Consequently, the function $\int_\Omega G_\alpha(x,y)dy$ may not belong to the Sobolev space $W^{1,1}(\Omega)$. Hence we cannot prove Theorem 1.1 by using the method in [7].

Now, we study the property of the solution of equation (1.5).

Lemma 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ $(n \geq 2)$ is an open bounded C^2 domain and $\nu \in \mathfrak{M}_+(\Omega, \delta^\beta)$ with $0 \leq \beta \leq \alpha$. Then for $p \in \left(1, \frac{N}{N-2\alpha+\beta}\right)$, there exists $C = C(p)$ such that for any $\nu \in L^1(\Omega, \delta^\beta dx)$*

$$\|\mathbb{G}[\nu]\|_{W_\rho^{2\alpha-\gamma,p}(\Omega)} \leq C\|\nu\|_{L^1(\Omega,\delta^\beta dx)},$$

where $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$, and $p' = \frac{p}{p-1}$.

Proof. Similar to [7, Proposition 2.5], we use Stampacchia's duality method [21]. Let $u = \mathbb{G}[\nu]$, then $(-\Delta)^\alpha u = \nu$. For $\psi \in C_c^\infty(\bar{\Omega})$, we have

$$\begin{aligned}\Big|\int_\Omega \psi(-\Delta)^\alpha u dx\Big| = \Big|\int_\Omega \psi\nu dx\Big| &\leq \int_\Omega |\psi||\nu| dx \\ &\leq \|\psi\|_{C^\beta(\bar{\Omega})}\|\nu\|_{L^1(\Omega,\delta^\beta dx)}.\end{aligned}$$

By Sobolev–Morrey embedding type theorem, we have for any $p \in \left(1, \frac{N}{N-2\alpha+\beta}\right)$

$$\|\psi\|_{C^\beta(\bar{\Omega})} \leq C\|\psi\|_{W^{\gamma,p'}(\Omega)},$$

where $p' = \frac{p}{p-1}$, and $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$. Noting that $W_\rho^{\gamma,p'}(\Omega) \subset W^{\gamma,p'}(\Omega)$ and $\|u\|_{W^{\gamma,p'}(\Omega)} \leq C\|u\|_{W_\rho^{\gamma,p'}(\Omega)}$ (C only depends on Ω), we have

$$\begin{aligned}\Big|\int_\Omega \psi(-\Delta)^\alpha u dx\Big| &\leq C\|\psi\|_{W^{\gamma,p'}(\Omega)}\|\nu\|_{L^1(\Omega,\delta^\beta dx)} \\ &\leq C\|\psi\|_{W_\rho^{\gamma,p'}(\Omega)}\|\nu\|_{L^1(\Omega,\delta^\beta dx)},\end{aligned}$$

which implies that the mapping $\psi \to \int_\Omega \psi(-\Delta)^\alpha u dx$ is continuous on $W_\rho^{\gamma,p'}(\Omega)$ and thus

$$\|(-\Delta)^\alpha u\|_{W_\rho^{-\gamma,p}(\Omega)} \leq C\|\nu\|_{L^1(\Omega,\delta^\beta dx)}.$$

Since $(-\Delta)^\alpha$ is an isomorphism from $W_\rho^{2\alpha-\gamma,p}(\Omega)$ into $W_\rho^{-\gamma,p}(\Omega)$, it follows that

$$\|u\|_{W_\rho^{2\alpha-\gamma,p}(\Omega)} \leq C\|\nu\|_{L^1(\Omega,\delta^\beta dx)}.$$

This completes the proof. $\square$

It follows from Lemma 2.1 that for $\nu \in L^1(\Omega, \delta^\beta dx)$ with $0 \leq \beta \leq \alpha$, we have $u \in W_\rho^{2\alpha-\gamma,p}(\Omega)$, where $(-\Delta)^\alpha u = \nu$. Noting that

$$W_\rho^{2\alpha-\gamma,p}(\Omega) \subset W^{2\alpha-\gamma,p}(\Omega) \hookrightarrow^{compact} L^q(\Omega), \tag{2.3}$$

where $q \in [1, \frac{Np}{N-(2\alpha-\gamma)p})$, we can obtain the compactness of the sequence $\{u_n\}$ if $\|\nu_n\|_{L^1(\Omega,\delta^\beta dx)} \leq C$, $(-\Delta)^\alpha u_n = \nu_n$ and C does not depend on n. Additionally, when $\beta > 0$, using $\gamma = \beta + N - \frac{N}{p}$, we have

$$\frac{Np}{N-(2\alpha-\gamma)p}=\frac{Np}{N-(2\alpha-\beta-N+\frac{N}{p})p}=\frac{N}{N-2\alpha+\beta}.$$

When $\beta=0$, $\frac{Np}{N-(2\alpha-\gamma)p}=\frac{N}{N-2\alpha}$.

Before we end this section, we consider the following problem:

$$\begin{cases}(-\Delta)^\alpha u=f(u)+g, & \text{in } \Omega,\\ u=0, & \text{in } \Omega^c.\end{cases}\tag{2.4}$$

Lemma 2.2. *Assume that $f\in C^\theta(\mathbb{R})\cap L^\infty(\mathbb{R})$ for $\theta\in(0,1]$ and $g\in C^\theta(\bar\Omega)$. The problem (2.4) admits a unique classical solution u. Moreover, $u\ge 0$ if $g+f(0)\ge 0$.*

The proof of Lemma 2.2 is standard and omitted here, also see [7, Theorem 2.1] for similar proof.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using Schauder's fixed point theorem.

Proof of Theorem 1.1. Let $\{\nu_n\}\subset C^1(\Omega)$ be a sequence of non-negative functions such that $\nu_n\to\nu$ in the sense of duality with $C_\beta(\bar\Omega):=\{u\in C(\bar\Omega): \delta^{-\beta}u\in C(\bar\Omega)\}$, that is,

$$\lim_{n\to\infty}\int_\Omega u\nu_n dx=\int_\Omega u\nu dx,\qquad \forall u\in C_\beta(\bar\Omega).$$

By the Banach–Steinhaus Theorem, $\|\nu_n\|_{\mathfrak{M}_+(\Omega,\delta^\beta)}$ is bounded independently of n. Let $\{f_n\}$ be a sequence of C^1 non-negative functions defined on $\mathbb{R}_+$ such that

$$f_n\le f_{n+1}\le f,\qquad \sup_{s\in\mathbb{R}_+}f_n(s)=n,\qquad \lim_{n\to\infty}\|f_n-f\|_{L^\infty_{loc}(\mathbb{R}_+)}=0.$$

Let $p_0=\frac{p+p_{\alpha,\beta}}{2}\in(p,p_{\alpha,\beta})$, where $p_{\alpha,\beta}$ is given by (1.6) and $p<p_{\alpha,\beta}$ is the growth rate of f which satisfies (H), and

$$M(v)=\left(\int_\Omega |v|^{p_0}dx\right)^{\frac{1}{p_0}}.$$

We assume that

$$\|\nu_n\|_{L(\Omega,\delta^\beta dx)}\le 2\|\nu\|_{L(\Omega,\delta^\beta dx)}$$

for all n. We divide the proof into two steps.

Step 1. We claim that for $n\ge 1$

$$\begin{cases}(-\Delta)^\alpha u_n=\lambda f_n(u_n)+\nu_n, & \text{in } \Omega,\\ u_n=0, & \text{in } \Omega^c\end{cases}$$

admits a non-negative solution u_n such that $M(u_n)\le\bar\mu$, where $\bar\mu>0$ is independent of n.

Define the operator T_n by

$$T_n u = \mathbb{G}[\lambda f_n(u) + \nu_n], \qquad \forall u \in L^{p_0}(\Omega).$$

Using (2.1), we have

$$\begin{aligned}\|T_n u\|_{L^{p_0}(\Omega)} &\le \lambda \|\mathbb{G}[\lambda f_n(u)]\|_{L^{p_0}(\Omega)} + \|\mathbb{G}_\alpha[\nu_n]\|_{L^{p_0}(\Omega)} \\ &\le C_1 \left(\lambda \|f_n\|_{L^\infty(\mathbb{R}_+)} + \|\nu_n\|_{L^\infty(\Omega)}\right),\end{aligned} \tag{3.1}$$

where $C_1 = \| \int_\Omega G_\alpha(\cdot, y)dy\|_{L^{p_0}(\Omega)} < \infty$. Actually, by using polar transform, we have

$$\begin{aligned}\left|\int_\Omega \left|\int_\Omega G_\alpha(x,y)dy\right|^{p_0} dx\right| &\le C\left|\int_\Omega \left|\int_\Omega \frac{1}{|x-y|^{N-2\alpha}}dy\right|^{p_0} dx\right| \\ &\le C\left|\int_\Omega \left|\int_{B_R(x)} \frac{1}{|x-y|^{N-2\alpha}}dy\right|^{p_0} dx\right| \\ &\le C|\Omega| \int_0^R \frac{1}{r^{N-2\alpha}} r^{N-1} dr \\ &\le C|\Omega| R^{2\alpha},\end{aligned}$$

where $R > 0$ and $\Omega \subset B_R(x)$, $\forall x \in \Omega$. It follows from Lemma 2.1 that

$$\begin{aligned}\|T_n u\|_{L^{p_0}(\Omega)} \le C\|T_n u\|_{W^{2\alpha-\gamma,p}_\rho(\Omega)} &= C\|\mathbb{G}[\lambda f_n(u) + \nu_n]\|_{W^{2\alpha-\gamma,p}_\rho(\Omega)} \\ &\le C_2 \|\lambda f_n(u) + \nu_n\|_{L^1(\Omega,\delta^\beta dx)} \\ &\le aC_2\lambda \|u^p\|_{L^1(\Omega,\delta^\beta dx)} + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} \\ &\le aC_2\lambda \|u\|^p_{L^{p_0}(\Omega)} \left(\int_\Omega \delta^{\frac{\beta p_0}{p_0 - p}} dx\right)^{\frac{1}{p} - \frac{1}{p_0}} \\ &\quad + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)},\end{aligned} \tag{3.2}$$

which implies

$$M(T_n u) \le aC_3\lambda M(u)^p + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)},$$

where $C_3 = C_2 \left(\int_\Omega \delta^{\frac{\beta p_0}{p_0-p}} dx\right)^{\frac{1}{p} - \frac{1}{p_0}}$ is independent of n.

Therefore, if we assume that $M(u) \le \mu$, inequality (3.2) yields

$$M(T_n u) \le aC_3\lambda\mu^p + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)}.$$

Let $\bar{\mu} > 0$ be the largest root of the equation

$$aC_3\lambda\mu^p + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} = \mu. \tag{3.3}$$

We have the following 4 cases:

(i) If $p \in (0, 1)$, (3.3) admits a unique positive root;
(ii) If $p = 1$ and $aC_3\lambda < 1$, (3.3) admits a unique positive root;

(iii) If $p \in (1, p_{\alpha,\beta})$ and $bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)}$ is small enough, (3.3) admits two positive roots;
(iv) If $p \in (1, p_{\alpha,\beta})$ and $\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)}$ fixed, there exists a constant $\lambda^* > 0$ such that when $\lambda \in (0, \lambda^*)$, (3.3) admits two positive roots. When $\lambda = \lambda^*$, (3.3) admits a unique positive root. (3.3) admits no positive root when $\lambda > \lambda^*$.

If we suppose the one of the above 4 cases holds, the definition of $\bar{\mu} > 0$ implies that it is the largest $\mu > 0$ such that

$$aC_3\lambda\mu^p + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} \le \mu.$$

For $M(u) \le \bar{\mu}$, we obtain

$$M(T_n u) \le aC_3\lambda\bar{\mu}^p + bC_2\lambda + 2C_2\|\nu\|_{\mathfrak{M}_+(\Omega,\delta^\beta)} = \bar{\mu}.$$

By the assumption of Theorem 1.1, $\bar{\mu}$ exists and it is larger than $M(u_n)$, i.e.,

$$\left(\int_\Omega |T_n u|^{p_0} dx\right)^{\frac{1}{p_0}} \le \bar{\mu}.$$

Thus T_n maps $L^{p_0}(\Omega)$ into itself. By the condition (H), if $u_n \to u$ in $L^{p_0}(\Omega)$ as $n \to \infty$, then $f(u_n) \to f(u)$ in $L^1(\Omega)$, and thus T is continuous. We claim that T is a compact operator. In fact, for $u \in L^{p_0}(\Omega)$, we have $f_n(u) + \nu_n \in L^1(\Omega)$ and then, by Lemma 2.1, it implies that $T_n u \in W_\rho^{2\alpha-\gamma,p}(\Omega)$, where $p \in (1, p_{\alpha,\beta})$ and $\gamma \in (\frac{N(p-1)}{p}, 2\alpha - \beta)$. Note that

$$\frac{N(p-1)}{p} = N\left(1 - \frac{1}{p}\right) < N\left(1 - \frac{1}{p_{\alpha,\beta}}\right) = 2\alpha - \beta.$$

Since the embedding $W_\rho^{2\alpha-\gamma,p}(\Omega) \hookrightarrow L^{p_0}(\Omega)$ is compact, T_n is a compact operator.

Let

$$\mathcal{M} = \left\{u \in L^{p_0}(\Omega) : \ \|u\|_{L^{p_0}(\Omega)} \le C_1\left(\lambda\|f_n\|_{L^\infty(\mathbb{R}_+)} + \|\nu_n\|_{L^\infty(\Omega)}\right) \text{ and } M(u) \le \bar{\mu}\right\},$$

which is a closed and convex set of $L^{p_0}(\Omega)$. Combining with (3.1), we have

$$T_n(\mathcal{M}) \subseteq \mathcal{M}.$$

It follows by Schauder's fixed point theorem that there exists some $u_n \in L^{p_0}(\Omega)$ such that $T_n u_n = u_n$ and $M(u_n) \le \bar{\mu}$, where $\bar{\mu}$ is independent of n. By Lemma 2.2, u_n belongs to $C^{2\alpha+\varepsilon}$ locally in Ω ($\varepsilon > 0$ small enough) and

$$\int_\Omega u_n(-\Delta)^\alpha\phi dx = \int_\Omega f_n(u_n)\phi dx + \int_\Omega \phi\nu_n dx, \quad \forall\phi \in \mathbb{X}_\alpha.$$

Step 2. Taking limits.

Noting that $M(u_n) \le \bar{\mu}$ and using the condition (H), we have $f_n(u_n)$ is uniformly bounded in $L^1(\Omega, \delta^\beta dx)$. By Lemma 2.1, $\{u_n\}$ is bounded in $W_\rho^{2\alpha-\gamma,p}(\Omega)$, where $p \in (1, p_{\alpha,\beta})$ and $0 < \gamma < 2\alpha - \beta$. By (2.3), there exist a subsequence $\{u_{n_k}\}$ and u such that $u_{n_k} \to u$ a.e. in Ω and in $L^q(\Omega)$ for any $q \in [1, p_{\alpha,\beta})$. By condition (H), $f_{n_k}(u_{n_k}) \to f(u)$ in $L^1(\Omega)$. Letting $n_k \to \infty$, we have

$$\int_\Omega u(-\Delta)^\alpha \phi dx = \int_\Omega f(u)\phi dx + \int_\Omega \phi \nu_n dx, \quad \forall \phi \in \mathbb{X}_\alpha.$$

Thus u is a weak solution of (1.5), which is non-negative as $\{u_n\}$ are non-negative.

Lastly, we prove that when $f(u) \geq u$, there exists $\lambda_* > 0$ such that when $\lambda > \lambda_*$, problem (1.5) admits no non-negative weak solution. We can verify it by using eigenvalue and eigenfunction. Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta)^\alpha$ in Ω and $\phi_1 > 0$ the corresponding eigenfunction, that is,

$$\begin{cases} (-\Delta)^\alpha \phi_1 = \lambda_1 \phi_1, & \text{in } \Omega, \\ \phi_1 > 0, & \text{in } \Omega, \\ \phi_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The existence, simplicity, and boundedness of the first eigenvalue are proved in [19, Proposition 5] and [20, Proposition 4]. Assume that u is a non-negative weak solution of (1.5). Then, taking ϕ_1 as a test function for problem (1.5), we get

$$\begin{aligned} \lambda_1 \int_\Omega u\phi_1 = \int_\Omega u(-\Delta)^\alpha \phi_1 &= \lambda \int_\Omega f(u)\phi_1 + \int_\Omega \phi_1 d\nu \\ &\geq \lambda \int_\Omega f(u)\phi_1. \end{aligned}$$

Therefore, if $f(u) \geq u \geq 0$ and $u \not\equiv 0$, we will get a contradiction when $\lambda > \lambda_1$. Hence there exists $\lambda_* > 0$ such that when $\lambda > \lambda_*$, problem (1.5) admits no non-negative weak solution. □

4. Symmetry of solution: the method of moving plane

In this section, we consider the symmetry of the solution to the following equation:

$$\begin{cases} (-\Delta)^\alpha u = f(u), & x \in B_R(0)\backslash\{0\}, \\ u \equiv 0, & \text{in } B_R(0)^c, \end{cases} \tag{4.1}$$

where $R > 0$, $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function and $\alpha \in (0, 1)$.

Theorem 4.1. *Assume that f satisfies the condition* (H)*. Then any non-negative solution to (4.1) is symmetric about the origin. Moreover, $u > 0$ in $B_R(0)\backslash\{0\}$.*

In order to prove Theorem 4.1, we need the following lemmas.

Lemma 4.1. *(See [2, Lemma 4.2].) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose $w : \mathbb{R}^N \to \mathbb{R}_+$ is continuous, $(-\Delta)^\alpha w \geq 0$ on Ω, and satisfies $w \equiv 0$ on Ω^c. Then either $w \equiv 0$ or $w > 0$ on Ω.*

Lemma 4.2. *(See [2, Lemma 4.3].) Let $\Omega \subset \mathbb{R}^N$ be open, $w : \mathbb{R}^N \to \mathbb{R}_+$ continuous with $w \equiv 0$ on Ω^c, $(-\Delta)^\alpha w \geq 0$ on Ω, w not identically zero. Let $x_0 \in \partial\Omega$ satisfy an interior sphere condition, i.e., there exists a ball $B_\rho(x_1)$ with $\overline{B_\rho(x_1)} \cap \Omega^c = \{x_0\}$, and let ν be an outward pointing unit vector at x_0. Then*

$$\frac{\partial}{\partial \nu} w(x_0) < 0$$

(in fact, $\lim_{\varepsilon \downarrow 0} (w(x_0) - w(x_0 - \varepsilon))/\varepsilon = -\infty$).

Proof of Theorem 4.1. The existence of non-negative solutions follows from Theorem 1.1 with $\beta = 0$ and $\nu = \delta_0$, where δ_0 is the Dirac measure at the origin.

We first prove $u > 0$ in $B_R(0)\backslash\{0\}$. Suppose $u(x_0) = \min_{x\in B_R(0)} u(x) = 0$. Then

$$(-\Delta)^\alpha u(x_0) = C_{N,\alpha} \int\limits_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2\alpha}} dy < 0.$$

And thus

$$0 = (-\Delta)^\alpha u(x_0) - f(u(x_0)) < 0,$$

which yields a contradiction. On the other hand, it follows from Lemma 4.1 that either $w \equiv 0$ or $w > 0$ in $B_R(0)\backslash\{0\}$. It follows from the above discussion that $u > 0$ in $B_R(0)\backslash\{0\}$.

Now, we apply the method of moving plane to prove the symmetry of the solution. Choose any direction in $\mathbb{R}^N$, without loss of generality the x_1-direction, and show that u is mirror symmetric with respect to the hyperplane through the origin with this given direction as a normal vector. For this, let us define for $0 < \sigma < R$,

$$\sum_\sigma = \{x \in B_R : \ x_1 > \sigma\}, \quad T_\sigma = \{x \in B_R : \ x_1 = \sigma\},$$

and $u_\sigma(x) = u(x^\sigma)$ for $x \in \sum_\sigma$, where x^σ is the reflection of x with respect to the line $x_1 = \sigma$, i.e., $x^\sigma = (2\sigma - x_1, x_2, \cdots, x_N)$.

Set $w_\sigma(x) = u_\sigma(x) - u(x)$ for $x \in \sum_\sigma$. Then w_σ satisfies the following equation:

$$\begin{cases} (-\Delta)^\alpha w_\sigma = f(u_\sigma) - f(u) + \delta(2\sigma, \mathbf{0}), & \text{in} \sum_\sigma, \\ w_\sigma \geq 0, \quad \text{on } \partial \sum_\sigma, \end{cases}$$

where $\delta(2\sigma, \mathbf{0})$ is the Dirac measure at the point $(2\sigma, \mathbf{0})$. Define

$$S_u = \left\{ \rho \in (0, R) : \ w_\sigma > 0 \text{ in } \sum_\sigma \ \text{ for } \sigma \in (\rho, R) \right\},$$
$$\rho_u = \inf_{\rho \in S_u} \{\rho\}.$$

First, we show that $S_u \neq \emptyset$. By Lemma 4.2 and $u(x) > 0$ in $B_R\backslash\{0\}$, we have $\frac{\partial}{\partial \nu} u(x) < 0$ for $|x| = R$, where ν is the unit outer normal to ∂B_R at x. In particular,

$$u_\sigma(x) = u(x^\sigma) > u(x) \qquad \text{for } x \in \sum_\sigma,$$

if σ is sufficiently close to R. This shows that $w_\sigma > 0$ in $\sum_\sigma$ if σ is sufficiently close to R. Hence S_u is nonempty.

Next, we prove that $\rho_u = 0$. Suppose this is not true. Then, by continuity, we have $w_{\rho_u}(x) \geq 0$ in $\sum_\sigma$. By the non-decreasing of f, it is easy to see that

$$\begin{cases} (-\Delta)^\alpha w_{\rho_u} \geq \delta(2\sigma, \mathbf{0}), & \text{in } \sum_\sigma, \\ w_{\rho_u} \geq 0, & \text{in } \sum_{\rho_u} \cup \partial \sum_{\rho_u}. \end{cases} \tag{4.2}$$

Thus, if $w_{\rho_u}(x_0) = 0$ for some $x_0 \in \sum_{\rho_u}$, then by Lemma 4.1, we have $w_{\rho_u} \equiv 0$ in $\overline{\sum}_{\rho_u}$. However, this contradicts the fact that $w_{\rho_u} = u(x^{\rho_u}) - u(x) = u(x^{\rho_u}) > 0$ for $x \in \partial \sum_{\rho_u} \setminus \{x_1 = \rho_u\}$. Therefore, we obtain that

$$\begin{cases} w_{\rho_u} > 0, & \text{for any } \overline{\sum}_\sigma \setminus T_{\rho_u}, \\ w_{\rho_u} = 0, & \text{on } \sum_{\rho_u} \cap T_{\rho_u}. \end{cases} \tag{4.3}$$

By (4.2), (4.3) and Lemma 4.2, we obtain

$$\frac{\partial}{\partial x_1} w_{\rho_u} < 0, \quad \text{on } \sum_{\rho_u} \cap T_{\rho_u}. \tag{4.4}$$

On the other hand, since $\rho_u > 0$, there exists a positive sequence ε_k such that $\rho_u - \varepsilon_k > 0$ and $(\rho_u - \varepsilon_k) \to \rho_u$ as $k \to \infty$. By the definition of ρ_u, for each ε_k, we obtain that $w_{\rho_u - \varepsilon_k}$ is non-positive somewhere in $\sum_{\rho_u - \varepsilon_k}$. By the way, we have $w_{\rho_u - \varepsilon_k} > 0$ on $\partial \sum_{\rho_u - \varepsilon_k} \setminus T_{\rho_u - \varepsilon_k}$ and $w_{\rho_u - \varepsilon_k} = 0$ on $\partial \sum_{\rho_u - \varepsilon_k} \cap T_{\rho_u - \varepsilon_k}$. Hence, for each ε_k there exists $x_k \in \sum_{\rho_u - \varepsilon_k}$ such that

$$\begin{cases} w_{\rho_u - \varepsilon_k}(x_k) \le 0, \\ \nabla w_{\rho_u - \varepsilon_k}(x_k) = \mathbf{0}. \end{cases} \tag{4.5}$$

Since $\{x_k\}$ is a bounded sequence, there exists a convergence subsequence, we still denote it by x_k, such that $x_k \to x_0$. By (4.5), we obtain that

$$0 \ge \lim_{k \to \infty} w_{\rho_u - \varepsilon_k}(x_k) = \lim_{k \to \infty} [u(x^{\rho_u - \varepsilon_k}) - u(x_k)] = u(x_0^{\rho_u}) - u(x_0) = w_{\rho_u}(x_0).$$

Hence, by the above inequality and (4.3), we conclude that $x_0 \in \sum_{\rho_u} \cap T_{\rho_u}$ and, by (4.5),

$$0 = \lim_{k \to \infty} \frac{\partial w_{\rho_u - \varepsilon_k}}{\partial x_1}(x_k) = \lim_{k \to \infty} [\frac{\partial u(x_k^{\rho_u - \varepsilon_k})}{\partial x_1}(x_k) - \frac{\partial u(x_k)}{\partial x_1}(x_k)] = \frac{\partial u}{\partial x_1}(x_0^{\rho_u}) - \frac{\partial u}{\partial x_1}(x_0) = \frac{\partial w_{\rho_u}}{\partial x_1}(x_0).$$

This contradicts (4.4). Thus $\rho_u = 0$ and u is radially symmetric. $\square$

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References

[1] D. Applebaum, Lévy Process and Stochastic Calculus, second edition, Cambridge University Press, Cambridge, 2009.
[2] M. Birkner, J.A. López-Mimbela, A. Wakolbinger, Comparison results and steady states for the Fujita equation with fractional Laplacian, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005) 83–97.
[3] J. Blackledge, Application of the fractional diffusion equation for predicting market behavior, Int. J. Appl. Math. 41 (2010) 130–158.
[4] K. Bogdan, T. Kulczycki, A. Nowak, Gradient estimates for harmonic and q-harmonic functions of symmetric stable process, Illinois J. Math. 46 (2002) 541–556.
[5] Z. Chen, R. Song, Estimates on Green functions and Poisson kernels for symmetric stable processes, Math. Ann. 312 (1998) 465–501.
[6] H. Chen, L. Véron, Singular solutions of fractional elliptic equations with absorption, arXiv:1302.1247v1 [math.AP], 6 Feb 2013.

[7] H. Chen, L. Véron, Semilinear fractional elliptic equations with gradient nonlinearity involving measures, J. Funct. Anal. 266 (2014) 5467–5492.
[8] H. Chen, L. Véron, Semilinear fractional elliptic equations involving measures, J. Differential Equations 257 (2014) 1457–1486.
[9] Q. Du, M. Gunzburger, R.B. Lehoucq, K. Zhou, A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, Technical report 2010-8353J, Sandia National Laboratories, 2010.
[10] Q. Du, M. Gunzburger, R.B. Lehoucq, K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev. 54 (2012) 667–696.
[11] M. Kwaśnicki, Eigenvalues of fractional Laplace operator in the interval, J. Funct. Anal. 262 (2012) 2379–2402.
[12] G. Lv, J. Duan, Martingale and weak solutions for a stochastic nonlocal Burgers equation on finite intervals, submitted for publication.
[13] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A: Math. Gen. 37 (2004) R161.
[14] S.P. Neuman, D.M. Tartakovsky, Perspective on theories of non-Fickian transport in heterogeneous media, Adv. Water Resour. 32 (2009) 670–680.
[15] X. Ros-Oton, The Regularity for the fractional Gelfand problem up to dimension 7, J. Math. Anal. Appl. 419 (1) (2014) 10–19.
[16] X. Ros-Oton, J. Serra, The extremal solution for the fractional Laplacian, Calc. Var. Partial Differential Equations 50 (2014) 723–750.
[17] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. 101 (2014) 275–302.
[18] R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, J. Math. Anal. Appl. 389 (2011) 887–898.
[19] R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst. 33 (2013) 2105–2137.
[20] R. Servadei, E. Valdinoci, A Brezis–Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal. 12 (2013) 2445–2464.
[21] G. Stampacchia, Some limit cases of L^p-estimates for solutions of second order elliptic, Comm. Pure Appl. Math. 16 (1963) 505–510.