J. Math. Anal. Appl. 433 (2016) 1220–1229

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two porosity theorems for nonexpansive mappings in hyperbolic spaces

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ARTICLE INFO

Article history:
Received 16 April 2015
Available online 20 August 2015
Submitted by T. Domínguez Benavides

Keywords:
Complete metric space
Contractive mapping
Fixed point
Hyperbolic space
Nonexpansive mapping
Porous set

ABSTRACT

In a previous paper of ours we used the notion of porosity to show that most of the nonexpansive self-mappings of bounded, closed and convex subsets of a Banach space are contractive and possess a unique fixed point which is the uniform limit of all iterates. In this paper we extend this result to nonexpansive self-mappings of closed and convex sets in a Banach space which are not necessarily bounded. As a matter of fact, it turns out that our results are true for all complete hyperbolic metric spaces.

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1. Introduction and statement of main results

For more than fifty years now, there has been considerable interest in the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings in metric and Banach spaces. See, for instance, the papers and books by de Blasi and Myjak [2,3], Goebel and Kirk [4], Goebel and Reich [5], and Kirk [6], as well as the references mentioned therein. This interest originates in Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. Since that seminal result, many developments have taken place in this area. We mention, for example, existence results for fixed points of nonexpansive mappings which are not strictly contractive [4,5]. Such results were obtained for general nonexpansive mappings in special Banach space, while for self-mappings of general complete metric spaces existence results were established for, the so-called, contractive mappings [7]. For general nonexpansive mappings in general Banach spaces the existence of a unique fixed point was established in the generic sense, using the Baire category approach [2,3,11,14]. More precisely, in these works the space $\mathcal{A}$ of all nonexpansive self-mappings of a closed and convex set K in a Banach space is endowed with the natural metric of uniform convergence on bounded

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http://dx.doi.org/10.1016/j.jmaa.2015.08.043
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subsets, and it is shown that there exists a subset $\mathcal{A}' \subset \mathcal{A}$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that every mapping in $\mathcal{A}'$ has a unique fixed point. Note that in [2,3] the set K was assumed to be bounded, while in [11] this assumption was removed.

As a matter of fact, it turns out that all these results are also true for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, an important class of metric spaces the definition of which we now recall.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding is called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1 - t)x \oplus ty$. We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. This inequality, in its turn, implies that

$$\rho((1 - t)x \oplus ty, (1 - t)w \oplus tz) \le (1 - t)\rho(x, w) + t\rho(y, z)$$

for all x, y, z and w in X, and all $0 \le t \le 1$.

A set $K \subset X$ is called *ρ-convex* if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [5,8,9].

A property of elements of a complete metric space Z is said to be *generic* (typical) in Z if the set of all elements of Z which possess this property contains an everywhere dense G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z has this property.

As we have already mentioned, a typical nonexpansive self-mapping of a bounded, closed and ρ-convex subset of a complete hyperbolic metric space has a unique fixed points which is the uniform limit of all its iterates. As a matter of fact, the subset of all those nonexpansive mappings which lack this property is small not only in the sense of Baire category, but also in the sense of porosity, a concept which we now recall.

Let Z be a complete metric space. We denote by $B_Z(y, r)$ the closed ball of center $y \in Z$ and radius $r > 0$. A subset $E \subset Z$ is called *porous* in Z if there exist numbers $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each number $r \in (0, r_0]$ and each point $y \in Z$, there exists a point $z \in Z$ for which

$$B_Z(z, \alpha r) \subset B_Z(y, r) \setminus E.$$

A subset of the space Z is called *σ-porous* in Z if it is a countable union of porous subsets of Z.

Note that in the definition of a porous set we may assume that the point y belongs to E.

Other notions of porosity can be found in the literature. We use this rather strong concept of porosity which has already found application in, for example, approximation theory, the calculus of variations and nonlinear analysis. See, for example, [14,16] and the references mentioned therein.

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Z is a finite-dimensional Euclidean space, then σ-porous sets are also of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have Lebesgue measure zero and are of the first Baire category.

Now let (X, ρ, M) be a complete hyperbolic metric space and let K be a nonempty, closed and ρ-convex subset of X.

Fix a point $\theta \in K$. Denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ such that

$$\rho(A(x), A(y)) \le \rho(x, y) \text{ for all } x, y \in K. \tag{1.1}$$

By (1.1), for each pair $A, B \in \mathcal{A}$ and each $x \in K$,

$$\begin{aligned}\rho(A(x), B(x)) &\le \rho(A(x), A(\theta)) + \rho(A(\theta), B(\theta)) + \rho(B(\theta), B(x)) \\ &\le \rho(x, \theta) + \rho(A(\theta), B(\theta)) + \rho(x, \theta)\end{aligned}$$

and

$$\rho(A(x), B(x)) \le 2\rho(x, \theta) + \rho(A(\theta), B(\theta)). \tag{1.2}$$

For each $A, B \in \mathcal{A}$, set

$$d(A, B) = \inf\{\lambda > 0 : \ \rho(A(x), B(x)) \le \lambda(\rho(x, \theta) + 1) \text{ for all } x \in K\}. \tag{1.3}$$

In view of (1.2) and (1.3), for each $A, B \in \mathcal{A}$, $d(A, B)$ is well defined,

$$d(A, B) = \sup\{\rho(A(x), B(x))(\rho(x, \theta) + 1)^{-1} : \ x \in K\}, \tag{1.4}$$

$d : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a metric on $\mathcal{A}$ and the metric space $(\mathcal{A}, d)$ is complete. Clearly, the topology induced by the metric d is stronger than the topology of uniform convergence on bounded sets, but weaker than the topology of uniform convergence on all of K. It does not depend on the choice of θ. More precisely, if we take $\theta_1 \ne \theta$, then we get Lipschitz-equivalent metrics. To see this, let $\theta_1 \in K \setminus \{\theta\}$ and for each $A, B \in \mathcal{A}$, set

$$d_1(A, B) = \sup\{\rho(A(x), B(x))(\rho(x, \theta_1) + 1)^{-1} : \ x \in K\}.$$

Let $A, B \in \mathcal{A}$. Then for each $x \in K$,

$$\begin{aligned}\rho(A(x), B(x)) &\le d(A, B)(\rho(x, \theta) + 1) \\ &\le d(A, B)(\rho(x, \theta_1) + 1 + \rho(\theta_1, \theta)) \le d(A, B)(\rho(x, \theta_1) + 1)(\rho(\theta_1, \theta) + 1)\end{aligned}$$

and

$$\rho(A(x), B(x))(\rho(x, \theta_1) + 1)^{-1} \le d(A, B)(\rho(\theta_1, \theta) + 1).$$

This implies that

$$d_1(A, B) \le d(A, B)(\rho(\theta_1, \theta) + 1),$$

as asserted.

For each $x \in K$ and each $r > 0$, set

$$B(x,r) = \{y \in K : \ \rho(x,y) \le r\}.$$

Let $A \in \mathcal{A}$ and D be a nonempty subset of K. We say that the mapping A is *contractive* in the sense of Rakotch on D [7] if there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(A(x), A(y)) \le \phi(\rho(x,y))\rho(x,y) \text{ for all } x, y \in D.$$

It is not difficult to see that A is Rakotch contractive on D if and only if for every $\beta > 0$,

$$\sup\{\rho(A(x), A(y))\rho(x,y)^{-1} : \ x, y \in D \text{ and } \rho(x,y) \ge \beta\} < 1.$$

In previous papers of ours [10,12,13] we studied the space $\mathcal{A}$ in the case where the set K is bounded. Using the Baire category approach and the notion of porosity, we show there that most elements of the space $\mathcal{A}$ are Rakotch contractive. When the set K is unbounded, it is known [15] that our results no longer hold. Nevertheless, in the present paper we prove a variant of our results for *unbounded* sets by showing that most mappings in this space are Rakotch contractive on all *bounded* subsets.

More precisely, we establish in Sections 2 and 4 the following two theorems. An auxiliary result is proved in Section 3.

Theorem 1.1. *There exists a set $\mathcal{F}_* \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{F}_*$ is σ-porous in $(\mathcal{A}, d)$ and for each $A \in \mathcal{F}_*$, there exists $M_A > 0$ such that for each $M \ge M_A$, we have the inclusion*

$$A(B(\theta, M)) \subset B(\theta, M)$$

and the mapping A is Rakotch contractive on $B(\theta, M)$.

In particular, each $A \in \mathcal{F}_*$ has a unique fixed point in K. This fixed point is the limit of each sequence of iterates of A.

Theorem 1.2. *There exists a set $\mathcal{G} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{G}$ is σ-porous in $(\mathcal{A}, d)$ and each $A \in \mathcal{G}$ has the following two properties:*

1. *There exists a unique fixed point $x_A \in K$ of A, that is, $A(x_A) = x_A$;*
2. *There exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that*

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(A(x), x_A) \le \phi(\rho(x, x_A))\rho(x, x_A) \text{ for all } x \in K.$$

In the sequel we assume without any loss of generality that the set K is not a singleton and we fix a number $\kappa \in (0,1)$ for which there exist two points $u, v \in K$ satisfying

$$\rho(u,v) \ge \kappa.$$

2. Proof of Theorem 1.1

Proposition 2.1. *There exists a set $\mathcal{F} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{F}$ is porous in $(\mathcal{A}, d)$ and each $A \in \mathcal{F}$ has the following property:*

(P1) *there exists $M_A > 0$ such that for each $M \geq M_A$,*

$$\cup\{B(A(z), 1) : \ z \in B(\theta, M)\} \subset B(\theta, M).$$

Proof. Denote by $\mathcal{F}$ the set of all $A \in \mathcal{A}$ for which (P1) holds. We claim that its complement $\mathcal{A} \setminus \mathcal{F}$ is porous in $(\mathcal{A}, d)$.

Fix

$$\alpha = 8^{-1}. \tag{2.1}$$

Let

$$A \in \mathcal{A} \setminus \mathcal{F} \ \text{ and } \ r \in (0, 1]. \tag{2.2}$$

Fix a natural number n such that

$$\rho(\theta, A(\theta)) \leq n, \tag{2.3}$$

set

$$\gamma = r/4 \tag{2.4}$$

and

$$M_A = 8(n + 2)r^{-1}. \tag{2.5}$$

Define

$$A_\gamma(x) := (1 - \gamma)A(x) \oplus \gamma A(\theta), \ \ x \in K. \tag{2.6}$$

By (1.1) and (2.6), for all $x, y \in K$, we have

$$\begin{aligned} \rho(A_\gamma(x), A_\gamma(y)) &= \rho((1 - \gamma)A(x) \oplus \gamma A(\theta), (1 - \gamma)A(y) \oplus \gamma A(\theta)) \\ &\leq (1 - \gamma)\rho(A(x), A(y)) \leq (1 - \gamma)\rho(x, y). \end{aligned} \tag{2.7}$$

In view of (1.1), (1.4) and (2.6), we also have

$$\begin{aligned} d(A, A_\gamma) &= \sup\{\rho(A(x), A_\gamma(x))(\rho(x, \theta) + 1)^{-1} : \ x \in K\} \\ &\leq \sup\{\gamma\rho(A(x), A(\theta))(\rho(x, \theta) + 1)^{-1} : \ x \in K\} \leq \gamma. \end{aligned} \tag{2.8}$$

Assume now that $C \in \mathcal{A}$ satisfies

$$d(C, A_\gamma) \leq \alpha r. \tag{2.9}$$

Then it follows from (2.1), (2.4), (2.8) and (2.9) that

$$d(C,A) \le d(C,A_\gamma) + d(A_\gamma,A) \le \alpha r + \gamma \le r/2. \tag{2.10}$$

Assume that

$$M \ge M_A,\ z \in B(\theta,M),\ u \in B(C(z),1). \tag{2.11}$$

In view of (2.11),

$$\rho(\theta,u) \le \rho(\theta,C(z)) + \rho(C(z),u) \le \rho(\theta,C(z)) + 1. \tag{2.12}$$

By (1.4), (2.9) and (2.11),

$$\begin{aligned}\rho(\theta,C(z)) &\le \rho(\theta,A_\gamma(z)) + \rho(A_\gamma(z),C(z))\\ &\le \rho(\theta,A_\gamma(z)) + d(C,A_\gamma)(\rho(z,\theta)+1)\\ &\le \rho(\theta,A_\gamma(z)) + \alpha r(M+1).\end{aligned} \tag{2.13}$$

It follows from (1.1), (2.3), (2.2) and (2.6) that

$$\begin{aligned}\rho(\theta,A_\gamma(z)) &\le \rho(\theta,A(\theta)) + \rho(A(\theta),A_\gamma(z))\\ &\le n + \rho(A(\theta),(1-\gamma)A(z) \oplus \gamma A(\theta))\\ &\le n + (1-\gamma)\rho(A(\theta),A(z))\\ &\le n + (1-\gamma)\rho(\theta,z).\end{aligned} \tag{2.14}$$

By (2.1), (2.2), (2.4), (2.5) and (2.11)–(2.14),

$$\begin{aligned}\rho(\theta,u) &\le 1 + \alpha r(M+1) + n + (1-\gamma)\rho(\theta,z)\\ &\le 1 + \alpha r + n + M(1-\gamma+\alpha r)\\ &\le 2 + n + M(1-\gamma/2)\\ &\le M + (2+n) - \gamma M/2\\ &\le M + 2 + n - (r/8)8(n+2)r^{-1} = M\end{aligned}$$

and

$$\rho(\theta,u) \le M. \tag{2.15}$$

Thus we have shown that (2.11) implies (2.15) and

$$B(C(z),1) \subset B(\theta,M) \text{ for all } z \in B(\theta,M).$$

Hence $C \in \mathcal{F}$ and when combined with (2.10), this implies that

$$\{C \in \mathcal{A} :\ d(C,A_\gamma) \le \alpha r\} \subset \mathcal{F} \cap \{C \in \mathcal{A} :\ d(C,A) \le r\}.$$

Therefore the set $\mathcal{A} \setminus \mathcal{F}$ is porous in $(\mathcal{A},d)$. Proposition 2.1 is proved. □

Before proceeding, recall that in Section 1 we fixed a number $\kappa \in (0,1)$ for which there exist two points $u, v \in K$ satisfying

$$\rho(u,v) \geq \kappa. \tag{2.16}$$

We now choose and fix a natural number

$$n_0 > \max\{\rho(\theta,u),\ \rho(\theta,v)\}. \tag{2.17}$$

For each natural number $n \geq n_0$, we denote by $\mathcal{G}_n$ the set of all mappings $A \in \mathcal{A}$ such that

$$\sup\{\rho(A(x),A(y))\rho(x,y)^{-1} :\ x,y \in B(\theta,n) \text{ and } \rho(x,y) \geq \kappa n^{-1}\} < 1. \tag{2.18}$$

Proposition 2.2. *For each integer $n \geq n_0$, the set $\mathcal{A} \setminus \mathcal{G}_n$ is porous in $(\mathcal{A},d)$.*

Proposition 2.2 follows from Lemma 3.1, which is proved in the next section.
Theorem 1.1 follows easily from Propositions 2.1 and 2.2.

3. Main lemma

Recall that $u, v \in K$ and $\kappa \in (0,1)$ satisfy (2.16) and that the natural number n_0 satisfies (2.17). For every natural number $n \geq n_0$, denote by $\mathcal{F}_n$ the set of all mappings $A \in \mathcal{A}$ such that

$$\sup\{\rho(A(x),A(y))\rho(x,y)^{-1} :\ x \in K,\ y \in B(\theta,n) \text{ and } \rho(x,y) \geq \kappa n^{-1}\} < 1. \tag{3.1}$$

Evidently,

$$\mathcal{F}_n \subset \mathcal{G}_n \text{ for all natural numbers } n \geq n_0. \tag{3.2}$$

Lemma 3.1. *For any integer $n \geq n_0$, the set $\mathcal{A} \setminus \mathcal{F}_n$ is σ-porous in $(\mathcal{A},d)$.*

Proof. For each natural number p, set

$$\mathcal{A}_p := \{A \in \mathcal{A} :\ \rho(\theta,A(\theta)) \leq p\}. \tag{3.3}$$

Let $n \geq n_0$ be an integer. Clearly,

$$\mathcal{A} \setminus \mathcal{F}_n = \cup_{p=1}^{\infty}(\mathcal{A}_p \setminus \mathcal{F}_n). \tag{3.4}$$

In order to complete the proof of the lemma it is sufficient to show that for any natural number p, the set $\mathcal{A}_p \setminus \mathcal{F}_n$ is porous in $(\mathcal{A},d)$.

To this end, let $p \geq 1$ be an integer. Set

$$\alpha = (64n^2p)^{-1}\kappa. \tag{3.5}$$

Assume that

$$A \in \mathcal{A}_p \setminus \mathcal{F}_n \ \text{ and } \ r \in (0,1]. \tag{3.6}$$

Set

$$\gamma = (4p)^{-1}r. \tag{3.7}$$

Define

$$A_\gamma(x) := (1-\gamma)A(x) \oplus \gamma\theta, \quad x \in K. \tag{3.8}$$

By (1.1) and (3.8), for all $x, y \in K$,

$$\begin{aligned}\rho(A_\gamma(x), A_\gamma(y)) &= \rho((1-\gamma)A(x) \oplus \gamma\theta, (1-\gamma)A(y) \oplus \gamma\theta) \\ &\le (1-\gamma)\rho(A(x), A(y)) \le (1-\gamma)\rho(x,y).\end{aligned} \tag{3.9}$$

In view of (1.1), (3.3), (3.6) and (3.8), for all $x \in K$,

$$\begin{aligned}\rho(A(x), A_\gamma(x)) &= \rho((1-\gamma)A(x) \oplus \gamma\theta, A(x)) \\ &\le \gamma\rho(A(x), \theta) \le \gamma(\rho(A(x), A(\theta)) + \rho(A(\theta), \theta)) \\ &\le \gamma(\rho(x,\theta) + \rho(A(\theta), \theta)) \\ &\le \gamma(\rho(x,\theta) + p) \le p\gamma(\rho(x,\theta) + 1).\end{aligned} \tag{3.10}$$

Relations (1.3), (3.7) and (3.10) imply that

$$d(A, A_\gamma) \le p\gamma = r/4. \tag{3.11}$$

Assume that $B \in \mathcal{A}$ satisfies

$$d(B, A_\gamma) \le \alpha r. \tag{3.12}$$

It follows from (3.4), (3.11) and (3.12) that

$$d(B, A) \le d(B, A_\gamma) + d(A_\gamma, A) \le \alpha r + r/4 < r. \tag{3.13}$$

Assume that

$$x \in K, \ y \in B(\theta, n), \ \rho(x,y) \ge \kappa n^{-1}. \tag{3.14}$$

It follows from (1.4), (3.9), (3.12) and (3.14) that

$$\begin{aligned}\rho(B(x), B(y)) &\le \rho(B(x), A_\gamma(x)) + \rho(A_\gamma(x), A_\gamma(y)) + \rho(A_\gamma(y), B(y)) \\ &\le \alpha r(\rho(x,\theta) + 1) + (1-\gamma)\rho(x,y) + \alpha r(\rho(y,\theta) + 1) \\ &\le \alpha r(\rho(x,y) + \rho(y,\theta) + 1) + (1-\gamma)\rho(x,y) + \alpha r(\rho(y,\theta) + 1) \\ &= (1-\gamma)\rho(x,y) + \alpha r(\rho(x,y) + 2\rho(y,\theta) + 2) \\ &\le (1-\gamma)\rho(x,y) + \alpha r\rho(x,y) + \alpha r(2n+2).\end{aligned}$$

When combined with (3.4), (3.7) and (3.14), this implies that

$$\begin{aligned}\rho(B(x),B(y))\rho(x,y)^{-1} &\le 1-\gamma+\alpha r+\alpha r(2n+2)\kappa^{-1}n\\ &\le 1-\gamma+\alpha r(2n^2+2n+1)\kappa^{-1}\\ &\le 1-\gamma+\alpha r(8n^2)\kappa^{-1}=1-(8p)^{-1}r. \end{aligned} \tag{3.15}$$

Therefore (3.14) implies (3.15),

$$\sup\{\rho(B(x),B(y))\rho(x,y)^{-1}:\ x\in K,\ y\in B(\theta,n) \text{ and } \rho(x,y)\ge \kappa n^{-1}\}\le 1-(8p)^{-1}r$$

and in view of (3.1),

$$B\in\mathcal{F}_n. \tag{3.16}$$

Therefore (3.12) implies (3.13) and (3.16) and

$$\{B\in\mathcal{A}:\ d(B,A_\gamma)\le \alpha r\}\subset \mathcal{F}_n\cap\{B\in\mathcal{A}:\ d(B,A)\le r\}.$$

Thus $\mathcal{A}_p\setminus\mathcal{F}_n$ is indeed porous in $(\mathcal{A},d)$. Lemma 3.1 is proved. □

Proposition 2.2 follows from Lemma 3.1 and (3.2).

4. Proof of Theorem 1.2

Recall that in Section 1 we fixed a number $\kappa\in(0,1)$ for which there exist two points $u,v\in K$ satisfying

$$\rho(u,v)\ge\kappa. \tag{4.1}$$

In Section 2 we also fixed a natural number

$$n_0>\max\{\rho(\theta,u),\ \rho(\theta,v)\}. \tag{4.2}$$

Recall that for every natural number $n\ge n_0$, we denote by $\mathcal{F}_n$ the set of all mappings $A\in\mathcal{A}$ such that

$$\sup\{\rho(A(x),A(y))\rho(x,y)^{-1}:\ x\in K,\ y\in B(\theta,n) \text{ and } \rho(x,y)\ge\kappa n^{-1}\}<1$$

(see (3.1)).

Let the set $\mathcal{F}_*$ be as guaranteed by Theorem 1.1. Define

$$\mathcal{G}:=\mathcal{F}_*\cap(\cap_{n=n_0}^{\infty}\mathcal{F}_n). \tag{4.3}$$

By (4.3), Theorem 1.1 and Lemma 3.1, $\mathcal{A}\setminus\mathcal{G}$ is a σ-porous set in $(\mathcal{A},d)$.

Let

$$A\in\mathcal{G}. \tag{4.4}$$

In view of (4.3), (4.4) and Theorem 1.1, there exists a unique point $x_A\in K$ such that

$$A(x_A)=x_A. \tag{4.5}$$

Let a natural number $n>n_0$ satisfy

$$x_A\in B(\theta,n). \tag{4.6}$$

Since $A \in \mathcal{F}_n$, it follows from (3.1), (4.5) and (4.4) that

$$1 > \sup\{\rho(A(x), A(y))\rho(x,y)^{-1} : \ x \in K, \ y \in B(\theta, n) \text{ and } \rho(x,y) \geq \kappa n^{-1}\}$$
$$\geq \sup\{\rho(A(x), x_A)\rho(x, x_A)^{-1} : \ x \in K \text{ and } \rho(x, x_A) \geq \kappa n^{-1}\}.$$

Since the above inequality holds for any integer $n > n_0$, there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\rho(A(x), x_A) \leq \phi(\rho(x, x_A))\rho(x, x_A) \text{ for all } x \in K.$$

This completes the proof of Theorem 1.2. □

Acknowledgments

The research of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund. Both authors are very grateful to an anonymous referee for many pertinent comments and helpful suggestions, which have substantially improved this paper.

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