



Self-inversive Hilbert space operator polynomials with spectrum on the unit circle

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Abstract

Hilbert space operator polynomials with self-inversive structure are studied. If the inner numerical radius of an associated polynomial is greater than or equal to one then the spectrum lies on the unit circle and consists of normal approximate characteristic values.

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1. Introduction

Let

$$f(z) = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} + f_n z^n \in \mathbb{C}[z]$$

with $f_0 f_n \neq 0$. Define

$$\widehat{f}(z) = \bar{f}_n + \bar{f}_{n-1} z + \cdots + \bar{f}_1 z^{n-1} + \bar{f}_0 z^n. \quad (1.1)$$

If

$$f(z) = \gamma \widehat{f}(z) \text{ and } |\gamma| = 1, \quad (1.2)$$

then $f(z)$ is said to be γ -self-inversive (see e.g. Marden [36, p. 201], Sheil-Small [49, p. 149], and Rahman and Schmeisser [45]). Self-inversive polynomials can be found in the literature under various names such as self-reciprocal [1, 15, 31], reciprocal [2, 32, 34], palindromic [9, 53, 11], conjugate symmetric [8], symmetric [51], or conjugate reciprocal [42]. If all zeros of a complex polynomial $f(z)$ lie on the unit circle then there exists a unimodular γ such that $f(z)$ is γ -self-inversive [16]. Polynomials with roots of modulus

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one have a wide range applications. We mention Lie algebras [30], kinematics [20], quantum chaotic dynamics [10], signal and speech processing [50, 51] and self-dual codes [15]. The starting point of our note is a constructive
 25 result of Schur [48, XII]. We state it in a modified form as follows (see also [13], [29]).

Theorem 1.1. *Let $p(z) \in \mathbb{C}[z]$ a polynomial of degree k , let $r \in \mathbb{N}_0$ and $|\gamma| = 1$. Then $f(z) = p(z) + \gamma z^r \hat{p}(z)$ is γ -self-inversive. Suppose $p(z) = c \prod_{j=1}^k (z - \omega_j)$. If $|\omega_j| \geq 1$, $j = 1, \dots, k$, then the zeros of $f(z)$ lie on the
 30 unit circle.*

In this paper we consider Hilbert space operator polynomials with γ -self-inversive structure. Let H be a complex Hilbert space and let $\mathcal{L}(H)$ be the algebra of bounded linear operators on H . If $T \in \mathcal{L}(H)$ then T^* shall denote the adjoint of T . Let

$$F(z) = F_0 + F_1 z + \dots + F_{m-1} z^{m-1} + F_m z^m \in \mathcal{L}(H)[z]$$

35 with $F_0 \neq 0$, $F_m \neq 0$. In accordance with (1.1) and (1.2) we associate to $F(z)$ an operator polynomial

$$\hat{F}(z) = F_m^* + F_{m-1}^* z + \dots + F_1^* z^{m-1} + F_0^* z^m,$$

and we say that $F(z)$ is γ -self-inversive if

$$F(z) = \gamma \hat{F}(z) \quad \text{and} \quad |\gamma| = 1.$$

Applications of γ -self-inversive matrix polynomials are cited in [39]. We mention vibrational analysis of railroad tracks excited by high speed trains [38],
 40 modelling and numerical simulation of periodic surface acoustic wave filters [28], computation of the Crawford number of two hermitian matrices [26], and the solution of discrete-time linear-quadratic optimal control problems via structured eigenvalue problems [12].

Let $P(z) = \sum_{j=0}^k A_j z^j \in \mathcal{L}(H)[z]$ be given such that $A_k \neq 0$, $A_0 \neq 0$. It
 45 is easy to see that

$$F(z) = P(z) + \gamma z^r \hat{P}(z)$$

is γ -self-inversive, if $|\gamma| = 1$ and $r \geq 0$. Using the inner numerical radius of operator polynomials $P(z)$ we shall extend Schur's theorem. Our main result is Theorem 3.1. It involves the spectrum of $F(z)$ and its approximate and residual parts, and therefore goes beyond results on matrix polynomials in
 50 [29]. In Section 2 below we gather together basic facts on the spectrum and the approximate numerical range of operator polynomials in Hilbert space. In Section 4 we deal with a special case of $F(z)$ assuming that $P(z)$ is of degree 1.

2. Operator polynomials

2.1. The spectrum

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Let $S_H = \{x \in H; \|x\|^2 = \langle x, x \rangle = 1\}$ be the unit sphere of a complex Hilbert space H . If $v \in H$ then $v^* \in H^*$ is defined by $v^*(u) = \langle v, u \rangle$, $u \in H$. Let $u = (u_\nu) \in H^\mathbb{N}$. We write

$$u \hat{=} 0 \quad \text{if} \quad \lim_{\nu \rightarrow \infty} u_\nu = 0,$$

and $u \not\hat{=} 0$ if (u_ν) is not a null sequence. Let $T \in \mathcal{L}(H)$, and let $\sigma(T)$ denote the spectrum of the bounded linear operator T . A complex number λ is called an *approximate eigenvalue* of T , if for all $\epsilon > 0$ there exists a $y \in H$ such that

$$\|(\lambda I - T)y\| < \epsilon \|y\|. \quad (2.1)$$

The set $\sigma_A(T)$ of approximate eigenvalues of T is the *approximate point spectrum* of T (see [6], [27, p. 54] [5, p. 241], [41, p. 413], [22, p. 81]). We say that a sequence $v = (v_\nu) \in H^\mathbb{N}$ is an *approximate eigenvector* corresponding to λ if

$$(\lambda I - T)v \hat{=} 0 \quad \text{and} \quad v \not\hat{=} 0. \quad (2.2)$$

If convenient, one can assume $v_\nu \in S_H$, $\nu \in \mathbb{N}$, in (2.2). We define

$$\text{Ker}_A(\lambda I - T) = \{v \in H^\mathbb{N}; (\lambda I - T)v \hat{=} 0\}.$$

Suppose $\lambda \in \sigma_A(T)$. If there exists a corresponding approximate eigenvector $v = (v_\nu)$ such that $(\lambda I - T)^*v \hat{=} 0$, then λ is called a *normal approximate eigenvalue* of T (see e.g. [17], [21], [35]). Thus λ is normal if

$$\text{Ker}_A(\lambda I - T) = \text{Ker}_A(\lambda I - T)^*.$$

The set

$$\sigma_P(T) = \{\lambda \in \mathbb{C}; \lambda I - T \text{ is not injective}\}$$

is the *point spectrum* of T . From (2.1) follows $\sigma_P(T) \subseteq \sigma_A(T)$. Let

$$\sigma_R(T) = \{\lambda \in \mathbb{C}; \lambda I - T \text{ is injective and } \overline{\text{range}(\lambda I - T)} \neq H\}$$

be the *residual spectrum* of T . Then (see e.g. [41, p. 413])

$$\sigma(T) = \sigma_A(T) \cup \sigma_R(T),$$

and (see e.g. [3, p. 298], [41, p. 412])

$$\sigma_P(T) \cap \sigma_R(T) = \emptyset. \quad (2.3)$$

75 It is known ([47, p. 194], [18, p. 161]) that

$$\sigma_R(T) \subseteq \sigma_P(T^*). \quad (2.4)$$

We extend the notion of spectrum from operators $T \in \mathcal{L}(H)$ to operator polynomials

$$B(z) = \sum_{j=0}^m B_j z^j \in \mathcal{L}(H)[z]. \quad (2.5)$$

We assume that $B(\mu) \in \mathcal{L}(H)$ is invertible for some $\mu \in \mathbb{C}$. If H is finite dimensional, say $H = \mathbb{C}^n$, then $B(z) \in \mathbb{C}^{n \times n}[z]$, and in that case the preceding
80 assumption means that $\det B(z)$ is not the zero polynomial. We define

$$\sigma(B) = \{\lambda \in \mathbb{C}; B(\lambda) \text{ is not invertible}\}.$$

Thus $\sigma(B) = \{\lambda \in \mathbb{C}; 0 \in \sigma(B(\lambda))\}$. Similarly, we define

$$\sigma_M(B) = \{\lambda; 0 \in \sigma_M(B(\lambda))\} \quad \text{for } M \in \{P, A, R\}.$$

Thus $\lambda \in \sigma_A(B)$ if and only if

$$B(\lambda)v \hat{=} 0, \quad v \neq 0, \quad (2.6)$$

for some sequence $v = (v_\nu) \in H^\mathbb{N}$. Adapting a notion of [4] we call the elements of $\sigma_A(B)$ *approximate characteristic values* of $B(z)$. If (2.6) holds
85 then $v = (v_\nu)$ is said to be an *approximate eigenvector* of $B(z)$ corresponding to λ . Moreover, $\lambda \in \sigma_A(B)$ is called *normal* if there exists an approximate eigenvector v such that $B(\lambda)^*v \hat{=} 0$. In particular, λ is normal, if $\text{Ker}_A B(\lambda) = \text{Ker}_A B(\lambda)^*$. We note a preliminary result.

Lemma 2.1. *Let $F(z) = \sum_{j=0}^m F_j z^j \in \mathcal{L}(H)[z]$ be γ -self-inversive. If $\lambda \in$
90 $\sigma_A(F)$ and $|\lambda| = 1$, then λ is normal, and we have*

$$\text{Ker}_A F(\lambda) = \text{Ker}_A F(\lambda)^*. \quad (2.7)$$

Proof. If $|\lambda| = 1$ and $y \in H$ then $F = \gamma \hat{F}$ implies

$$\begin{aligned} (F(\lambda)y)^* &= y^* \left(\sum_{j=0}^m F_j^* \bar{\lambda}^j \right) = \bar{\gamma} y^* \left(\gamma \sum_{j=0}^m F_j^* \lambda^{-j} \right) = \\ &= (\bar{\gamma} \lambda^{-m}) (y^* \gamma \hat{F}(\lambda)) = (\bar{\gamma} \lambda^{-m}) y^* F(\lambda). \end{aligned} \quad (2.8)$$

Now let $\lambda \in \sigma_A(F)$ and $v = (v_\nu)$ be a corresponding approximate eigenvector. If $|\lambda| = 1$, then it follows from (2.8) that $\lim_{\nu \rightarrow \infty} F(\lambda)v_\nu = 0$ is equivalent to $\lim_{\nu \rightarrow \infty} v_\nu^* F(\lambda) = 0$. \square

2.2. The approximate numerical range

Let $B(z) \in \mathcal{L}(H)[z]$. The set

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$$W(B) = \{\lambda \in \mathbb{C}; y^* B(\lambda) y = 0 \text{ for some } y \in H, y \neq 0\} \quad (2.9)$$

is the *numerical range* of $B(z)$. We refer to [33] for a detailed study of the numerical range of matrix polynomials and to [37], [19], [40], [43], [14] for further investigations. In addition to (2.9) we consider the *approximate numerical range* $W_A(B)$ of $B(z)$. We define

$$W_A(B) = \{\lambda \in \mathbb{C}; \lim_{\nu \rightarrow \infty} y_\nu^* B(\lambda) y_\nu = 0 \text{ for some } y = (y_\nu) \in H^\mathbb{N}, y \neq 0\}.$$

Then $\sigma_A(B) \subseteq W_A(B)$. In [52] it was shown that

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$$W_A(B) = \overline{W(B)}. \quad (2.10)$$

If $B(z) = zI - T$ and $T \in \mathcal{L}(H)$, then $W_A(B)$ and $W(B)$ are equal to

$$N_A(T) = \{\lambda \in \mathbb{C}; \lambda = \lim_{\nu \rightarrow \infty} x_\nu^* T x_\nu \text{ for some } (x_\nu) \in H^\mathbb{N}, x_\nu \in S_H, \nu \in \mathbb{N}\}$$

and

$$N(T) = \{\lambda \in \mathbb{C}; \lambda = x^* T x \text{ for some } x \in S_H\} = \{x^* T x; x \in S_H\},$$

respectively. The set $N(T)$ is the numerical range (or field of values) of T . Let

$$w(B) = \sup\{|\lambda|; \lambda \in W(B)\} \text{ and } \nu(T) = \sup\{|\lambda|; \lambda \in N(T)\}$$

be the *numerical radius* of $B(z)$ and T , respectively, and let

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$$w_i(B) = \inf\{|\lambda|; \lambda \in W(B)\}$$

be the *inner numerical radius* of $B(z)$. From (2.10) follows

$$w(B) = \sup\{|\lambda|; \lambda \in W_A(B)\}$$

and $\nu(T) = \sup\{|\lambda|; \lambda \in N_A(T)\}$ and $w_i(B) = \min\{|\lambda|; \lambda \in W_A(B)\}$.

3. The main result

Let $\mathbb{D} = \{\lambda; |\lambda| < 1\}$ be the open unit disc and $\partial\mathbb{D} = \{\lambda; |\lambda| = 1\}$ be
 110 the unit circle.

Theorem 3.1. *Let $P(z) = \sum_{j=0}^k A_j z^j \in \mathcal{L}(H)[z]$ be given with $A_k \neq 0$, $A_0 \neq 0$. If $|\gamma| = 1$ and $r \geq 0$ then*

$$F(z) = P(z) + \gamma z^r \hat{P}(z)$$

is γ -self-inversive. If $w_i(P) \geq 1$ then

$$\sigma(F) \subseteq W_A(F) \subseteq \partial\mathbb{D}.$$

The residual spectrum of $F(z)$ is empty, that is $\sigma(F) = \sigma_A(F)$. The charac-
 115 *teristic values λ of $F(z)$ are approximately normal, satisfying (2.7).*

Proof. Let us show first that $\lambda \in \partial\mathbb{D}$ if $\lambda \in W(F)$. Suppose $v \in H$, $v \neq 0$, and $v^* F(\lambda)v = 0$. Set

$$p_v(z) = \sum_{j=0}^k v^* A_j v z^j = v^* P(z)v \quad \text{and} \quad f_v(z) = v^* F(z)v.$$

Then $f_v(z) = p_v(z) + \gamma z^r \hat{p}_v(z)$. The assumption $w_i(P) \geq 1$, that is $W(P) \subseteq \{\lambda; |\lambda| \geq 1\}$, implies that $|\omega| \geq 1$ if $p_v(\omega) = 0$. Hence Theorem 1.1 yields
 120 $\lambda \in \partial\mathbb{D}$, and we obtain $W(F) \subseteq \partial\mathbb{D}$. Then $\overline{W}(F) \subseteq \partial\mathbb{D}$. Therefore we conclude from $W_A(F) = \overline{W}(F)$ and $\sigma_A(F) \subseteq W_A(F)$ that

$$\sigma_A(F) \subseteq W_A(F) \subseteq \partial\mathbb{D}.$$

We now show that $\sigma_R(F) = \emptyset$. Suppose $\lambda \in \sigma_R(F)$, that is $0 \in \sigma_R(F(\lambda))$. Then (2.4) implies $0 \in \sigma_P(F(\lambda)^*)$. If $\lambda = 0$ then $0 \in \sigma_P(F(0)^*) = \sigma_P(F_0^*)$. Then $0 = y^* F_0^* y = y^* F_0 y$ for some $y \in S_H$. Thus $0 \in W(F)$, in contradiction to $W(F) \subseteq \partial\mathbb{D}$. If $\lambda \neq 0$ then

$$\begin{aligned} (F(\lambda))^* &= F_0^* + F_1^* \bar{\lambda} + \cdots + F_m^* \bar{\lambda}^m = \\ &\bar{\lambda}^m (F_0^* \bar{\lambda}^{-m} + F_1^* \bar{\lambda}^{-(m-1)} + \cdots + F_m^*) = \bar{\lambda}^m \hat{F}(\bar{\lambda}^{-1}) = \bar{\lambda}^m \gamma F(\bar{\lambda}^{-1}). \end{aligned}$$

Hence $0 \in \sigma_P(F(\lambda)^*)$ is equivalent to $0 \in \sigma_P(F(\bar{\lambda}^{-1}))$, that is to $\bar{\lambda}^{-1} \in \sigma_P(F)$. Then $\sigma_P(F) \subseteq \sigma_A(F) \subseteq \partial\mathbb{D}$ implies $\bar{\lambda}^{-1} = \lambda \in \sigma_P(F)$. Hence $\lambda \in \sigma_R(F) \cap \sigma_P(F)$, in contradiction to (2.3). The last statement follows
 125 from Lemma 2.1. \square

4. $P(z) = I - Tz$

In this section we deal with self-inversive operator polynomials

$$F(z) = P(z) + z\hat{P}(z)$$

where $P(z) = I - Tz$ and $T \in \mathcal{L}(H)$. Let $\mathcal{R}(T) = \frac{1}{2}(T + T^*)$ be the real part of T . Then

$$F(z) = (1 + z^2)I - 2z\mathcal{R}(T).$$

It is easy to see that $w_i(P) = \nu(T)$. Hence $\sigma(F) \subseteq \partial\mathbb{D}$ if $\nu(T) \leq 1$ (by Theorem 3.1). A sharper result is the following. 130

Proposition 4.1. (i) *The spectrum of $F(z) = (1 + z^2)I - 2z\mathcal{R}(T)$ is*

$$\sigma(F) = \{\lambda; \lambda^2 - 2\lambda\mu + 1 = 0, \mu \in \sigma(\mathcal{R}(T))\}. \quad (4.1)$$

(ii) *Suppose $\sigma(\mathcal{R}(T)) = [-a, a]$, $a \in \mathbb{R}$, $a > 0$. Then $\sigma(F) \subseteq \partial\mathbb{D}$ if and only if $a \leq 1$.*

(iii) *If $\sigma(\mathcal{R}(T)) = [-a, a] \subseteq [-1, 1]$, and $a = \cos \alpha$, with $0 \leq \alpha < \frac{\pi}{2}$, then* 135

$$\sigma(F) = W_A(F) = A_+ \cup A_-,$$

where

$$A_+ = \{e^{i\tau}; \alpha \leq \tau \leq \pi - \alpha\} \quad \text{and} \quad A_- = \{e^{-i\tau}; \alpha \leq \tau \leq \pi - \alpha\},$$

are circular arcs with midpoints i and $-i$, respectively. In particular, if $a = 1$ then $\sigma(F) = W_A(F) = \partial\mathbb{D}$.

Proof. (i) The spectral mapping theorem for polynomials (see e.g. [5, p. 226], [24, p. 53]) implies 140

$$\sigma(F(\lambda)) = F[\sigma((\lambda^2 + 1)I - 2\lambda\mathcal{R}(T))].$$

Hence $0 \in \sigma(F(\lambda))$ if and only if

$$(\lambda^2 + 1) - 2\lambda\mu = 0 \quad (4.2)$$

for some $\mu \in \mathcal{R}(T)$.

(ii) If $\mu \in \mathbb{R}$, then the roots of the quadratic equation (4.2) are

$$\lambda_{\pm}(\mu) = \mu \pm \sqrt{\mu^2 - 1}.$$

If $|\mu| \leq 1$, $\mu = \cos \phi$, then $\lambda_{\pm}(\mu) = e^{\pm i\phi}$. If $|\mu| \geq 1$, $|\mu| = \cosh \psi$, then
 145 $\lambda_{\pm}(\mu) \in \{\pm e^{\pm i\psi}\}$. Therefore $|\lambda_{\pm}(\mu)| = 1$ is equivalent to $|\mu| \leq 1$. Hence we
 have $|\mu| \leq 1$ for all $\mu \in \mathbb{R}$ if and only if $a \leq 1$.

(iii) We have $\lambda_+(a) = e^{i\alpha}$ and $\lambda_+(-a) = e^{i(\pi-\alpha)}$. Hence, if μ varies
 between a and $-a$ then the characteristic values $\lambda_+(\mu)$ of $F(z)$ yield the arc
 A_+ . Similarly, $\lambda_-(a) = e^{-i\alpha}$ and $\lambda_-(-a) = e^{-i(\pi-\alpha)}$ are the end points of the
 150 arc A_- . If $a = 1$ then $\alpha = 0$, and $A_+ \cup A_- = \partial\mathbb{D}$.

It remains to show that $\sigma(F) = W_A(F)$. We have $\lambda \in W(F)$ if and only
 if

$$v^*((\lambda^2 + 1)I - 2\lambda\mathcal{R}(T))v = 0 \text{ for some } v \in S_H. \quad (4.3)$$

Then $0 \notin W(F)$, and we obtain

$$W(F) = \{\lambda; (1 + \lambda^2)(2\lambda)^{-1} \in N(\mathcal{R}(T))\}. \quad (4.4)$$

Let $\text{conv}(M)$ denote the convex hull of a set $M \subseteq \mathbb{C}$. The operator $\mathcal{R}(T) \in$
 155 $\mathcal{L}(H)$ is normal, and therefore (see [23, p. 16], [25, Problem 216])

$$\overline{N}(\mathcal{R}(T)) = \text{conv}(\sigma(\mathcal{R}(T))).$$

Hence (4.4) implies

$$W_A(F) = \{\lambda; (1 + \lambda^2)(2\lambda)^{-1} = \mu \text{ for some } \mu \in [-a, a]\}.$$

Then (4.1) completes the proof. \square

To illustrate the preceding observations we choose $H = \ell_2(0, \infty)$ and
 $T = aS$, where S is the left shift on H ,

$$S : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots),$$

160 and $a \in \mathbb{C}$, $a \neq 0$. If $P(z, a) = I - zaS$ and $F(z, a) = P(z, a) + \widehat{P}(z, a)$ then

$$F(z, a) = (z^2 + 1)I - z(aS + \bar{a}S^*).$$

The adjoint S^* is the right shift on $\ell_2(0, \infty)$,

$$S^* : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots).$$

The numerical radius of S is equal to 1 (see e.g. [25, Problem 112]). Hence
 $\nu(aS) = |a|$, and therefore the condition $|a| \leq 1$ is sufficient for $\sigma(F(z, a)) \subseteq$
 $\partial\mathbb{D}$. If $|\eta| = 1$ and $D(\eta) = \text{diag}(1, \eta, \eta^2, \dots)$, then $D(\eta)^{-1} = D(\eta)^*$ and

$$D(\eta)(\eta S)D(\eta)^{-1} = S$$

Hence (see also [46, p.7]), if $a = |a|e^{i\theta}$, then

$$\begin{aligned} F(z, a) &= (1 + z^2)I - z(aS + \bar{a}S^*) = \\ &D(e^{-i\theta}) \left[(1 + z^2)I - z|a|(S + S^*) \right] D(e^{i\theta}) = D(e^{-i\theta})F(z, |a|)D(e^{i\theta}). \end{aligned}$$

Therefore only the case $a = |a|$ is of interest. Since S is hyponormal, that is $S^*S - SS^* \geq I$, it follows from [44], [7] that 165

$$\sigma(\mathcal{R}(S)) = [-1, 1] \text{ and } \sigma(\mathcal{R}(S)) \cap \sigma_P(\mathcal{R}(S)) = \emptyset. \quad (4.5)$$

We also refer to [25, Problem 144]. Hence Proposition 4.1 implies that $\sigma(F(z, a)) \subseteq \partial\mathbb{D}$ is equivalent to $|a| \leq 1$. Moreover, if $0 < |a| \leq 1$ and $|a| = \cos \alpha$, $0 \leq \alpha < \frac{\pi}{2}$, then

$$\sigma(F(z, a)) = W_A(F(z, a)) = \{e^{i\theta}; \alpha \leq \theta \leq \pi - \alpha\} \cup \{e^{i\theta}; \pi + \alpha \leq \theta \leq 2\pi - \alpha\}.$$

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