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Continued fraction inequalities related to $\left(1+\frac{1}{x}\right)^x$

Xu You [a,*], Di-Rong Chen [b,c], Hong Shi [a]

[a] *Department of Mathematics and Physics, Beijing Institute of Petrochemical Technology, Beijing 102617, PR China*
[b] *Department of Mathematics, Wuhan Textile University, Hubei, Wuhan 430200, PR China*
[c] *School of Mathematics and System Science, Beihang University, Beijing 100191, PR China*

A R T I C L E I N F O

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A B S T R A C T

In this paper, we provide some new continued fraction inequalities related to $(1+\frac{1}{x})^x$.

1. Introduction

Yang and Debnath [10] presented the following double inequality for every x in $0 < x \le \frac{1}{5}$:

$$e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 < (1+x)^{1/x} < e - \frac{e}{2}x + \frac{11e}{24}x^2. \tag{1.1}$$

Such inequalities were proven to be of great interest through the researchers, especially in the recent past, due to many practical problems where they can be applied. As example, we refer to inequality (1.1) which is the main tool for improving Carleman's inequality in [10].

Recently, Mortici and Yang [6] proposed an improvement of (1.1) and provided a simple, direct proof of (1.1) for every real number $x \in (0; 1]$.

$$a(x) < (1+x)^{1/x} < b(x), \tag{1.2}$$

where $a(x) = e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 + \frac{2447e}{5760}x^4 - \frac{959e}{2304}x^5$ and $b(x) = a(x) + \frac{959e}{2304}x^5$.

* Corresponding author.
E-mail addresses: youxu@bipt.edu.cn (X. You), drchen@buaa.edu.cn (D.-R. Chen), shihong@bipt.edu.cn (H. Shi).

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Motivated by the work of Yang and Debnath [10], Mortici and Yang [6], Hu and Mortici [5], in this paper we will continue our previous works, and apply the multiple-correction method to construct some new continued fraction inequalities related to $\left(1+\frac{1}{x}\right)^x$, which have faster rate of convergence.

This paper is organized as follows. In Section 2, we explain how to find the continued fraction approximation for $\left(1+\frac{1}{x}\right)^x$. In Section 3, we give the improved continued fraction approximation for $\left(1+\frac{1}{x}\right)^x$.

Throughout the paper, the notation $\Psi(k;x)$ means a polynomial of degree k in x with all of its non-zero coefficients positive, which may be different at each occurrence.

2. Continued fraction approximation for $\left(1+\frac{1}{x}\right)^x$

Theorem 1. *For every real number $x \geq 1$, we have*

$$\left(1+\frac{1}{x}\right)^x < e \cdot a_i(x) = e\left(1+\cfrac{s_1}{x+t_1+\cfrac{s_2}{x+t_2+\cfrac{s_3}{x+t_3+\cfrac{s_4}{x+t_4+\cdots}}}}\right), i=1,2,3,..., \tag{2.1}$$

where $s_1 = -\frac{1}{2}$, $t_1 = \frac{11}{12}$; $s_2 = -\frac{5}{144}$, $t_2 = \frac{34}{75}$; $s_3 = -\frac{481}{10000}$, $t_3 = \frac{357866}{757575}$; $s_4 = -\frac{792876605}{14692348944}$, $t_4 = \frac{2317657460602}{4805307952263}$; $\ldots$.

Proof. Inequalities (2.1) are equivalent to $f_i < 0$ on $[1, +\infty)$, where

$$f_i(x) = x\ln\left(1+\frac{1}{x}\right) - \ln a_i(x) - 1.$$

Based on our previous works [7,11–13], we will apply *multiple-correction method* to study the estimate for $\left(1+\frac{1}{x}\right)^x$. Now, we prove the estimate for $\left(1+\frac{1}{x}\right)^x$ by *multiple-correction method* [1–3] as follows:

(Step 1) The first-correction. Because $\left(x\ln\left(1+\frac{1}{x}\right)\right)'' = -\frac{1}{x(1+x)^2}$, so we choose $a_1(x) = 1+\frac{s_1}{x+t_1}$. Then letting the coefficient of x^4, x^3 of the molecule in the following fractions equal to zero, we have $s_1 = -\frac{1}{2}$, $t_1 = \frac{11}{12}$ and

$$\begin{aligned} f_1''(x) &= \left(x\ln\left(1+\frac{1}{x}\right) - \ln a_1(x) - 1\right)'' \\ &= \frac{P_1(x)}{x(1+x)^2(5+12x)^2(11+12x)^2}. \end{aligned}$$

As $P_1(x) = -3025 - 7296x - 4320x^2$ has all coefficients negative, it results that $f_1(x)$ is strictly concave on $[1,\infty)$ with $f_1(\infty) = 0$, we deduce that $f_1(x) < 0$ on $[1,\infty)$. As $f_1(1) = -1+\ln\frac{23}{17}+\ln 2 = -0.00457195... < 0$.

(Step 2) The second-correction. We let $a_2(x) = 1+\frac{s_1}{x+t_1+\frac{s_2}{x+t_2}}$. Then letting the coefficient of x^6, x^5 of the molecule in the following fractions equal to zero, we have $s_2 = -\frac{5}{144}$, $t_2 = \frac{34}{75}$ and

$$\begin{aligned} f_2''(x) &= \left(x\ln\left(1+\frac{1}{x}\right) - \ln a_2(x) - 1\right)'' \\ &= \frac{P_2(x)}{x(1+x)^2(185+1044x+1200x^2)^2(457+1644x+1200x^2)^2}. \end{aligned}$$

As $P_2(x) = -7147857025 - 52161296256x - 132444066720x^2 - 139358976000x^3 - 51948000000x^4$ has all coefficients negative, it results that $f_2(x)$ is strictly concave on $[1,\infty)$ with $f_2(\infty) = 0$, we deduce that $f_2(x) < 0$ on $[1,\infty)$. As $f_2(1) = -1+\ln\frac{3301}{2429}+\ln 2 = -0.000107017... < 0$.

(Step 3) The third-correction. Similarly, we let $a_3(x) = 1 + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2+\frac{s_3}{x+t_3}}}$. Then letting the coefficient of x^8, x^7 of the molecule in the following fractions equal to zero, we have $s_3 = -\frac{481}{10000}$, $t_3 = \frac{357866}{757575}$ and

$$
\begin{aligned}
f_3''(x) &= \left(x \ln\left(1+\frac{1}{x}\right) - \ln a_3(x) - 1\right)'' \\
&= \frac{P_3(x)}{x(1+x)^2\Psi_1^2(3;x)\Psi_2^2(3;x)},
\end{aligned}
$$

where $\Psi_1(3;x) = 1535537 + 15041160x + 39051120x^2 + 29090880x^3$, $\Psi_2(3;x) = 3950767 + 28506120x + 53596560x^2 + 29090880x^3$. As $P_3(x) = -3680301563955478981352064 1 - 5102400501329393409750950 40x - 2693418207575323688814043200x^2 - 7005074374845717299901265920x^3 - 9583241661488326994121772800x^4 - 6605509732240923396484300800x^5 - 1807392287181467915782963200x^6$ has all coefficients negative, it results that $f_3(x)$ is strictly concave on $[1,\infty)$ with $f_3(\infty) = 0$, we deduce that $f_3(x) < 0$ on $[1,\infty)$. As $f_3(1) = -1 + \ln\frac{115144327}{84718697} + \ln 2 = -2.78166... \times 10^{-6} < 0$.

Similarly, repeat the above approach, we can get $a_4(x) = 1 + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2+\frac{s_3}{x+t_3+\frac{s_4}{x+t_4}}}}$, where $s_4 = -\frac{792876605}{14692348944}$, $t_4 = \frac{2317657460602}{4805307952263}$, and

$$
f_4''(x) = \left(x \ln\left(1+\frac{1}{x}\right) - \ln a_4(x) - 1\right)'' = \frac{P_4(x)}{x^2(1+x)^2\Psi_1^2(4;x)\Psi_2^2(4;x)}.
$$

As $P_4(x) = -\Psi(8;x)$ has all coefficients negative, it results that $f_4(x)$ is strictly concave on $[1,\infty)$ with $f_4(\infty) = 0$, we deduce that $f_4(x) < 0$ on $[1,\infty)$. As $f_4(1) = -7.57111... \times 10^{-8} < 0$. This is the end of Theorem 1.

Remark 1. It is worth to point out that Theorem 1 provides some continued fraction estimates of $\left(1+\frac{1}{x}\right)^x$ by multiple-correction method. Similarly, repeating the above approach step by step, we can get more sharp estimates. But this may cause some computational increase, the details are omitted here.

3. Improved continued fraction approximation for $\left(1+\frac{1}{x}\right)^x$

In this section, we show the following improvement of Theorem 1.

(Step 1) The first-correction. We choose $b_1(x) = 1 + \frac{k_1}{x}$. Then letting the coefficient of x^3 of the molecule in the following fractions equal to zero, we have $k_1 = -\frac{1}{2}$ and

$$
\begin{aligned}
g_1''(x) &= \left(x \ln\left(1+\frac{1}{x}\right) - \ln b_1(x) - 1\right)'' \\
&= \frac{Q_1(x)}{x^2(1+x)^2(-1+2x)^2}.
\end{aligned}
$$

As $Q_1(x+1) = 11 + 23x + 11x^2$ has all coefficients positive, it results that $g_1(x)$ is strictly convex on $[1,\infty)$ with $g_1(\infty) = 0$, we deduce that $g_1(x) > 0$ on $[1,\infty)$.

(Step 2) The second-correction. We let $b_2(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0}$. Then letting the coefficient of x^{10}, x^9, x^8 of the molecule in the following fractions equal to zero, we have $k_2 = \frac{11}{24}$, $t_1 = \frac{21}{22}$, $t_0 = \frac{457}{29040}$ and

$$g_2''(x) = \left(x\ln\left(1+\frac{1}{x}\right) - \ln b_2(x) - 1\right)''$$
$$= \frac{Q_2(x)}{x^2(1+x)^2(-457+27720x+29040x^2)^2(457-2014x+26400x^2+58080x^3)^2}.$$

As $Q_2(x+1) = -1383118582943652269 - 8185800606735613017x - 20543391880395185989x^2 - 28305926099272779320x^3 - 23095991476134434880x^4 - 11144489439639916800x^5 - 2940717766115500800x^6 - 327003516322560000x^7$ has all coefficients negative, it results that $g_2(x)$ is strictly concave on $[1,\infty)$ with $g_2(\infty)=0$, we deduce that $g_2(x)<0$ on $[1,\infty)$.

(Step 3) The third-correction. We let $b_3(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0+\frac{u_1}{x+v_1}}$. Then letting the coefficient of x^{11}, x^{10} of the molecule in the following fractions equal to zero, we have $u_1 = \frac{5341}{638880}$, $v_1 = \frac{90156641}{148052520}$ and

$$g_3''(x) = \left(x\ln\left(1+\frac{1}{x}\right) - \ln b_3(x) - 1\right)''$$
$$= \frac{Q_3(x)}{x^2(1+x)^2\Psi_3^2(3;x)\Psi_3^2(4;x)}.$$

As $Q_3(x+1) = -\Psi(9;x)$ has all coefficients negative, it results that $g_3(x)$ is strictly concave on $[1,\infty)$ with $g_3(\infty)=0$, we deduce that $g_3(x)<0$ on $[1,\infty)$.

Similarly, repeat the above approach, we can get $b_4(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0+\frac{u_1}{x+\frac{v_1}{x+v_2}}}$, where $v_2 = -\frac{66627667815049}{1213447081344120}$, and

$$g_4''(x) = \left(x\ln\left(1+\frac{1}{x}\right) - \ln b_4(x) - 1\right)'' = \frac{Q_4(x)}{x^2(1+x)^2\Psi_4^2(4;x)\Psi_1^2(5;x)}.$$

As $Q_4(x+1) = \Psi(12;x)$ has all coefficients positive, it results that $g_4(x)$ is strictly convex on $[1,\infty)$ with $g_4(\infty)=0$, we deduce that $g_4(x)>0$ on $[1,\infty)$.

So we can get the following theorem:

Theorem 2. *For every real number $x \ge 1$, we have*

$$e\cdot b_4(x) < \left(1+\frac{1}{x}\right)^x < e\cdot b_3(x), \tag{3.1}$$

where

$$b_4(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0+\frac{u_1}{x+\frac{v_1}{x+v_2}}} = 1 - \frac{\frac{1}{2}}{x} + \frac{\frac{11}{24}}{x^2+\frac{21}{22}x+\frac{457}{29040}+\frac{\frac{5341}{638880}}{x+\frac{\frac{90156641}{148052520}}{x-\frac{66627667815049}{1213447081344120}}}},$$

and

$$b_3(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0+\frac{u_1}{x+v_1}} = 1 - \frac{\frac{1}{2}}{x} + \frac{\frac{11}{24}}{x^2+\frac{21}{22}x+\frac{457}{29040}+\frac{\frac{5341}{638880}}{x+\frac{90156641}{148052520}}}.$$

Remark 2. Different from the right-hand inequalities in Theorem 1, Theorem 2 gives the double-hand inequalities for $\left(1+\frac{1}{x}\right)^x$.

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Remark 3. In the proof of Theorem 2, if we let $c_3(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0} + \frac{k_3}{x^5+t_4x^4+t_3x^3+t_2x^2+t_1x+t_0}$ from Step 3, then letting the coefficient of $x^{27}, x^{26}...$ of the molecule in the following fractions equal to zero, we have $k_3 = -\frac{5341}{1393920}$, $b_4 = \frac{372802361}{148052520}, ...$ and

$$h_3''(x) = \left(x\ln\left(1+\frac{1}{x}\right) - \ln c_3(x) - 1\right)''$$
$$= \frac{R_3(x)}{x^2(1+x)^2(-457+27720x+29040x^2)^2\Psi_2^2(5;x)\Psi^2(8;x)}.$$

As $R_3(x+1) = -\Psi(21;x)$ has all coefficients negative, it results that $h_3(x)$ is strictly concave on $[1,\infty)$ with $h_3(\infty) = 0$, we deduce that $h_3(x) < 0$ on $[1,\infty)$. So we can get the similar inequalities for every real number $x \geq 1$,

$$\left(1+\frac{1}{x}\right)^x < e \cdot c_3(x),$$

where $c_3(x) = 1 + \frac{k_1}{x} + \frac{k_2}{x^2+t_1x+t_0} + \frac{k_3}{x^5+t_4x^4+t_3x^3+t_2x^2+t_1x+t_0}$.

Finally, we are convinced that the inequalities presented in Theorem 1 and Theorem 2 can be successfully used to obtain other new results, such as in the problem of proving the Keller's limit [6], but also in the problem of improving inequalities of Carleman's inequality [4,8,9].

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