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A proof of Wang–Kooij's conjectures for a cubic Liénard system with a cusp ☆

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ABSTRACT

In this paper the global dynamics of a cubic Liénard system with a cusp is studied to follow Wang and Kooij (1992) [13], who proved that the maximum number of limit cycles is 2 and stated two conjectures about the curves of the cuspidal loop bifurcation and the double limit cycle bifurcation. We give positive answers to those two conjectures and further properties of those bifurcation curves such as monotonicity and smoothness. Finally, associated with previous results we obtain the complete bifurcation diagram and all phase portraits, and demonstrate some numerical examples.

1. Introduction

The planar Liénard system, a representation in the two-dimensional form of the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, is one of the classical mechanical systems. The research of its dynamical behaviors can be found in many monographs (see, e.g., [4,8,14]) and many interesting results are given in journal papers (see, e.g., [5,7,10,12,13]). A cubic Liénard system

$$\begin{cases} \dot{x} = y + \mu_1 x^2 + x^3, \\ \dot{y} = \mu_2 x^2 - x^3 \end{cases} \tag{1.1}$$

has been introduced in [1,11,13] to study the viscous flow structures of a three-dimensional system near a planar wall. The origin O is the unique equilibrium when $\mu_2 = 0$. Besides O, system (1.1) has another equilibrium $E : (\mu_2, -\mu_1\mu_2^2 - \mu_2^3)$ and O is a cusp when $\mu_2 \neq 0$. Since the form of (1.1) is invariant under

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the change $(x, y, \mu_1, \mu_2) \to (-x, -y, -\mu_1, -\mu_2)$, we only need to consider (μ_1, μ_2) in $\mathcal{G} := \{(\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_1 \geq 0\}$.

In [13] the global dynamical analysis of system (1.1) is done for $(\mu_1, \mu_2) \in \mathcal{G}$. It is proved that the maximum number of limit cycles is 2. The existence of the cuspidal loop bifurcation curves is given as well as the existence of the double limit cycle bifurcation curves. Moreover, the uniqueness of the cuspidal loop bifurcation curves is also proved and the unique one is denoted by $\mu_1 = \varphi(\mu_2)$. However, there is no answer to the uniqueness of the double limit cycle bifurcation curves. On the other hand, as stated in [13, Theorem 5] $\varphi(\mu_2) \geq \psi_1(\mu_2) := \max\{\mu_1 : (\mu_1, \mu_2)$ lies on the double limit cycle bifurcation curves$\}$ for any fixed μ_2. But we do not know if there exists a point (μ_1, μ_2) lying on both the cuspidal loop bifurcation curve and one of the double limit cycle bifurcation curves, i.e., the location relation of the cuspidal loop bifurcation curve and the double limit cycle bifurcation curves is another unsolved question. Hence, in [13] there are two conjectures:

Conjecture (a) $\varphi(\mu_2) > \psi_1(\mu_2)$.
Conjecture (b) *The double limit cycle bifurcation curve is unique.*

Note that the bifurcation diagram, shown in [13, Figure 5], is given based on that both these conjectures have positive answers. As indicated in the proof of [13, Theorem 5], the stability of the cuspidal loop if it exists is equivalent to $\varphi(\mu_2) > \psi_1(\mu_2)$ because the semistability of the cuspidal loop means $\varphi(\mu_2) = \psi_1(\mu_2)$. Thus, **Conjecture (a)** is actually equivalent to conjecture that the cuspidal loop is stable.

Following the work of [13], we continue to study the global dynamical behaviors of system (1.1). Our main purpose is to answer **Conjectures (a)** and **(b)** so that the bifurcation diagram can be given strictly and to investigate the monotonicity of those bifurcation curves as well as their smoothness. To help the readers and keep the completeness of results, associated with some results of [13, Theorem 5] we give our main result in the following theorem, where large (resp. small) limit cycles mean periodic orbits surrounding two equilibria (resp. a single equilibrium).

Theorem 1.1. *As shown in Fig. 1, the global bifurcation diagram of (1.1) consists of the following bifurcation curves:*

(1) *generalized transcritical bifurcation curve* $GT = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = 0\}$;
(2) *Hopf bifurcation curve* $H = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_1 = -3\mu_2/2 > 0\}$ *for* E;
(3) *cuspidal loop bifurcation curve* $CL = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_1 = \varphi(\mu_2) > 0\}$;
(4) *double limit cycle bifurcation curve* $DL = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = \psi(\mu_1) < 0\}$;

where $\varphi \in C^\infty(\mathbb{R}^-, \mathbb{R}^+)$ *is decreasing,* $\psi \in C^0(\mathbb{R}^+, \mathbb{R}^-)$ *and*

$$-\mu_2 < \min\{\mu_1 : \mu_2 = \psi(\mu_1)\} \leq \max\{\mu_1 : \mu_2 = \psi(\mu_1)\} < \varphi(\mu_2) < -3\mu_2/2. \tag{1.2}$$

The complete classification of phase portraits is also given in Fig. 1, where

$$I := \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 > 0\};$$
$$II := \left\{(\mu_1, \mu_2) \in \mathcal{G} : 0 < \frac{-3\mu_2}{2} < \mu_1\right\};$$
$$III := \left\{(\mu_1, \mu_2) \in \mathcal{G} : \varphi(\mu_2) < \mu_1 < \frac{-3\mu_2}{2}\right\};$$
$$IV := \{(\mu_1, \mu_2) \in \mathcal{G} : \psi(\mu_1) < \mu_2 < \varphi^{-1}(\mu_1)\};$$

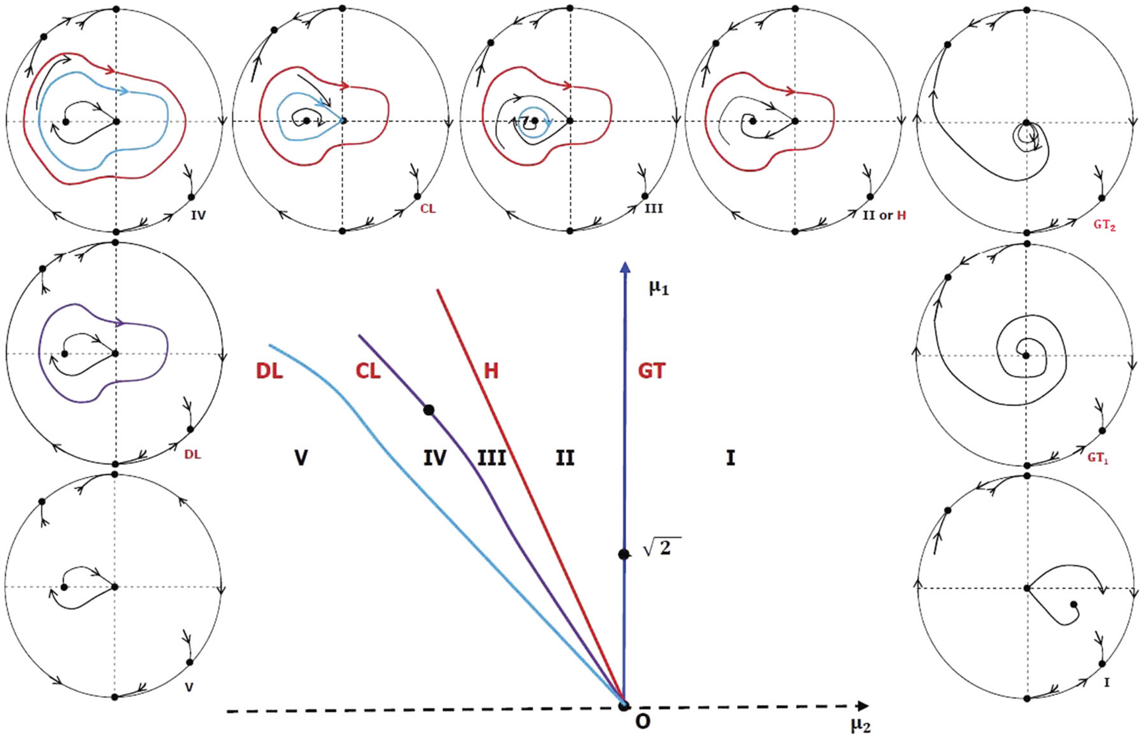


Fig. 1. The global bifurcation diagram and global phase portraits of system (1.1).

Table 1
Limit cycles and cuspidal loops of system (1.1).

Subsets of $\mathcal{G}$	Large limit cycles	Small limit cycles	Cuspidal loops
I, V, GT_1, GT_2	0	0	0
II, H	1, unstable	0	0
III	1, unstable	1, stable	0
IV	2, the inner one is stable, the outer one is unstable	0	0
CL	1, unstable	0	1, stable
DL	1, semistable	0	0

$$V := \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 < \psi(\mu_1)\};$$

$$GT_1 := \left\{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = 0,\ \mu_1 < \sqrt{2}\right\};$$

$$GT_2 := \left\{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = 0,\ \mu_1 \geq \sqrt{2}\right\}.$$

Moreover, all results about limit cycles and cuspidal loops are listed in Table 1*.*

By Theorem 1.1 we give positive answers to **Conjectures (a)** and **(b)** because (1.2) holds and the unique double limit cycle bifurcation curve is exactly determined by $\mu_2 = \psi(\mu_1)$. Remark that function $\psi(\mu_1)$ may be not monotonic. That is, the double limit cycle bifurcation curve may be as shown in Fig. 2, i.e., $\psi^{-1}(\mu_2)$ may be multi-valued. On the other hand, by the phase portraits shown in Fig. 1 a large limit cycle occurs when (μ_1, μ_2) crosses GT from I to II because of the change of the stability of E. Thus, in such sense GT can also be called *a generalized Hopf bifurcation curve*.

This paper is organized as follows. In Section 2, we recall some qualitative results of (1.1) given in [13] including the qualitative properties of equilibria and limit cycles, which are used in following sections. In Section 3, we prove the stability of the cuspidal loop if it exists, the monotonicity and C^∞ smoothness of the cuspidal loop bifurcation curve, the uniqueness and continuity of the double limit cycle bifurcation curve.

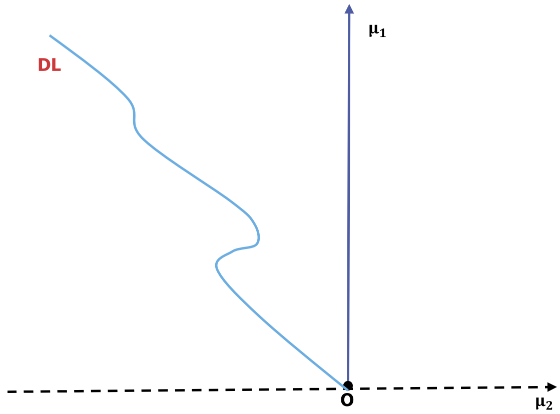


Fig. 2. The double limit cycle bifurcation curve is not monotonic.

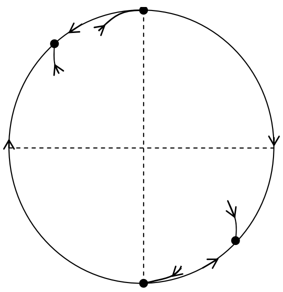


Fig. 3. Equilibria at infinity.

In Section 4, Theorem 1.1 is proved by those results given in Section 3 and in [13], and some numerical examples are given to show our theoretical results. In the last section, we give some concluding remarks about **Conjectures (a)**, **(b)** and the orbits connecting the cusp O and the equilibria at infinity.

2. Preliminaries

In order to investigate the global dynamics of (1.1), we need to know the qualitative properties of equilibria, the location and number of limit cycles and cuspidal loops. In [13, Section 3] the information about all equilibria is given as follows.

Proposition 2.1. *When $\mu_2 = 0$, the unique equilibrium O of (1.1) is an unstable degenerate focus if $\mu_1 < \sqrt{2}$, a degenerate equilibrium with an elliptic sector if $\mu_1 \geq \sqrt{2}$; When $\mu_2 \neq 0$, system (1.1) has two equilibria O and E, where O is a cusp and E is an antisaddle. The qualitative behavior of (1.1) at infinity is given in Fig. 3, where the equilibria lie on y-axis and the line $y = -x$.*

Actually, further analysis shows that E is either a focus or a node as given in Table 2 when $\mu_2 \neq 0$. In [13, Theorem 5], the results about limit cycles and cuspidal loops are given as follows.

Proposition 2.2. *When $\mu_2 \geq 0$, system (1.1) has no closed orbits. When $\mu_2 < 0$, there exist functions $\varphi(\mu_2)$, $\psi_1(\mu_2)$, $\psi_2(\mu_2)$ such that $-\mu_2 < \psi_2(\mu_2) \leq \psi_1(\mu_2) \leq \varphi(\mu_2)$, $-4\mu_2/3 \leq \varphi(\mu_2) < -3\mu_2/2$ and*

(1) *for $\mu_1 \geq -3\mu_2/2$ system (1.1) has no small limit cycles and exactly one large limit cycle, which is unstable;*

Table 2
The type of E.

$\mu_2 \neq 0$	$\mu_1 \geq 0$	Type of E
$\mu_2 > 0$	$\mu_1 > -3\mu_2/2 + 1$	unstable node
	$\mu_1 = -3\mu_2/2 + 1$	unstable proper node
	$-3\mu_2/2 < \mu_1 < -3\mu_2/2 + 1$	unstable focus
$\mu_2 < 0$	$\mu_1 > -3\mu_2/2 + 1$	stable node
	$\mu_1 = -3\mu_2/2 + 1$	stable proper node
	$-3\mu_2/2 < \mu_1 < -3\mu_2/2 + 1$	stable focus
	$\mu_1 = -3\mu_2/2$	stable weak focus of order one
	$-3\mu_2/2 - 1 < \mu_1 < -3\mu_2/2$	unstable focus
	$\mu_1 = -3\mu_2/2 - 1$	unstable proper node
	$\mu_1 < -3\mu_2/2 - 1$	unstable node

(2) *for $\varphi(\mu_2) < \mu_1 < -3\mu_2/2$ (resp. $\mu_1 = \varphi(\mu_2)$) system (1.1) has exactly one large limit cycle and exactly one small limit cycle (resp. one cuspidal loop), where the large limit cycle is unstable and the small one (resp. cuspidal loop) is stable;*

(3) *for $\psi_1(\mu_2) < \mu_1 < \varphi(\mu_2)$ system (1.1) has no small limit cycles and exactly two large limit cycles, where outer one is unstable and the inner one is stable;*

(4) *for $\psi_2(\mu_2) < \mu_1 < \psi_1(\mu_2)$ (resp. either $\mu_1 = \psi_1(\mu_2)$ or $\mu_1 = \psi_2(\mu_2)$) system (1.1) has no small limit cycles and at most two large limit cycles (resp. a unique large limit cycle, which is semistable);*

(5) *for $\mu_1 < \psi_2(\mu_2)$ system (1.1) has no limit cycles.*

Note that, as indicated in the proof of [13, Theorem 5], the stability of the cuspidal loop given in **(2)** of Proposition 2.2 (denoted by Theorem 5 in [13]) is based on the fact that **Conjecture (a)** has a positive answer, i.e., $\varphi(\mu_2) > \psi_1(\mu_2)$.

3. Bifurcations of cuspidal loops and double limit cycles

The existence and the uniqueness of the cuspidal loop bifurcation curve are given in Proposition 2.2. The existence of the double limit cycle bifurcation curves is also given in Proposition 2.2 but, there is no result about its uniqueness. In this section we study the monotonicity and smoothness of those curves, prove the stability of the cuspidal loop and the uniqueness of the double limit cycle bifurcation curve.

Lemma 3.1. *The cuspidal loop of (1.1) is stable if it exists.*

Proof. From Proposition 2.2, we obtain $0 < -4\mu_2/3 \leq \mu_1 < -3\mu_2/2$ if there exists a cuspidal loop. By $(x, y, t) \to (x + \mu_2, -y - \mu_1\mu_2^2 - \mu_2^3, -t)$, system (1.1) is transformed into

$$\begin{cases} \dot{x} = y + \mu_1\mu_2^2 + \mu_2^3 - \mu_1(x + \mu_2)^2 - (x + \mu_2)^3 =: y - F(x), \\ \dot{y} = -(\mu_2 + x)^2 x =: -g(x). \end{cases} \tag{3.1}$$

Clearly, $xg(x) > 0$, both $F(x), F'(x)$ have unique zeros respectively for all $x \in (-\infty, 0) \cup (0, -\mu_2)$ and $F'(-2\mu_1/3 - \mu_2) = 0$. Thus, (i) and (ii) of [3, Proposition 2.3] hold. Let

$$s := \hat{x}_1 + \hat{x}_2 + 2\mu_2, \tag{3.2}$$

where $\hat{x}_1 < 0 < \hat{x}_2 \leq -\mu_2$. If

$$F(\hat{x}_1) = F(\hat{x}_2), \tag{3.3}$$

then

$$0 \le (\hat{x}_1 + \mu_2)(\hat{x}_2 + \mu_2) = \mu_1 s + s^2, \tag{3.4}$$

implying $s \le -\mu_1$. If

$$\frac{g(\hat{x}_1)}{F'(\hat{x}_1)} = \frac{g(\hat{x}_2)}{F'(\hat{x}_2)}, \tag{3.5}$$

then $3s^2 + 5\mu_1 s - 2\mu_1\mu_2 = 0$, which has a unique root in $(-\infty, -\mu_1]$. Further, by (3.2) and (3.4) both $\hat{x}_1 + \hat{x}_2$ and $\hat{x}_1\hat{x}_2$ have unique values respectively, implying that equations (3.3) and (3.5) have at most one solution $(\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1 < 0 < \hat{x}_2 \le -\mu_2$. Thus, (iii) of [3, Proposition 2.3] holds. Straight computations show that

$$\frac{F(x)F'(x)}{g(x)} = \frac{[2\mu_1 + 3(x+\mu_2)][2\mu_1\mu_2 + 3\mu_2^2 + (\mu_1 + 3\mu_2)x + x^2]}{x + \mu_2},$$

which is decreasing for $x \in (-\infty, -2\mu_1/3 - \mu_2)$ because both $[2\mu_1 + 3(x+\mu_2)]/(x+\mu_2)$ and $2\mu_1\mu_2 + 3\mu_2^2 + (\mu_1 + 3\mu_2)x + x^2$ are decreasing and positive. Thus, (iv) of [3, Proposition 2.3] holds. Therefore, the cuspidal loop is stable if it exists by [3, Proposition 2.3]. □

As mentioned in the last paragraph of Section 2, the stability of the cuspidal loop given in Proposition 2.2 is based on the fact that **Conjecture (a)** has a positive answer. If **Conjecture (a)** has a negative answer, then the cuspidal loop is semi-stable. In Lemma 3.1 we prove the stability of the cuspidal loop strictly. Thus, Lemma 3.1 actually gives a positive answer to **Conjecture (a)**.

By Proposition 2.2, the cuspidal loop and limit cycles do not exist when $\mu_2 \ge 0$. Thus, in the following we only consider the case that $\mu_2 < 0$. By the global homeomorphism transformation and time rescaling

$$x \to -\mu_2 x, \quad y \to \mu_2^2 y - \mu_1\mu_2^2 x^2 + \mu_2^3 x^3, \quad t \to -\frac{t}{\mu_2}, \tag{3.6}$$

system (1.1) can be rewritten as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^2 - x^3 + (2\mu_1 x - 3\mu_2 x^2)y. \end{cases} \tag{3.7}$$

Transformation (3.6) changes the curve $y = F(x)$ of (1.1) to the x-axis of (3.7). As indicated in [14, Chapter 4], the phase portrait of (1.1) is homeomorphic globally with that of (3.7). It is easy to check that (3.7) has exactly two equilibria at $(-1, 0)$ and $(0, 0)$. Here $(0, 0)$ is a cusp. In the following lemma, for (3.7) we give the changes of the invariant manifolds of the cusp and orbits crossing through y-axis with respect to μ_1, μ_2.

Lemma 3.2. *For fixed μ_1 (resp. μ_2) in (3.7), x_A decreases continuously and x_B increases continuously as μ_2 (resp. μ_1) increases, where x_A, x_B are the abscissas of points A, B respectively as shown in Fig. 4.*

Proof. The idea of the proof follows [2] but, we still write the whole proof here for its completeness. Assume that μ_1 is fixed. Let W_0^s and W_0^u be the stable manifold and the unstable one of system (3.7) at the cusp $(0, 0)$, respectively; W_ϵ^s and W_ϵ^u be the stable manifold and the unstable one of system $(3.7)|_{\mu_2 \to \mu_2 + \epsilon}$ (a perturbation of (3.7)) at the cusp $(0, 0)$, respectively, where $0 < |\epsilon| \ll 1$; $x_A(\epsilon), x_B(\epsilon)$ be the abscissas of intersection points of W_ϵ^s and W_ϵ^u on the negative x-axis as x_A, x_B, respectively. Denote the points on W_0^s, W_ϵ^s for $x \in (\delta, 0)$ as $(x, y_0^s(x))$, $(x, y_\epsilon^s(x))$ respectively, where $\delta := \max(x_A, x_A(\epsilon))$. Let $z_1(x)$ be the difference between $y_\epsilon^s(x)$ and $y_0^s(x)$, i.e., $z_1(x) := y_\epsilon^s(x) - y_0^s(x)$. Clearly, $z_1(0) = 0$. For (3.7) and all x in $(\delta, 0)$,

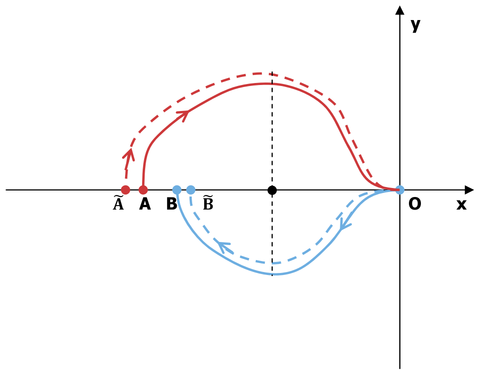


Fig. 4. The stable and unstable manifolds depend on μ_1 and μ_2.

$$\begin{aligned} z_1(x) &= z_1(x) - z_1(0) = \{y_\epsilon^s(\tau) - y_0^s(\tau)\}\,|_{\tau=0}^{\tau=x} \\ &= \int_0^x \left\{ \frac{-\tau^2 - \tau^3 + \left(2\mu_1\tau - 3(\mu_2+\epsilon)\tau^2\right) y_\epsilon^s(\tau)}{y_\epsilon^s(\tau)} + \frac{\tau^2 + \tau^3 - \left(2\mu_1\tau - 3\mu_2\tau^2\right) y_0^s(\tau)}{y_0^s(\tau)} \right\} d\tau \\ &= H_1(x) + \int_0^x z_1(\tau)H_2(\tau)d\tau, \end{aligned} \tag{3.8}$$

where $H_1(x) := -\epsilon x^3$, $H_2(\tau) := (\tau^2+\tau^3)/(y_0^s(\tau)y_\epsilon^s(\tau))$. It follows from (3.8) that

$$H_2(x)z_1(x) = H_1(x)H_2(x) + H_2(x)\int_0^x z_1(\tau)H_2(\tau)d\tau.$$

Then,

$$\frac{dH_3(x)}{dx} - H_2(x)H_3(x) = H_1(x)H_2(x), \tag{3.9}$$

where $H_3(x) := \int_0^x z_1(\tau)H_2(\tau)d\tau$. Solving H_3 from (3.9) we obtain

$$H_3(x) = \int_0^x H_1(\tau)H_2(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau. \tag{3.10}$$

Hence, by (3.8) and (3.10),

$$\begin{aligned} z_1(x) &= H_1(x) + \int_0^x H_1(\tau)H_2(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau \\ &= H_1(0)\exp\left\{\int_0^x H_2(\eta)d\eta\right\} + \int_0^x H_1'(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau \\ &= -3\epsilon\int_0^x \tau^2\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau \end{aligned}$$

$$= \begin{cases} > 0, & \text{if } \epsilon > 0, \\ < 0, & \text{if } \epsilon < 0, \end{cases} \tag{3.11}$$

implying that W_ϵ^s lies above W_0^s when $\epsilon > 0$. Thus, for (3.7) x_A decreases as μ_2 increases.

Denote the points on W_0^u, W_ϵ^u for $x \in (\tilde{\delta}, 0)$ as $(x, y_0^u(x))$, $(x, y_\epsilon^u(x))$ respectively, where $\tilde{\delta} := \max(x_B, x_B(\epsilon))$. Let $z_2(x)$ be the difference between $y_\epsilon^u(x)$ and $y_0^u(x)$, i.e., $z_2(x) := y_\epsilon^u(x) - y_0^u(x)$. Similarly to $z_1(x)$, we obtain

$$z_2(x) = -3\epsilon \int_0^x \tau^2 \exp\left\{ \int_\tau^x \hat{H}_2(\eta) d\eta \right\} d\tau = \begin{cases} > 0, & \text{if } \epsilon > 0, \\ < 0, & \text{if } \epsilon < 0 \end{cases} \tag{3.12}$$

for $x \in (\tilde{\delta}, 0)$, where $\hat{H}_2(x) := (x^2 + x^3)/(y_0^u(x) y_\epsilon^u(x))$. Thus, x_B increases as μ_2 increases.

We can similarly study how x_A and x_B change with respect to μ_1 for fixed μ_2. Let $\tilde{W}_\gamma^s$ and $\tilde{W}_\gamma^u$ be the stable and unstable manifolds of system $(3.7)|_{\mu_1 \to \mu_1 + \gamma}$ at the cusp $(0, 0)$, respectively, where $0 < |\gamma| \ll 1$. Denote the point on $\tilde{W}_\gamma^s$ for $x \in (\hat{\delta}, 0)$ by $(x, y_\gamma^s(x))$, where $\hat{\delta} := \max(x_A, x_A(\gamma))$. Let $\tilde{z}_1(x) := y_\gamma^s(x) - y_0^s(x)$ be the difference between $y_\gamma^s(x)$ and $y_0^s(x)$. Similarly to $z_1(x)$ given in (3.11), we obtain

$$\tilde{z}_1(x) = 2\gamma \int_0^x s \exp\left\{ \int_s^x \tilde{H}_2(\tau) d\tau \right\} ds = \begin{cases} > 0, & \text{if } \gamma > 0, \\ < 0, & \text{if } \gamma < 0, \end{cases} \tag{3.13}$$

where $\tilde{H}_2(x) := (x^2 + x^3)/(y_0^s(x) y_\gamma^s(x))$. Thus, x_A decreases as μ_1 increases. Denote the point on $\tilde{W}_\gamma^u$ for $x \in (\bar{\delta}, 0)$ by $(x, y_\gamma^u(x))$, where $\bar{\delta} := \max(x_A, x_A(\gamma))$. Let $\tilde{z}_2(x) := y_\gamma^u(x) - y_0^u(x)$ be the difference between $y_\gamma^u(x)$ and $y_0^u(x)$. Similarly to (3.13), we obtain

$$\tilde{z}_2(x) = 2\gamma \int_0^x s \exp\left\{ \int_s^x \bar{H}_2(\tau) d\tau \right\} ds = \begin{cases} > 0, & \text{if } \gamma > 0, \\ < 0, & \text{if } \gamma < 0 \end{cases} \tag{3.14}$$

for $x \in (\bar{\delta}, 0)$. Here $\bar{H}_2(x) := (x^2 + x^3)/(y_0^u(x) y_\gamma^u(x))$. Thus, x_B increases as μ_1 increases. □

Assume that points P, Q lie on the positive y-axis and the negative y-axis for system (3.7) respectively and points C, D are the first intersection points on the positive x-axis of the orbits starting from P, Q as $t \to +\infty$ and $t \to -\infty$ respectively. Let x_C, x_D be the abscissas of C, D respectively. We remark that for fixed μ_1 the value of x_C (resp. x_D) decreases (resp. increases) continuously as μ_2 increases as shown in Fig. 5(a) by the same method used in the proof of Lemma 3.2 or by the Comparison Theorem (see [9, Corollary 6.3 of Chapter 1]). The changes of intersection points on the negative x-axis also can be obtained as shown in Fig. 5(a). Similarly, we get their changes depending on μ_1 as shown in Fig. 5(b) when μ_2 is fixed.

Proposition 2.2 states the existence of the cuspidal loop bifurcation curve, determined by $\mu_1 = \varphi(\mu_2)$. In the following lemma we consider its monotonicity and smoothness.

Lemma 3.3. *The cuspidal loop bifurcation curve $\mu_1 = \varphi(\mu_2)$ is decreasing and C^∞.*

Proof. To prove the smoothness of $\varphi(\mu_2)$, we consider the equivalent system (3.7) of (1.1) when $\mu_1 = \varphi(\mu_2)$, i.e., there is a cuspidal loop. Let $(x_1, 0)$ be the intersection point of the cuspidal loop on the negative x-axis. Taking $(\mu_1, \mu_2) \to (\mu_1, \mu_2 + \epsilon)$, we assume that $x_1 - \delta_1$ and $x_1 + \delta_2$ are abscissas of the closest intersection points near $(x_1, 0)$ of the stable manifold and the unstable one on the negative x-axis. Furthermore, taking

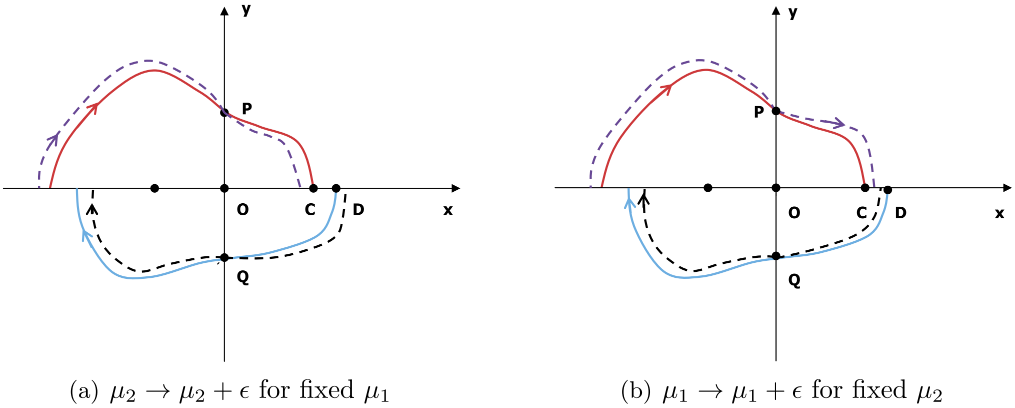


Fig. 5. Orbit changing for perturbations of system (3.7).

$(\mu_1, \mu_2 + \epsilon) \to (\mu_1 - \gamma, \mu_2 + \epsilon)$ we assume that $x_1 - \delta_1 + \delta_3$ and $x_1 + \delta_2 - \delta_4$ are abscissas of the closest intersection points near $(x_1, 0)$ of the stable manifold and the unstable one on the negative x-axis, and $\mu_1 - \gamma = \varphi(\mu_2 + \epsilon)$. We firstly consider that $\epsilon > 0$. Consequently, $\gamma > 0$ by Lemma 3.2. From (3.11)–(3.14), we obtain

$$\begin{aligned}
z_1(x) &= K_1(x, \mu_1, \mu_2)\epsilon + o(\epsilon), \\
z_2(x) &= K_2(x, \mu_1, \mu_2)\epsilon + o(\epsilon), \\
\tilde{z}_1(x) &= (K_3(x, \mu_1, \mu_2) + O(\epsilon))\,\gamma + o(\gamma), \\
\tilde{z}_2(x) &= (K_4(x, \mu_1, \mu_2) + O(\epsilon))\,\gamma + o(\gamma),
\end{aligned}$$

where

$$\begin{aligned}
K_1(x, \mu_1, \mu_2) &:= -3\int_0^x \tau^2 \exp\left\{\int_\tau^x H_2^*(\eta)d\eta\right\} d\tau, \\
K_2(x, \mu_1, \mu_2) &:= -3\int_0^x \tau^2 \exp\left\{\int_\tau^x H_2^{**}(\eta)d\eta\right\} d\tau, \\
K_3(x, \mu_1, \mu_2) &:= 2\int_0^x s\exp\left\{\int_s^x H_2^*(\tau)d\tau\right\} ds, \\
K_4(x, \mu_1, \mu_2) &:= 2\int_0^x s\exp\left\{\int_s^x H_2^{**}(\tau)d\tau\right\} ds,
\end{aligned} \tag{3.15}$$

$H_2^*(x) := (x^2 + x^3)/(y_0^s(x))^2, H_2^{**}(x) := (x^2 + x^3)/(y_0^u(x))^2$ and y_0^s, y_0^u correspond to the orbits on the stable manifold and on the unstable one of (3.7) respectively.

In the following we compute δ_1. It is easy to obtain

$$\delta_1 = \int_{x_1-\delta_1}^{x_1} dx = \int_0^{z_1(x_1)} \frac{dx}{dy} dy = \int_0^{z_1(x_1)} \frac{y}{\bar{g}(x) - \bar{f}(x)y} dy,$$

where $\bar{g}(x) := -x^2 - x^3, \bar{f}(x) := -2\mu_1 x + 3\mu_2 x^2$. Then

$$\delta_1 = \int_0^{z_1(x_1)} \frac{y}{\bar{g}(x_1-\delta_1)+O(y)-\bar{f}(x_1-\delta_1)y-o(y)}dy = \left[\frac{y^2}{2\bar{g}(x_1-\delta_1)}+\Gamma_1(y)\right]\Bigg|_{y=0}^{y=z_1(x_1)},$$

where $\Gamma_1(y)=o(y^2)$. Thus,

$$\delta_1 = \frac{z_1^2(x_1)}{2\bar{g}(x_1-\delta_1)}+\Gamma_1(z_1(x_1)) = \frac{K_1^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1-\delta_1)}\epsilon^2+o(\epsilon^2). \tag{3.16}$$

Similarly to δ_1, we compute δ_i $(i=2,3,4)$ and obtain

$$\begin{aligned}
\delta_2 &= \frac{K_2^2(x_1+\delta_2,\mu_1,\mu_2)}{2\bar{g}(x_1)}\epsilon^2+o(\epsilon^2),\\
\delta_3 &= \frac{K_3^2(x_1-\delta_1+\delta_3,\mu_1,\mu_2+\epsilon)}{2\bar{g}(x_1-\delta_1)}\gamma^2+o(\gamma^2),\\
\delta_4 &= \frac{K_4^2(x_1+\delta_2,\mu_1,\mu_2+\epsilon)}{2\bar{g}(x_1+\delta_2-\delta_4)}\gamma^2+o(\gamma^2).
\end{aligned} \tag{3.17}$$

From $\varphi(\mu_2+\epsilon)=\mu_1-\gamma$, we obtain $\delta_1-\delta_3=-\delta_2+\delta_4$. Then, from (3.16) and (3.17)

$$\begin{aligned}
&\left(\frac{K_1^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1-\delta_1)}+\frac{K_2^2(x_1+\delta_2,\mu_1,\mu_2)}{2\bar{g}(x_1)}\right)\epsilon^2+o(\epsilon^2)\\
&=\left(\frac{K_3^2(x_1-\delta_1+\delta_3,\mu_1,\mu_2+\epsilon)}{2\bar{g}(x_1-\delta_1)}+\frac{K_4^2(x_1+\delta_2,\mu_1,\mu_2+\epsilon)}{2\bar{g}(x_1+\delta_2-\delta_4)}\right)\gamma^2+o(\gamma^2),
\end{aligned}$$

which implies

$$\begin{aligned}
\lim_{\epsilon\to0^+}\frac{\varphi(\mu_2+\epsilon)-\varphi(\mu_2)}{\epsilon} &= \lim_{\epsilon\to0^+}\frac{-\gamma}{\epsilon}\\
&= -\lim_{\epsilon\to0^+}\frac{\sqrt{\frac{K_1^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1)}+\frac{K_2^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1)}}}{\sqrt{\frac{K_3^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1)}+\frac{K_4^2(x_1,\mu_1,\mu_2)}{2\bar{g}(x_1)}}}+\hat{\Gamma}(\epsilon)\\
&= -\sqrt{\frac{K_1^2(x_1,\mu_1,\mu_2)+K_2^2(x_1,\mu_1,\mu_2)}{K_3^2(x_1,\mu_1,\mu_2)+K_4^2(x_1,\mu_1,\mu_2)}}.
\end{aligned} \tag{3.18}$$

Here $\hat{\Gamma}(\epsilon)\to0$ as $\epsilon\to0$ and all K_i's are given in (3.15). Note that $\bar{g}(x_1)>0$, x_1 is analytic in μ_1,μ_2 and $\mu_1=\varphi(\mu_2)$. It is easy to check that the expression of the right-hand side of (3.18) is continuous in μ_2. Similarly to (3.18),

$$\lim_{\epsilon\to0^-}\frac{\varphi(\mu_2+\epsilon)-\varphi(\mu_2)}{\epsilon} = -\sqrt{\frac{K_1^2(x_1,\mu_1,\mu_2)+K_2^2(x_1,\mu_1,\mu_2)}{K_3^2(x_1,\mu_1,\mu_2)+K_4^2(x_1,\mu_1,\mu_2)}} = \lim_{\epsilon\to0^+}\frac{\varphi(\mu_2+\epsilon)-\varphi(\mu_2)}{\epsilon}.$$

Thus, $\varphi(\mu_2)$ is C^1 and decreasing. Continuing this process, we find that $\varphi(\mu_2)$ and $\varphi'(\mu_2)$ have the smoothness, implying the C^∞ smoothness of φ. □

The results for $\mu_1>\varphi(\mu_2)$ given in Proposition 2.2 are proved in [13, Theorem 5]. As mentioned in the last paragraph of Section 2, when $\mu_1=\varphi(\mu_2)$, the stability of the cuspidal loop given in Proposition 2.2 is based on the fact that **Conjecture (a)** has a positive answer and is proved strictly in Lemma 3.1. However the discussion for the case $\mu_1<\varphi(\mu_2)$ in Proposition 2.2 is not complete because it is unknown if there exist two double limit cycle bifurcation curves or a unique one, i.e., no answer to **Conjecture (b)**. In order to study the dynamical behaviors clearly of system (1.1) for the case $\mu_1 < \varphi(\mu_2)$ we give the following lemma, while a positive answer to **Conjecture (b)** is given.

Lemma 3.4. *There exists a continuous function $\psi(\mu_1)$ for $\mu_1 > 0$ such that $-\mu_1 < \psi(\mu_1) < \varphi^{-1}(\mu_1)$ and*

(1) *for $\psi(\mu_1) < \mu_2 < \varphi^{-1}(\mu_1)$ system (1.1) has no small limit cycles and exactly two large limit cycles, where the outer one is unstable and the inner one is stable;*
(2) *for $\mu_2 = \psi(\mu_1)$ system (1.1) has a unique limit cycle, which is large and semistable;*
(3) *for $\mu_2 < \psi(\mu_1)$ system (1.1) has no limit cycles.*

Proof. In this proof we still consider the equivalent system (3.7) of (1.1) and assume that $\mu_1 = \varphi(\mu_2)$. Having Lemma 3.1, the result for $\mu_1 = \varphi(\mu_2)$ given in Proposition 2.2 holds strictly, i.e., (3.7) has exactly one large limit cycle and a cuspidal loop. Moreover, the large limit cycle is unstable and the cuspidal loop is stable. By the orbit changing indicated after the proof of Lemma 3.2 and Proposition 2.2, system $(3.7)|_{\mu_2 \to \varphi^{-1}(\mu_1)-\epsilon}$ ($\epsilon > 0$) has exactly one large unstable limit cycle Γ_1 near the original one and exactly one large stable limit cycle Γ_2 near the original cuspidal loop. Continuing this process, i.e., μ_2 continues to decrease, the distance between the two intersection points of the outer large limit cycle on x-axis is decreasing and the distance for the inner one is increasing by the orbit changing indicated after the proof of Lemma 3.2. On the other hand, there are no limit cycles when $\mu_2 = -\mu_1$ by Proposition 2.2 and $\varphi^{-1}(\mu_1) > -\mu_1$ by $-\mu_2 < \varphi(\mu_2)$ given in Proposition 2.2. Thus, there exist values of μ_2 in $(-\mu_1, \varphi^{-1}(\mu_1))$ such that there is exactly one large limit cycle, which is semi-stable. We claim the uniqueness of these values. In fact, let $\psi(\mu_1)$ be the maximum of these values, i.e., there are exactly two large limit cycles when $\mu_2 \in (\psi(\mu_1), \varphi^{-1}(\mu_1))$. Define the Poincaré return map $\Pi(\rho, \mu_1, \mu_2)$ for $\rho > 0$ by the orbit passing through $(\rho, 0)$ and its successor function $h(\rho, \mu_1, \mu_2) := \Pi(\rho, \mu_1, \mu_2) - \rho$. Thus, $h(\rho, \mu_1, \psi(\mu_1)) \geq 0$ because of the semi-stability of the unique large limit cycle and there exists a unique ρ^* such that $h(\rho^*, \mu_1, \psi(\mu_1)) = 0$, which corresponds to the unique large limit cycle. Clearly, $h(\rho, \mu_1, \mu_2)$ is decreasing strictly for $\mu_2 \in (-\mu_1, \varphi^{-1}(\mu_1))$ as a function of μ_2 by the orbit changing indicated after the proof of Lemma 3.2. Then, $h(\rho, \mu_1, \mu_2) > h(\rho, \mu_1, \psi(\mu_1)) \geq 0$ when $\mu_2 < \psi(\mu_1)$. Thus, there are no large limit cycles when $\mu_2 < \psi(\mu_1)$, implying $\psi(\mu_1)$ is the unique value for μ_2 such that there is exactly one large limit cycle. The results about large limit cycles of this lemma are proved.

Now we prove the continuity of $\psi(\mu_1)$. As mentioned in last paragraph, $h(\rho, \mu_1, \psi(\mu_1)) \geq 0$ and there exists a unique ρ^* such that $h(\rho^*, \mu_1, \psi(\mu_1)) = 0$. By the orbit changing depending on μ_1 indicated after the proof of Lemma 3.2, $h(\rho, \mu_1 + \gamma, \psi(\mu_1)) \geq \delta$, where $|\gamma|, |\delta|$ are sufficiently small and $\delta \to 0$ as $\gamma \to 0$. Further, for given γ there exists a unique ϵ such that $h(\rho, \mu_1 + \gamma, \psi(\mu_1) + \epsilon) \geq 0$ and $h(\rho^{**}, \mu_1 + \gamma, \psi(\mu_1) + \epsilon) = 0$ by the monotonic change of orbits depending on μ_2, where ρ^{**} is unique and near ρ^*, $\epsilon \to 0$ as $\gamma \to 0$. Thus, $\psi(\mu_1 + \gamma) = \psi(\mu_1) + \epsilon$, implying the continuity of ψ.

To finish this proof, we only need to prove that there are no small limit cycles for $\mu_2 < \varphi^{-1}(\mu_1)$. Since $\varphi^{-1}(\mu_1) < -2\mu_1/3$ by $\varphi(\mu_2) < -3\mu_2/2$ given in Proposition 2.2, equilibrium E is unstable by Table 2. On the other hand, the orbits between the stable manifold and the unstable one approach either a large limit cycle or an equilibrium at infinity when $t \to +\infty$ by the stability of the equilibria at infinity and the internal stability of the inner large limit cycle if it exists. Thus, the number of small limit cycles is even, where a n-multiple limit cycle is considered as n limit cycles. Therefore, there are no small limit cycles because the number of small limit cycles is at most 1 by [13, Theorem 5]. □

In Lemma 3.4 all dynamical behaviors are analyzed for the case that $\mu_1 < \varphi(\mu_2)$. On the other hand, from Lemma 3.4 there exists a semi-stable limit cycle if and only if $\mu_2 = \psi(\mu_1)$. Thus, the double limit cycle bifurcation curve is unique, i.e., the curve determined by $\mu_2 = \psi(\mu_1)$. That is, **Conjecture (b)** has a positive answer.

4. Proof of Theorem 1.1 and numerical examples

In this section, we prove Theorem 1.1 and demonstrate some numerical examples.

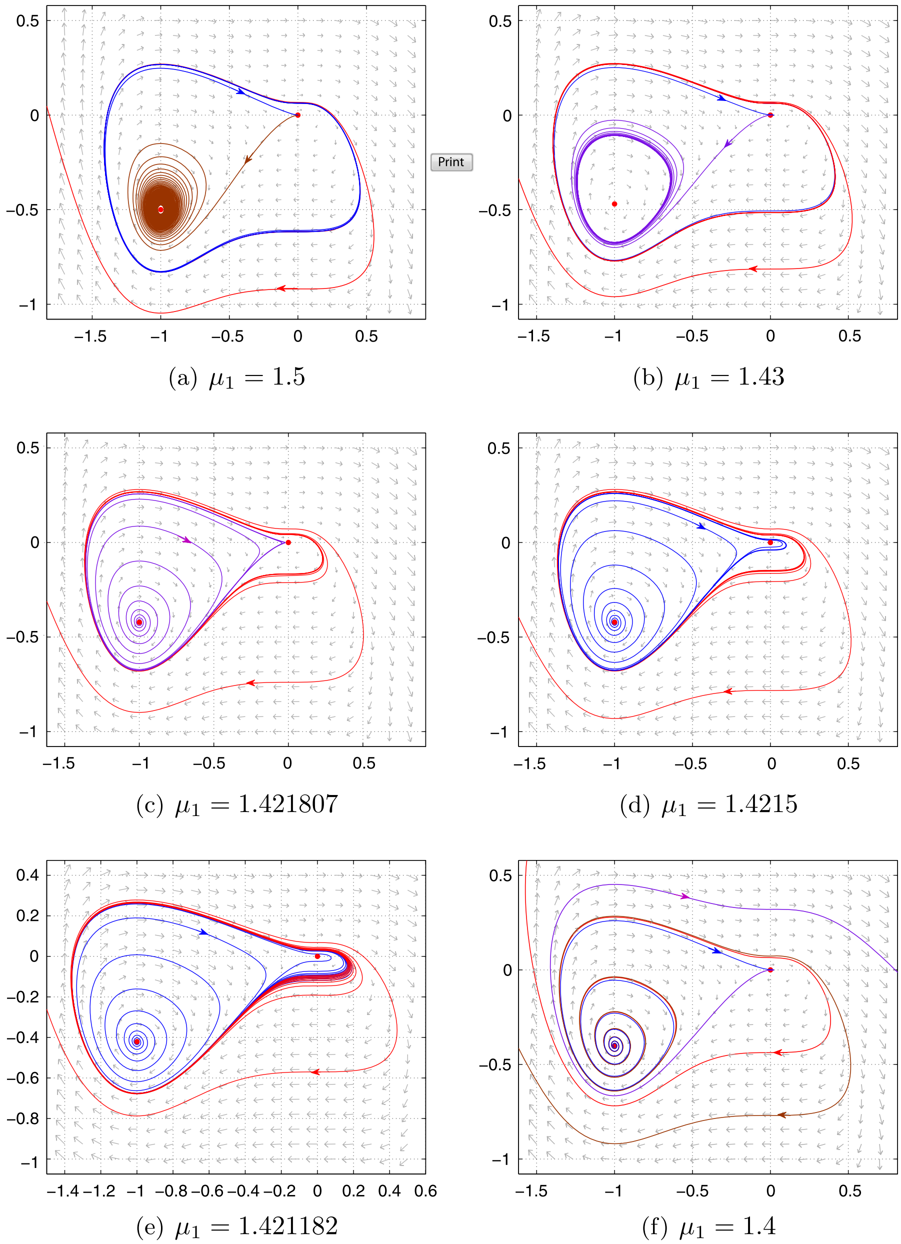
Proof of Theorem 1.1. All information about the equilibria of system (1.1) is obtained in Proposition 2.1 and Table 2, from which we see that a generalized transcritical bifurcation happens when μ_2 crosses 0. Thus, Conclusion **(1)** is proved. By Proposition 2.1 the stable weak focus E of order 1 becomes an unstable rough focus as μ_1 changes from $-3\mu_2/2$ to $-3\mu_2/2 - \epsilon$, where $\epsilon > 0$ is sufficiently small. Thus, a Hopf bifurcation happens in this process, i.e., H is the Hopf bifurcation curve for E and Conclusion **(2)** is proved. Conclusions **(3)** and **(4)** follow directly from Lemmas 3.3 and 3.4. By $-\mu_1 < \psi(\mu_1) < \varphi^{-1}(\mu_1)$ given in Lemma 3.4 and $\varphi(\mu_2) < -3\mu_2/2$ given in Proposition 2.2, we obtain (1.2). All phase portraits given in this theorem are obtained by **(1)**, **(2)** of Proposition 2.2 and Lemma 3.1 for $\mu_1 \geq \varphi(\mu_2)$ and by Lemma 3.4 for $\mu_1 < \varphi(\mu_2)$. □

In the following we give some numerical examples. Fixing $\mu_2 = -1$ and taking different values for μ_1, we draw the corresponding phase portraits of system (1.1) by Matlab and show them in Fig. 6. For $(\mu_1, \mu_2) = (1.5, -1) \in H$ there is a unique limit cycle and it is unstable and large as shown in Fig. 6(a), which is consistent with Fig. 1(II or H). For $(\mu_1, \mu_2) = (1.43, -1)$ there are exactly one small limit cycle and one large limit cycle as shown in Fig. 6(b), which is consistent with Fig. 1(III). Thus, $(1.43, -1) \in III$. For $(\mu_1, \mu_2) = (1.421807, -1)$ there are exactly one large limit cycle and one small limit cycle as shown in Fig. 6(c), which is very similar to Fig. 1(CL). Moreover, the small one crosses a small neighborhood of the cusp. Thus, $\varphi(-1) \approx 1.421807$. For $(\mu_1, \mu_2) = (1.4215, -1)$ there are exactly two large limit cycles as shown in Fig. 6(d), which is consistent with Fig. 1(IV). Thus, $(1.4215, -1) \in IV$. For $(\mu_1, \mu_2) = (1.421182, -1)$ there are exactly two large limit cycles and they are very close to each other as shown in Fig. 6(e), which is very similar to Fig. 1(DL). Thus, $\psi(1.421182) \approx -1$. For $(\mu_1, \mu_2) = (1.4, -1)$ there are no limit cycles as shown in Fig. 6(f), which is consistent with Fig. 1(V). Thus, $(1.4, -1) \in V$.

5. Concluding remarks

As mentioned in Section 1, **Conjecture (a)** is equivalent to the stability of the cuspidal loop. A positive answer is given to **Conjecture (a)** by Lemma 3.1. In the Remark given at the end of [13, Section 4], it is stated that **Conjecture (b)** is equivalent to $\psi_1(\mu_2) = \psi_2(\mu_2)$, where $\psi_1(\mu_2)$ is defined in Section 1 as the maximum of $\{\mu_1 : \ (\mu_1, \mu_2)$ lies on the double limit cycle bifurcation curves$\}$ for any fixed μ_2 and $\psi_2(\mu_2)$ being the minimum of this set. However, this equivalence is incorrect because the condition $\psi_1(\mu_2) = \psi_2(\mu_2)$ is sufficient, but not necessary, to the uniqueness of the double limit cycle bifurcation curves. Actually, by Theorem 1.1 there exists a unique double limit cycle bifurcation curve, which is exactly the graph of function $\psi(\mu_1)$. That is, we determine the unique double limit cycle bifurcation curve via a function of μ_1, not a function of μ_2 as tried in [13]. On the other hand, as mentioned in Section 1 the monotonicity of $\psi(\mu_1)$ is still open, which actually is equivalent to $\psi_1(\mu_2) = \psi_2(\mu_2)$. By [13, p. 1617], a possible way to prove this monotonicity is to prove system (3.1) (system (5) in [13]) has no limit cycles intersecting the vertical line $x = -2\mu_2$ because the vector field of system (3.1) is rotated with respect to μ_1 in the strip $-\infty < x < -2\mu_2$.

By Proposition 2.1 there are four equilibria at infinity. Let A^+ (resp. A^-) be the equilibria at infinity on the positive (resp. negative) y-axis and B^+ (resp. B^-) be the equilibria at infinity in the second (resp. forth) quadrant. Each phase portrait shown in Fig. 1(I,V) means four possibilities because of the connection between the unstable manifold of the cusp O and the equilibria at infinity. It is difficult to split region I



(a) $\mu_1 = 1.5$ (b) $\mu_1 = 1.43$

(c) $\mu_1 = 1.421807$ (d) $\mu_1 = 1.4215$

(e) $\mu_1 = 1.421182$ (f) $\mu_1 = 1.4$

Fig. 6. Phase portraits for different values of μ_1 and fixed $\mu_2 = -1$.

or V into four subsets such that the connection is fixed for all (μ_1, μ_2) in each one of them. The following examples show that different possibilities of those four ones can happen even for different points (μ_1, μ_2) on a line lying in V. By the P4 program (see [6, Chapter XI]), the ω-limit of the unstable manifold is B^+, B^-, A^- when $(\mu_1, \mu_2) = (0.5, -0.5), (0.6, -0.6), (0.555727, -0.555727)$ respectively.

Recently, in [3] the global dynamics of

$$\begin{cases} \dot{x} = y - (ax + 2bx^2 + bx^3), \\ \dot{y} = -x(x+1)^2 \end{cases} \tag{5.1}$$

is investigated. By $(x, y, t) \to (-x-1, y, -t)$, system (5.1)$|_{a=b>0}$ is rewritten as

$$\begin{cases} \dot{x} = y + b(-1+x)x^2, \\ \dot{y} = -x^2(x+1). \end{cases} \tag{5.2}$$

On the other hand, system (1.1)$|_{\mu_1=-\mu_2=b>0}$ is also transformed into (5.2) by $(x, y, t) \to (\mu_1 x, \mu_2^2 y, t/\mu_1)$. By Theorem 1.1 system (5.2) has a phase portrait, which is topologically equivalent to the phase portrait shown in Fig. 1(V). This is consistent with the phase portrait given in [3, Figure 2].

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