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Holomorphically projective mappings between generalized hyperbolic Kähler spaces ☆

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ABSTRACT

We define generalized hyperbolic Kähler spaces as a particular case of Eisenhart's generalized Riemannian spaces, with some additional conditions related to the almost product structure. Since a generalized hyperbolic Kähler space is equipped with a non-symmetric basic tensor, it admits five linearly independent curvature tensors. Some properties of these curvature tensors as well as those of the corresponding Ricci tensors are established. Also, we consider holomorphically projective mappings, as well as equitorsion holomorphically projective mappings between generalized hyperbolic Kähler spaces and find some invariant geometric objects with respect to these mappings.

1. Introduction

An n-dimensional manifold M is called a *locally product space* if it admits a separating coordinate system (see [25]). This means that the manifold M is covered by a system of coordinate neighbourhoods such that in any intersection of two coordinate neighbourhoods (U, u^h) and $(U', u^{h'})$ we have

$$u^{a'} = u^{a'}(u^a), \quad u^{x'} = u^{x'}(u^x), \quad \det|\partial_a u^{a'}| \neq 0, \quad \det|\partial_x u^{x'}| \neq 0, \tag{1.1}$$

where the indices a, b, c run over the range $1, 2, \dots, p$ and the indices x, y, z run over the range $p+1, p+2, \dots, p+q = n$.

A locally product space is said to be a *hyperbolic Kähler space* if there is given a positive definite Riemannian metric and an affinor structure $F_i^h \neq \delta_i^h$ satisfying the conditions

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$$F_p^h F_i^p = \delta_i^h,$$
$$g_{\alpha\beta} F_i^\alpha F_j^\beta = -g_{ij}, \tag{1.2}$$
$$\nabla_k F_i^h = 0,$$

where ∇ is the operator of covariant differentiation with respect to the Levi-Civita connection of the metric g_{ij}.

As is well-known, Kähler manifolds are related to the algebra of complex numbers. In 1948, Raševskij was the first to consider a similar kind of manifolds, this time related to the algebra of double numbers and such space is called a hyperbolic Kähler space. Latterly, in 1949 Rozenfeld gave the explicit definition of para-Kähler manifolds. He compared Raševskij's definition with Kähler's definition in the complex case and remarked that Raševskij's spaces are (local) real models of para-Kähler manifolds. Much more historical remarks on para-Kähler manifolds are given in the survey paper [1].

The theory of holomorphically projective mappings between classical Kähler manifolds was started by the Japanese geometers Otsuki and Tashiro, and for a certain period of time it was one of the main research directions of the Japanese and Soviet differential geometric schools. Among Soviet geometers, some of the significant contributions to this theory have been made by Mikeš [2,5,7,8,10,9]. The theory of holomorphically projective (HP) transformations between locally product spaces was started by Prvanović [19]. As a particular case one can consider such transformations between locally decomposable Riemannian spaces and hyperbolic Kähler spaces, [21]. Among other things Prvanović [19] introduced the paraholomorphic projective curvature tensor and gave the explicit expression of the curvature tensor for spaces with constant paraholomorphic sectional curvature. Note that we have respected Prvanović's terminology by using the word "holomorphically," but strictly speaking one should use the word "paraholomorphically," to avoid any possible confusion.

Eisenhart, in his contributions to general relativity, proposed a generalization of Riemannian spaces [3,4]. This generalization consisted in using a non-symmetric basic tensor and play a fundamental role in Moffat's non-symmetric gravitational field theory [18]. Although, as is well-known, Moffat's theory is a controversial one, some modifications have improved it. Thus some hopes, based among others on results on dark matter and dark energy, have been pointed out, see for instance Janssen and Prokopec [6]. Equitorsion geodesic mappings between Eisenhart's generalized Riemannian spaces were considered in the papers [15,26,27]. So far, generalized (classical) Kähler spaces as a particular case of Eisenhart's generalized Riemannian spaces were defined and holomorphically projective mappings between such spaces were considered in the papers [16,22–24].

In the present paper we define generalized hyperbolic Kähler spaces. Also, we consider equitorsion holomorphically projective mappings between generalized hyperbolic Kähler spaces and find some invariant geometric objects with respect to these mappings. Geometric objects analogous to the Thomas projective parameter in the theory of geodesic mappings and the paraholomorphic curvature tensor in the theory of holomorphically projective mappings are examined.

2. Generalized hyperbolic Kähler spaces

On a manifold M with non-symmetric linear connection $\underset{1}{\nabla}$ another non-symmetric linear connection $\underset{2}{\nabla}$ can be defined in the following way [20]

$$\underset{2}{\nabla}_X Y = \underset{1}{\nabla}_Y X + [X, Y],\ X, Y \in T_p(M),$$

where as usual $[\cdot,\cdot]$ denotes the Lie bracket.

M. Prvanović [20] considered four curvature tensors of a non-symmetric linear connection and explained the geometric meaning of them in terms of parallel displacement with respect to the non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{2}{\nabla}$, whereas S.M. Minčić examined various Ricci type identities on a space with non-symmetric affine connection [11] and recently he reobtained these identities considering curvature tensors as polylinear mappings [14]. S.M. Minčić [13] showed that among the twelve curvature tensors which appeared in the Ricci type identities, five of them are linearly independent:

$$\begin{aligned}
\underset{\theta}{R}(X,Y)Z &= \underset{\theta}{\nabla}_X \underset{\theta}{\nabla}_Y Z - \underset{\theta}{\nabla}_Y \underset{\theta}{\nabla}_X Z - \underset{\theta}{\nabla}_{[X,Y]} Z, \;\; \theta = 1,2;\\
\underset{3}{R}(X,Y)Z &= \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z;\\
\underset{4}{R}(X,Y)Z &= \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z; \qquad (2.1)\\
\underset{5}{R}(X,Y)Z &= \frac{1}{2}\big(\underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{1}{\nabla}_X Z + \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z\\
&\quad + \underset{1}{\nabla}_{[Y,X]} Z + \underset{2}{\nabla}_{[Y,X]} Z\big).
\end{aligned}$$

Let (U,u), $u = (u^1,\dots,u^n)$ be a local chart at the point $p \in M$. Local coordinates $u^1,\dots,u^n$ give rise to the vector fields

$$\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n},$$

which form a basis of the tangent space $T_p(M)$.

Throughout this paper we shall use the following notation

$$X = \frac{\partial}{\partial u^i},\;\; Y = \frac{\partial}{\partial u^j},\;\; Z = \frac{\partial}{\partial u^k},$$

therefore $[X,Y] = 0$, and consequently

$$\underset{2}{\nabla}_X Y = \underset{1}{\nabla}_Y X.$$

Also, a non-symmetric linear connection $\underset{1}{\nabla}$ can be described thorough its symmetric part ∇ and the torsion tensor $\underset{1}{T}$ as

$$\underset{1}{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\underset{1}{T}(X,Y),$$

where the symmetric part ∇ of a non-symmetric linear connection $\underset{1}{\nabla}$ is given by

$$\nabla_X Y = \frac{1}{2}\big(\underset{1}{\nabla}_X Y + \underset{1}{\nabla}_Y X\big),$$

and the torsion tensor $\underset{1}{T}$ is defined by

$$\underset{1}{T}(X,Y) = \underset{1}{\nabla}_X Y - \underset{1}{\nabla}_Y X.$$

A generalized Riemannian space in Eisenhart's sense [3] is a differentiable manifold M equipped with a non-symmetric metric g. Therefore the metric g can be written as

$$g(X,Y) = \underline{g}(X,Y) + \underset{\vee}{g}(X,Y),$$

where $\underline{g}$ denotes the symmetric part of the metric g and $\underset{\vee}{g}$ denotes the skew-symmetric part of g, i.e.

$$\underline{g}(X,Y)=\frac{1}{2}(g(X,Y)+g(Y,X)) \text{ and } \underset{\vee}{g}(X,Y)=\frac{1}{2}(g(X,Y)-g(Y,X)).$$

A non-symmetric linear connection $\underset{1}{\nabla}$ of a generalized Riemannian manifold with the metric g is explicitly defined by

$$\underline{g}(\underset{1}{\nabla}_X Y,Z)=\frac{1}{2}(Xg(Y,Z)+Yg(Z,X)-Zg(Y,X)), \tag{2.2}$$

or in local coordinates by

$$\Gamma_{i.jk}=g_{\underline{ip}}\Gamma^p_{jk}=\frac{1}{2}(g_{ji,k}-g_{jk,i}+g_{ik,j}). \tag{2.3}$$

Here the functions $\Gamma_{i.jk}$ and Γ^i_{jk} are called generalized Cristoffel symbols of the first kind and the second kind, respectively.

The symmetric linear connection ∇ and the non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{2}{\nabla}$ induce covariant derivatives of tensors:

$$\begin{aligned}
\nabla_m a^i_j &\equiv a^i_{j;m}=a^i_{j,m}+\Gamma^i_{\underline{mp}}a^p_j-\Gamma^p_{\underline{mj}}a^i_p,\\
\underset{1}{\nabla}_m a^i_j &\equiv a^i_{j\underset{1}{|}m}=a^i_{j,m}+\Gamma^i_{pm}a^p_j-\Gamma^p_{jm}a^i_p,\\
\underset{2}{\nabla}_m a^i_j &\equiv a^i_{j\underset{2}{|}m}=a^i_{j,m}+\Gamma^i_{mp}a^p_j-\Gamma^p_{mj}a^i_p.
\end{aligned}$$

Here $a^i_{j,m}$ denotes the partial derivative of a tensor a^i_j with respect to x^m and $\underline{mp}$ signifies a symmetrization with division, i.e. $\Gamma^i_{\underline{mp}}=\frac{1}{2}(\Gamma^i_{mp}+\Gamma^i_{pm})$.

Moreover, we can consider two more kinds of covariant differentiation [12]:

$$\begin{aligned}
\underset{3}{\nabla}_m a^i_j &\equiv a^i_{j\underset{1}{|}m}=a^i_{j,m}+\Gamma^i_{pm}a^p_j-\Gamma^p_{mj}a^i_p,\\
\underset{4}{\nabla}_m a^i_j &\equiv a^i_{j\underset{1}{|}m}=a^i_{j,m}+\Gamma^i_{mp}a^p_j-\Gamma^p_{jm}a^i_p.
\end{aligned}$$

Generalized classical (elliptic) Kähler spaces were introduced in [16]. Analogously, we define generalized hyperbolic Kähler spaces.

Definition 2.1. A generalized Riemannian space (M,g) is called a *generalized hyperbolic Kähler space* if there exists a $(1,1)$ tensor F on M such that

$$F^2=I, \tag{2.4}$$

$$\underline{g}(FX,FY)=-\underline{g}(X,Y), \tag{2.5}$$

$$\underset{1}{\nabla}F=0 \text{ and } \underset{2}{\nabla}F=0, \tag{2.6}$$

where I denotes the identity operator.

Theorem 2.1. *The curvature tensors $\underset{\theta}{R}$, $\theta=1,\dots,4$ and the torsion tensor $\underset{1}{T}$ of a generalized hyperbolic Kähler space (M,g,F) satisfy the next relations:*

(i) $\underset{1}{T}(X,Y)=F(\underset{1}{T}(FX,Y))$,
(ii) $\underset{1}{R}(X,Y)FZ=F(\underset{1}{R}(X,Y)Z)$,

(iii) $\underset{2}{R}(X,Y)FZ = F(\underset{2}{R}(X,Y)Z)$,
(iv) $\underset{3}{R}(X,Y)FZ = F(\underset{3}{R}(X,Y)Z)$,
(v) $\underset{4}{R}(X,Y)FZ + F(\underset{3}{R}(Y,X)Z) = F\big(\underset{4}{\nabla}_X\underset{1}{T}(Z,Y) - \underset{3}{\nabla}_Y\underset{1}{T}(X,Z) + \underset{1}{T}(X,\underset{1}{T}(Z,Y)\big) - \underset{1}{T}(\underset{1}{T}(X,Z),Y))$.

Proof. (i) From (2.6) we obtain that

$$\underset{1}{T}(FU,Y) = F(\underset{1}{T}(U,Y)).$$

By putting $U = FX$ in the previous relation and using (2.5) we get part (i).

(ii) From (2.6) we get

$$\underset{1}{\nabla}_Z\underset{1}{\nabla}_Y FX - \underset{1}{\nabla}_Y\underset{1}{\nabla}_Z FX = 0.$$

After applying the first Ricci type identity (Eq. (9) from [11]) we obtain that

$$-F(\underset{1}{R}(X,Y)Z) + \underset{1}{R}(X,Y)FZ - \underset{1}{\nabla}_{\underset{1}{T}(Y,X)}FZ = 0, \tag{2.7}$$

i.e.

$$-F(\underset{1}{R}(X,Y)Z) + \underset{1}{R}(X,Y)FZ = 0, \tag{2.8}$$

where we used $\underset{1}{\nabla}F = 0$.

(iii) To prove this part we use the second Ricci type identity (Eq. (13) from [11]), i.e.

$$\underset{2}{\nabla}_Z\underset{2}{\nabla}_Y FX - \underset{2}{\nabla}_Y\underset{2}{\nabla}_Z FX = \underset{2}{R}(Z,Y)FX - F(\underset{2}{R}(Z,Y)X) + \underset{2}{\nabla}_{\underset{1}{T}(Y,Z)}FX. \tag{2.9}$$

By taking into account that $\underset{2}{\nabla}F = 0$, and $\underset{2}{\nabla}_Z\underset{2}{\nabla}_Y FX - \underset{2}{\nabla}_Y\underset{2}{\nabla}_Z FX = 0$, equation (2.9) becomes

$$\underset{2}{R}(Z,Y)FX - F(\underset{2}{R}(Z,Y)X) = 0,$$

which completes the proof of part (iii).

(iv) By applying an appropriate Ricci type identity (Eq. (58′) from [11]), one has

$$\underset{2}{\nabla}_Z\underset{1}{\nabla}_Y FX - \underset{1}{\nabla}_Y\underset{2}{\nabla}_Z FX = \underset{3}{R}(Z,Y)FX - F(\underset{3}{R}(Z,Y)X),$$

and using (2.6) we get (iv).

(v) From (2.6) and the definition of the covariant derivative of the third and fourth kind we get

$$\underset{3}{\nabla}_Y FX = F(\underset{1}{T}(X,Y)) \ \text{ and } \ \underset{4}{\nabla}_Z FX = F(\underset{1}{T}(Z,X)),$$

which further implies

$$\underset{4}{\nabla}_Z\underset{3}{\nabla}_Y FX = F(\underset{4}{\nabla}_Z\underset{1}{T}(X,Y)) + F(\underset{1}{T}(Z,\underset{1}{T}(X,Y)))$$

and

$$\underset{3}{\nabla}_Y\underset{4}{\nabla}_Z FX = F(\underset{3}{\nabla}_Y\underset{1}{T}(Z,X)) + F(\underset{1}{T}(\underset{1}{T}(Z,X),Y)).$$

On the other hand, the Ricci type identity (Eq. (56′) from [12]) reads

$$\underset{4}{\nabla}_Z\underset{3}{\nabla}_Y FX - \underset{3}{\nabla}_Y\underset{4}{\nabla}_Z FX = \underset{4}{R}(Z,Y)FX + F(\underset{3}{R}(Y,Z)X).$$

From the last three relations we get part (v). □

We denote the curvature tensor of type $(0,4)$ by

$$\underset{\theta}{R}(X,Y,Z,W) := \underline{g}(\underset{\theta}{R}(X,Y)Z,W),\ \ \theta = 1,\dots,4,$$

and the torsion tensor of type $(0,3)$ by

$$\underset{1}{T}(X,Y,Z) := \underline{g}(X,\underset{1}{T}(Y,Z)).$$

Corollary 2.1. *The curvature $(0,4)$-tensors $\underset{\theta}{R}$, $\theta = 1,\dots,4$ and the torsion tensor $\underset{1}{T}$ of type $(0,3)$ of a generalized hyperbolic Kähler space (M,g,F) satisfy the next relations:*

(i) $\underset{1}{R}(X,Y,Z,FW) + \underset{1}{R}(X,Y,FZ,W) = 0,$
(ii) $\underset{2}{R}(X,Y,Z,FW) + \underset{2}{R}(X,Y,FZ,W) = 0,$
(iii) $\underset{3}{R}(X,Y,Z,FW) + \underset{3}{R}(X,Y,FZ,W) = 0,$
(iv) $\underset{4}{R}(X,Y,Z,FW) + \underset{3}{R}(Y,X,FZ,W) = \underset{3}{\nabla}_Y\underset{1}{T}(FW,X,Z) - \underset{4}{\nabla}_X\underset{1}{T}(FW,Z,Y) + \underset{1}{T}(FW,\underset{1}{T}(X,Z),Y) - \underset{1}{T}(FW,X,\underset{1}{T}(Z,Y)),$
(v) $\underset{4}{R}(X,Y,Z,FW) - \underset{3}{R}(Y,X,Z,FW) = \underset{3}{\nabla}_Y\underset{1}{T}(FW,X,Z) - \underset{4}{\nabla}_X\underset{1}{T}(FW,Z,Y) + \underset{1}{T}(FW,\underset{1}{T}(X,Z),Y) - \underset{1}{T}(FW,X,\underset{1}{T}(Z,Y)).$

Proof. The proof directly follows from Theorem 2.1. □

The relations between the curvature tensors $\underset{\theta}{R}(\theta = 1,\dots,5)$ and the Riemannian curvature tensor R corresponding to the symmetric linear connection $\nabla_X Y = \frac{1}{2}(\underset{1}{\nabla}_X Y + \underset{1}{\nabla}_Y X)$ are examined in [13], obtaining

$$\begin{aligned}
\underset{1}{R}(X,Y)Z &= R(X,Y)Z + \frac{1}{2}\nabla_X\underset{1}{T}(Z,Y) - \frac{1}{2}\nabla_Y\underset{1}{T}(Z,X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&\quad - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y);\\
\underset{2}{R}(X,Y)Z &= R(X,Y)Z - \frac{1}{2}\nabla_X\underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y\underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&\quad + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y);\\
\underset{3}{R}(X,Y)Z &= R(X,Y)Z + \frac{1}{2}\nabla_X\underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y\underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&\quad + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y) - \frac{1}{2}\underset{1}{T}(\underset{1}{T}(Y,X),Z);\\
\underset{4}{R}(X,Y)Z &= R(X,Y)Z + \frac{1}{2}\nabla_X\underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y\underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&\quad + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y) + \frac{1}{2}\underset{1}{T}(\underset{1}{T}(Y,X),Z);\\
\underset{5}{R}(X,Y)Z &= R(X,Y)Z + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y).
\end{aligned} \tag{2.10}$$

For an arbitrary tensor field B we will use the symbols $\sum_{CA(\cdot,\cdot)}$ and $\sum_{CS(\cdot,\cdot)}$ to denote

$$\sum_{CA(Y,Z)} B(X,Y,Z) = B(X,Y,Z) - B(X,Z,Y),$$

and

$$\sum_{CS(Y,Z)} B(X,Y,Z) = B(X,Y,Z) + B(X,Z,Y),$$

respectively.

Theorem 2.2. *The Ricci tensors* $\underset{\theta}{\mathrm{Ric}}(X,Y) = \mathrm{Tr}\big(U \to \underset{\theta}{R}(U,X)Y\big), \theta = 1,\dots,5$ *on a generalized hyperbolic Kähler space* (M,g,F)*, have the following properties:*

$$\begin{aligned}
\underset{1}{\mathrm{Ric}}(FX,FY) &= -\underset{1}{\mathrm{Ric}}(X,Y) + \mathrm{Tr}\Big(U \to \sum_{CA(X,U)} \Big(\frac{1}{2}\nabla_U \underset{1}{T}(Y,X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Y,X),U)\Big)\Big)\\
&\quad + \mathrm{Tr}\Big(U \to \sum_{CA(FX,U)} \Big(\frac{1}{2}\nabla_U \underset{1}{T}(FY,FX) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(FY,FX),U)\Big)\Big),\\
\underset{2}{\mathrm{Ric}}(FX,FY) &= -\underset{2}{\mathrm{Ric}}(X,Y) - \mathrm{Tr}\Big(U \to \sum_{CA(X,U)} \Big(\frac{1}{2}\nabla_U \underset{1}{T}(Y,X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Y,X),U)\Big)\Big)\\
&\quad - \mathrm{Tr}\Big(U \to \sum_{CA(FX,U)} \Big(\frac{1}{2}\nabla_U \underset{1}{T}(FY,FX) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(FY,FX),U)\Big)\Big),\\
\underset{3}{\mathrm{Ric}}(FX,FY) &= -\underset{3}{\mathrm{Ric}}(X,Y) + \frac{1}{2}\mathrm{Tr}\Big(U \to \sum_{CS(U,X)} \nabla_U \underset{1}{T}(Y,X)\Big)\\
&\quad - \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,X)} \underset{1}{T}(\underset{1}{T}(Y,X),U)\Big) + \frac{1}{2}\mathrm{Tr}\Big(U \to \sum_{CS(U,FX)} \nabla_U \underset{1}{T}(FY,FX)\Big)\\
&\quad - \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,FX)} \underset{1}{T}(\underset{1}{T}(FY,FX),U)\Big) - \frac{1}{2}\mathrm{Tr}\Big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\Big)\\
&\quad - \frac{1}{2}\mathrm{Tr}\Big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\Big),\\
\underset{4}{\mathrm{Ric}}(FX,FY) &= -\underset{4}{\mathrm{Ric}}(X,Y) + \frac{1}{2}\mathrm{Tr}\Big(U \to \sum_{CS(U,X)} \nabla_U \underset{1}{T}(Y,X)\Big)\\
&\quad - \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,X)} \underset{1}{T}(\underset{1}{T}(Y,X),U)\Big) + \frac{1}{2}\mathrm{Tr}\Big(U \to \sum_{CS(U,FX)} \nabla_U \underset{1}{T}(FY,FX)\Big)\\
&\quad - \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,FX)} \underset{1}{T}(\underset{1}{T}(FY,FX),U)\Big) + \frac{1}{2}\mathrm{Tr}\Big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\Big)\\
&\quad + \frac{1}{2}\mathrm{Tr}\Big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\Big),\\
\underset{5}{\mathrm{Ric}}(FX,FY) &= -\underset{5}{\mathrm{Ric}}(X,Y) + \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,X)} \underset{1}{T}(\underset{1}{T}(Y,X),U)\Big)\\
&\quad + \frac{1}{4}\mathrm{Tr}\Big(U \to \sum_{CA(U,FX)} \underset{1}{T}(\underset{1}{T}(FY,FX),U)\Big).
\end{aligned}$$

Proof. The main idea of the proof is to use the relations between the curvature tensors $\underset{\theta}{R}\,(\theta = 1, \ldots, 5)$ and the Riemannian curvature tensor R and the well-known property of the Ricci tensor on a usual hyperbolic Kähler space.

From (2.10) it follows that

$$\begin{aligned}\underset{1}{\mathrm{Ric}}(Y,Z) = \mathrm{Ric}(Y,Z) &+ \frac{1}{2}\mathrm{Tr}\big(U \to \sum_{CA(Y,U)} \nabla_U \underset{1}{T}(Z,Y)\big)\\ &+ \frac{1}{4}\mathrm{Tr}\big(U \to \sum_{CA(Y,U)} \underset{1}{T}(\underset{1}{T}(Z,Y),U)\big),\end{aligned}$$

hence

$$\begin{aligned}\underset{1}{\mathrm{Ric}}(FX,FY) = \mathrm{Ric}(FX,FY) &+ \frac{1}{2}\mathrm{Tr}\big(U \to \sum_{CA(FX,U)} \nabla_U \underset{1}{T}(FY,FX)\big)\\ &+ \frac{1}{4}\mathrm{Tr}\big(U \to \sum_{CA(FX,U)} \underset{1}{T}(\underset{1}{T}(FY,FX),U)\big).\end{aligned}$$

The Ricci tensor satisfies [19]

$$\begin{aligned}\mathrm{Ric}(FX,FY) = -\mathrm{Ric}(X,Y) = -\underset{1}{\mathrm{Ric}}(X,Y) &+ \frac{1}{2}\mathrm{Tr}\big(U \to \sum_{CA(X,U)} \nabla_U \underset{1}{T}(Y,X)\big)\\ &+ \frac{1}{4}\mathrm{Tr}\big(U \to \sum_{CA(X,U)} \underset{1}{T}(\underset{1}{T}(Y,X),U)\big).\end{aligned}$$

From the last two relations we obtain the proof of the first statement of this theorem. The verification of the other statements of this theorem is left to the reader. □

By taking the cyclic sum of the relations between Ricci tensors from Theorem 2.2 we obtain Corollary 2.2.

Corollary 2.2. *The Ricci tensors* $\underset{\theta}{\mathrm{Ric}}(X,Y) = \mathrm{Tr}\big(U \to \underset{\theta}{R}(U,X)Y\big), \theta = 1, \ldots, 5$ *on a generalized hyperbolic Kähler space* (M, g, F) *have the following properties:*

$$\begin{aligned}\sum_{CS(X,Y)} \underset{1}{\mathrm{Ric}}(FX,FY) &= -\sum_{CS(X,Y)} \underset{1}{\mathrm{Ric}}(X,Y) - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\big)\\ &\quad - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\big),\\ \sum_{CS(X,Y)} \underset{2}{\mathrm{Ric}}(FX,FY) &= -\sum_{CS(X,Y)} \underset{2}{\mathrm{Ric}}(X,Y) - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\big)\\ &\quad - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\big),\\ \sum_{CS(X,Y)} \underset{3}{\mathrm{Ric}}(FX,FY) &= -\sum_{CS(X,Y)} \underset{3}{\mathrm{Ric}}(X,Y) - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\big)\\ &\quad - \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\big),\\ \sum_{CS(X,Y)} \underset{4}{\mathrm{Ric}}(FX,FY) &= -\sum_{CS(X,Y)} \underset{4}{\mathrm{Ric}}(X,Y) + \frac{2}{3}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{3}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\big),\\
\sum_{CS(X,Y)} \underset{5}{\mathrm{Ric}}(FX,FY) = -&\sum_{CS(X,Y)} \underset{5}{\mathrm{Ric}}(X,Y) + \frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(FX,U),FY)\big)\\
&+\frac{1}{2}\mathrm{Tr}\big(U \to \underset{1}{T}(\underset{1}{T}(X,U),Y)\big).
\end{aligned}
$$

3. Holomorphically projective mappings between generalized hyperbolic Kähler spaces

As in the case of classical (elliptic) generalized Kähler spaces, a holomorphically planar curve in a generalized hyperbolic Kähler space is determined by an ODE of second order in the Euler form.

Definition 3.1. [19,21] A curve $l : I \to M$ in a generalized hyperbolic Kähler space (M, g, F) satisfying the regularity condition $\lambda(t) = \frac{\mathrm{d}l(t)}{\mathrm{d}t} \neq 0$, $t \in I$, is called a *holomorphically planar curve* if for some functions ρ_1 and ρ_2 of a parameter t the following ODE holds

$$\nabla_{\lambda(t)}\lambda(t) = \rho_1(t)\lambda(t) + \rho_2(t)F\lambda(t),$$

where ∇ denotes the Levi-Civita connection corresponding to the symmetric part $\underline{g}$ of the metric g.

Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces of dimension n $(n > 2)$, such that M and $\overline{M}$ are diffeomorphic under the diffeomorphism $f : M \to \overline{M}$. We can consider the common coordinate system on M and $\overline{M}$ with respect to f. In this coordinate system the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates. Therefore we can suppose $M \equiv \overline{M}$ and we can put

$$\underset{1}{P} = \underset{1}{\overline{\nabla}} - \underset{1}{\nabla},$$

where $\underset{1}{P}$ is a tensor field of type $(1, 2)$, called the *difference tensor* of the linear connections $\underset{1}{\overline{\nabla}}$ and $\underset{1}{\nabla}$ with respect to the mapping f.

Definition 3.2. [19,21] Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces of dimension n $(n > 2)$. A diffeomorphism $f : M \to \overline{M}$ is called a *holomorphically projective mapping* if each holomorphically planar curve in (M, g, F) is mapped onto a holomorphically planar curve in the space $(\overline{M}, \overline{g}, \overline{F})$.

It is not difficult to prove that if a mapping $f : M \to \overline{M}$ is holomorphically projective, then the structure F is preserved. Also, the next theorem holds.

Theorem 3.1. [19,21] *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces of dimension n $(n > 2)$. A diffeomorphism $f : M \to \overline{M}$ is a holomorphically projective mapping if and only if*

$$\underset{1}{P}(X, Y) = \psi(X)Y + \psi(Y)X + \psi(FX)FY + \psi(FY)FX + \xi(X, Y), \tag{3.1}$$

where $X, Y \in T_p(M)$, ψ is a linear form and ξ is an anti-symmetric tensor field of type $(1, 2)$.

In the tensor index notation, (3.1) reads

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \psi_p F^p_{(i}F^h_{j)} + \xi^h_{ij}, \tag{3.2}$$

where Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are generalized Cristoffel symbols of the second kind, ψ_i is a covector and ξ^h_{ij} is an anti-symmetric tensor.

Generalized Cristoffel symbols of the first kind satisfy the relations [17], page 11

$$\Gamma_{i.jk} + \Gamma_{j.ik} = g_{\underline{ij},k}\,, \;\; \Gamma_{i.jk} + \Gamma_{k.ij} = g_{\underline{ik},j}\,. \tag{3.3}$$

Therefore,

$$|\underline{g}|_{,k} = |\underline{g}| g^{\underline{ij}} g_{\underline{ij},k} = |\underline{g}| g^{\underline{ij}} (\Gamma_{i.jk} + \Gamma_{j.ik}) = |\underline{g}|(\Gamma^j_{jk} + \Gamma^i_{ik})$$

i.e.

$$\frac{|\underline{g}|_{,k}}{2|\underline{g}|} = \Gamma^p_{pk}.$$

Analogously, by using the second relation in (3.3) one can prove that [17]

$$\frac{|\underline{g}|_{,k}}{2|\underline{g}|} = \Gamma^p_{kp},$$

which further implies

$$\underset{1}{T}{}^p_{kp} = \Gamma^p_{kp} - \Gamma^p_{pk} = 0. \tag{3.4}$$

Anti-symmetrization in (3.2) with respect to i and j yields

$$\xi^h_{ij} = \frac{1}{2}(\overline{\Gamma}^h_{ij} - \overline{\Gamma}^h_{ji}) - \frac{1}{2}(\Gamma^h_{ij} - \Gamma^h_{ji}). \tag{3.5}$$

From the previous two relations one concludes that

$$\xi^p_{ip} = \frac{1}{2}(\underset{1}{\overline{T}}{}^p_{ip} - \underset{1}{T}{}^p_{ip}) = 0. \tag{3.6}$$

By contracting relation (3.2) with respect to h and j and using (3.6) and $F^p_p = 0$, we obtain that

$$\overline{\Gamma}^p_{ip} - \Gamma^p_{ip} = (n+2)\psi_i. \tag{3.7}$$

Now, from (3.2), (3.5) and (3.7) we have

$$\begin{aligned} &\Gamma^h_{\underline{ij}} - \frac{1}{n+2}(\Gamma^p_{pi}\delta^h_j + \Gamma^p_{pj}\delta^h_i + \Gamma^q_{qp}F^p_{(i}F^h_{j)}) \\ &\quad = \overline{\Gamma}^h_{\underline{ij}} - \frac{1}{n+2}(\overline{\Gamma}^p_{pi}\delta^h_j + \overline{\Gamma}^p_{pj}\delta^h_i + \overline{\Gamma}^q_{qp}F^p_{(i}F^h_{j)}), \end{aligned} \tag{3.8}$$

where $\underline{ij}$ signifies a symmetrization with division.

According to the fact that the affinor structure F is preserved under a holomorphically projective mapping, we have proved the next theorem.

Theorem 3.2. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces of dimension n $(n > 2)$ and $f : M \to \overline{M}$ be a holomorphically projective mapping, then the geometric object*

$$\Gamma^h_{\underline{ij}} - \frac{1}{n+2}(\Gamma^p_{pi}\delta^h_j + \Gamma^p_{pj}\delta^h_i + \Gamma^q_{qp}F^p_{(i}F^h_{j)}), \tag{3.9}$$

is invariant with respect to the mapping f.

Remark 3.1. The geometric object (3.9) is analogous to the Thomas projective parameter in the theory of geodesic mappings.

3.1. Equitorsion holomorphically projective mappings

Equitorsion holomorphically projective mappings between generalized (classical) Kähler spaces were considered in the papers [22–24]. In this subsection we shall consider equitorsion holomorphically projective mappings between generalized hyperbolic Kähler spaces.

Definition 3.3. A mapping $f : M \to \overline{M}$ between generalized hyperbolic Kähler spaces is said to be an *equitorsion mapping* if it preserves the torsion $\underset{1}{T}$, i.e. $\underset{1}{T} = \underset{1}{\overline{T}}$.

If $f : M \to \overline{M}$ is an equitorsion holomorphically projective mapping between generalized hyperbolic Kähler spaces, then the anti-symmetric tensor field ξ from (3.1) vanishes identically. Indeed,

$$\begin{aligned}
2\xi(X,Y) &= \underset{1}{P}(X,Y) - \underset{1}{P}(Y,X)\\
&= \underset{1}{\overline{\nabla}}(X,Y) - \underset{1}{\nabla}(X,Y) - \underset{1}{\overline{\nabla}}(Y,X) + \underset{1}{\nabla}(Y,X)\\
&= \underset{1}{\overline{T}}(X,Y) - \underset{1}{T}(X,Y) = 0.
\end{aligned}$$

Thus the difference tensor $\underset{1}{P}$ with respect to the mapping f is a symmetric bilinear form given by

$$\underset{1}{P}(X,Y) = \psi(X)Y + \psi(Y)X + \psi(FX)FY + \psi(FY)FX. \tag{3.10}$$

Theorem 3.3. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces and $f : M \to \overline{M}$ be an equitorsion holomorphically projective mapping, then the geometric objects given by*

$$\begin{aligned}
\underset{\theta}{P}{}^h_{ijk} = \underset{\theta}{R}{}^h_{ijk} + \frac{1}{n+2}\Big[&\delta^h_j(\underset{\theta}{R}_{ik} - \underset{\theta}{Q}_{ik}) - \delta^h_k(\underset{\theta}{R}_{ij} - \underset{\theta}{Q}_{ij}) + \delta^h_i(\underset{\theta}{R}_{[jk]} - \underset{\theta}{Q}_{[jk]})\big)\\
&- F^h_k(\underset{\theta}{R}_{pj} - \underset{\theta}{Q}_{pj})F^p_i + F^h_j(\underset{\theta}{R}_{pk} - \underset{\theta}{Q}_{pk})F^p_i\\
&- F^h_i\big((\underset{\theta}{R}_{pj} - \underset{\theta}{Q}_{pj})F^p_k - (\underset{\theta}{R}_{pk} - \underset{\theta}{Q}_{pk})F^p_j\big)\\
&+ (-1)^{\theta-1}\big(\underset{1}{T}{}^h_{jk}\Gamma^s_{is} + \delta^h_i\Gamma^s_{ps}\underset{1}{T}{}^p_{jk}\\
&+ F^h_p\underset{1}{T}{}^p_{jk}\Gamma^s_{qs}F^q_i + \Gamma^s_{ps}F^p_q\underset{1}{T}{}^q_{jk}F^h_i\big)\Big],\quad \theta = 1,2,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
\underset{\theta}{Q}_{ij} = \frac{(-1)^{\theta-1}}{n+2}\Big(&2\Gamma^s_{ps}\underset{1}{T}{}^p_{ji} + \frac{2n-2}{n-2}F^q_p\underset{1}{T}{}^p_{jq}\Gamma^s_{rs}F^r_i + \frac{2}{n-2}F^q_p\underset{1}{T}{}^p_{iq}\Gamma^s_{rs}F^r_j\\
&+ \frac{2n-2}{n-2}\Gamma^s_{ps}F^p_q\underset{1}{T}{}^q_{jr}F^r_i + \frac{2}{n-2}\Gamma^s_{ps}F^p_q\underset{1}{T}{}^q_{ir}F^r_j\\
&+ \frac{2}{n-2}\big(F^p_q\underset{1}{T}{}^q_{rp}F^r_j\Gamma^s_{is} + \Gamma^s_{ps}F^p_q\underset{1}{T}{}^q_{ri}F^r_j\big)_{(ij)}\Big),\quad \theta = 1,2,
\end{aligned}$$

are invariant with respect to the mapping f.

Proof. The curvature tensors $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ of the generalized hyperbolic Kähler spaces (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$, respectively, satisfy the relation [15]

$$\begin{aligned}\underset{1}{\overline{R}}(X,Y)Z = \underset{1}{R}(X,Y)Z + \underset{1}{\nabla}_X \underset{1}{P}(Z,Y) - \underset{1}{\nabla}_Y \underset{1}{P}(Z,X) + \underset{1}{P}(\underset{1}{P}(Z,Y),X)\\ - \underset{1}{P}(\underset{1}{P}(Z,X),Y) + \underset{1}{P}(Z,\underset{1}{T}(Y,X)).\end{aligned} \tag{3.12}$$

Let us denote

$$\underset{1}{\psi}(X,Y) = \underset{1}{\nabla}_Y \psi(X) + \psi(X)\psi(Y) + \psi(FX)\psi(FY).$$

Substituting (3.10) into (3.12) we obtain that

$$\begin{aligned}\underset{1}{\overline{R}}(X,Y)Z = \underset{1}{R}(X,Y)Z + Y\underset{1}{\psi}(Z,X) - X\underset{1}{\psi}(Z,Y) + Z\sum_{CA(X,Y)}\underset{1}{\psi}(X,Y)\\ - FX\underset{1}{\psi}(FZ,Y) + FY\underset{1}{\psi}(FZ,X) - FZ\big(\underset{1}{\psi}(FX,Y) - \underset{1}{\psi}(FY,X)\big)\\ + \underset{1}{T}(Y,X)\psi(Z) + Z\psi(\underset{1}{T}(Y,X)) + F(\underset{1}{T}(Y,X))\psi(FZ) + \psi(F(\underset{1}{T}(Y,X)))FZ.\end{aligned}$$

The last relation in local coordinates reads

$$\begin{aligned}\underset{1}{\overline{R}}{}^h_{ijk} = \underset{1}{R}{}^h_{ijk} + \delta^h_j\underset{1}{\psi}_{ik} - \delta^h_k\underset{1}{\psi}_{ij} + \delta^h_i\underset{1}{\psi}_{[jk]}\\ - F^h_k\underset{1}{\psi}_{pj}F^p_i + F^h_j\underset{1}{\psi}_{pk}F^p_i - F^h_i(\underset{1}{\psi}_{pj}F^p_k - \underset{1}{\psi}_{pk}F^p_j)\\ + \underset{1}{T}{}^h_{jk}\psi_i + \delta^h_i\psi_p\underset{1}{T}{}^p_{jk} + F^h_p\underset{1}{T}{}^p_{jk}\psi_qF^q_i + \psi_pF^p_q\underset{1}{T}{}^q_{jk}F^h_i.\end{aligned} \tag{3.13}$$

Contracting on the indices h and k in (3.13) and by using (3.4) we get

$$\begin{aligned}\underset{1}{\overline{R}}_{ij} = \underset{1}{R}_{ij} - n\underset{1}{\psi}_{ij} + \underset{1}{\psi}_{[ji]} + \underset{1}{\psi}_{(pq)}F^p_iF^q_j\\ + \psi_p\underset{1}{T}{}^p_{ji} + F^q_p\underset{1}{T}{}^p_{jq}\psi_rF^r_i + \psi_pF^p_q\underset{1}{T}{}^q_{jr}F^r_i,\end{aligned} \tag{3.14}$$

where $\underset{1}{R}_{ij} = \underset{1}{R}{}^p_{ijp}$ and $\underset{1}{\overline{R}}_{ij} = \underset{1}{\overline{R}}{}^p_{ijp}$ are components of the Ricci tensors $\underset{1}{\mathrm{Ric}}(X,Y)$ and $\underset{1}{\overline{\mathrm{Ric}}}(X,Y)$, respectively.

Anti-symmetrization in (3.14) with respect to the indices i and j gives

$$\begin{aligned}(n+2)\underset{1}{\psi}_{[ij]} = \underset{1}{R}_{[ij]} - \underset{1}{\overline{R}}_{[ij]} + 2\psi_p\underset{1}{T}{}^p_{ji} + F^q_p\underset{1}{T}{}^p_{jq}\psi_rF^r_i - F^q_p\underset{1}{T}{}^p_{iq}\psi_rF^r_j\\ + \psi_pF^p_q\underset{1}{T}{}^q_{jr}F^r_i - \psi_pF^p_q\underset{1}{T}{}^q_{ir}F^r_j.\end{aligned} \tag{3.15}$$

By symmetrization in (3.14) with respect to i and j we obtain that

$$\begin{aligned}\underset{1}{\overline{R}}_{(ij)} = \underset{1}{R}_{(ij)} - n\underset{1}{\psi}_{(ij)} + 2\underset{1}{\psi}_{(pq)}F^p_iF^q_j + F^q_p\underset{1}{T}{}^p_{jq}\psi_rF^r_i + F^q_p\underset{1}{T}{}^p_{iq}\psi_rF^r_j\\ + \psi_pF^p_q\underset{1}{T}{}^q_{jr}F^r_i + \psi_pF^p_q\underset{1}{T}{}^q_{ir}F^r_j,\end{aligned} \tag{3.16}$$

and by composing with F^i_p and F^j_q in the last relation we obtain that

$$\overline{\underset{1}{R}}_{(pq)} F^p_i F^q_j = \underset{1}{R}_{(pq)} F^p_i F^q_j - n\psi_{(pq)} F^p_i F^q_j + 2\psi_{(ij)} + F^p_q \underset{1}{T}{}^q_{rp} F^r_j \psi_i \\ + F^q_p \underset{1}{T}{}^p_{rq} F^r_i \psi_j + \psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j + \psi_p F^p_q \underset{1}{T}{}^q_{rj} F^r_i. \tag{3.17}$$

In local coordinates, the first relation from Corollary 2.2 reads

$$\underset{1}{R}_{(pq)} F^p_i F^q_j = -\underset{1}{R}_{(ij)} - \frac{1}{2}\underset{1}{T}{}^p_{rq}\underset{1}{T}{}^q_{ps} F^r_i F^s_j - \frac{1}{2}\underset{1}{T}{}^p_{iq}\underset{1}{T}{}^q_{pj}, \tag{3.18}$$

and the same relation is valid on the space $(\overline{M}, \overline{g}, \overline{F})$, that is

$$\overline{\underset{1}{R}}_{(pq)} \overline{F}^p_i \overline{F}^q_j = -\overline{\underset{1}{R}}_{(ij)} - \frac{1}{2}\overline{\underset{1}{T}}{}^p_{rq}\overline{\underset{1}{T}}{}^q_{ps} \overline{F}^r_i \overline{F}^s_j - \frac{1}{2}\overline{\underset{1}{T}}{}^p_{iq}\overline{\underset{1}{T}}{}^q_{pj}. \tag{3.19}$$

By using the fact that the torsion tensor and the affinor structure F are preserved under an equitorsion holomorphically projective mapping and by substituting (3.18) and (3.19) into (3.17) we obtain that

$$-\overline{\underset{1}{R}}_{(ij)} = -\underset{1}{R}_{(ij)} - n\underset{1}{\psi}_{(pq)} F^p_i F^q_j + 2\underset{1}{\psi}_{(ij)} + F^p_q \underset{1}{T}{}^q_{rp} F^r_j \psi_i + F^p_q \underset{1}{T}{}^q_{rp} F^r_i \psi_j \\ + \psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j + \psi_p F^p_q \underset{1}{T}{}^q_{rj} F^r_i. \tag{3.20}$$

Summing (3.16) and (3.20) yields

$$\underset{1}{\psi}_{(pq)} F^p_i F^q_j = -\underset{1}{\psi}_{(ij)} + \frac{1}{n-2}\big(F^q_p \underset{1}{T}{}^p_{jq}\psi_r F^r_i + \psi_p F^p_q \underset{1}{T}{}^q_{jr} F^r_i \\ + F^p_q \underset{1}{T}{}^q_{rp} F^r_j \psi_i + \psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j\big)_{(ij)}. \tag{3.21}$$

After substituting (3.21) into (3.20) we get

$$(n+2)\underset{1}{\psi}_{(ij)} = \underset{1}{R}_{(ij)} - \overline{\underset{1}{R}}_{(ij)} + \frac{n}{n-2}\big(F^q_p \underset{1}{T}{}^p_{jq}\psi_r F^r_i + \psi_p F^p_q \underset{1}{T}{}^q_{jr} F^r_i\big)_{(ij)} \\ + \frac{2}{n-2}\big(F^p_q \underset{1}{T}{}^q_{rp} F^r_j \psi_i + \psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j\big)_{(ij)}. \tag{3.22}$$

Now, by summing (3.15) and (3.22) we get

$$(n+2)\underset{1}{\psi}_{ij} = \underset{1}{R}_{ij} - \overline{\underset{1}{R}}_{ij} + 2\psi_p \underset{1}{T}{}^p_{ji} + \frac{2n-2}{n-2} F^q_p \underset{1}{T}{}^p_{jq}\psi_r F^r_i + \frac{2}{n-2} F^q_p \underset{1}{T}{}^p_{iq}\psi_r F^r_j \\ + \frac{2n-2}{n-2}\psi_p F^p_q \underset{1}{T}{}^q_{jr} F^r_i + \frac{2}{n-2}\psi_p F^p_q \underset{1}{T}{}^q_{ir} F^r_j \\ + \frac{2}{n-2}\big(F^p_q \underset{1}{T}{}^q_{rp} F^r_j \psi_i + \psi_p F^p_q \underset{1}{T}{}^q_{ri} F^r_j\big)_{(ij)}.$$

By using (3.7) the last relation can be written in the form

$$(n+2)\underset{1}{\psi}_{ij} = \underset{1}{R}_{ij} - \overline{\underset{1}{R}}_{ij} + \overline{\underset{1}{Q}}_{ij} - \underset{1}{Q}_{ij}, \tag{3.23}$$

where $\underset{1}{Q}_{ij}$ is defined by

$$
\begin{aligned}
\underset{1}{Q}_{ij} = \frac{1}{n+2}\Big(&2\Gamma^s_{ps}\underset{1}{T}^p_{ji} + \frac{2n-2}{n-2}F^q_p\underset{1}{T}^p_{jq}\Gamma^s_{rs}F^r_i + \frac{2}{n-2}F^q_p\underset{1}{T}^p_{iq}\Gamma^s_{rs}F^r_j \\
&+ \frac{2n-2}{n-2}\Gamma^s_{ps}F^p_q\underset{1}{T}^q_{jr}F^r_i + \frac{2}{n-2}\Gamma^s_{ps}F^p_q\underset{1}{T}^q_{ir}F^r_j \\
&+ \frac{2}{n-2}\big(F^p_q\underset{1}{T}^q_{rp}F^r_j\Gamma^s_{is} + \Gamma^s_{ps}F^p_q\underset{1}{T}^q_{ri}F^r_j\big)_{(ij)}\Big),
\end{aligned} \tag{3.24}
$$

and $\underset{1}{\overline{Q}}_{ij}$ is defined in the same manner for the space $(\overline{M}, \overline{g}, \overline{F})$.

Finally, after changing (3.7) and (3.23) in (3.13), we get

$$
\underset{1}{P}^h_{ijk} = \underset{1}{\overline{P}}^h_{ijk},
$$

where the geometric object $\underset{1}{P}^h_{ijk}$ is defined by (3.11) and $\underset{1}{\overline{P}}^h_{ijk}$ is defined in the same manner. Since the generalized Christoffel symbols are not tensors, this geometric object is not a tensor. Analogously, we can consider the case $\theta = 2$. □

Theorem 3.4. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be two generalized hyperbolic Kähler spaces and $f : M \to \overline{M}$ be an equitorsion holomorphically projective mapping, then the geometric objects given by*

$$
\begin{aligned}
\underset{\theta}{P}^h_{ijk} = \underset{\theta}{R}^h_{ijk} + \frac{1}{n+2}\Big[&\delta^h_j(\underset{\theta}{R}_{ik} - Q_{ik} - \underset{1}{T}^p_{ik}\Gamma^s_{ps}) + \delta^h_i(\underset{\theta}{R}_{[jk]} - Q_{[jk]} - \underset{1}{T}^p_{jk}\Gamma^s_{ps})\big] \\
&- \delta^h_k(\underset{\theta}{R}_{ij} - Q_{ij}) - F^h_k(\underset{\theta}{R}_{pj} - Q_{pj})F^p_i + F^h_j(\underset{\theta}{R}_{pk} - Q_{pk} - \underset{1}{T}^q_{pk}\Gamma^s_{qs})F^p_i \\
&- F^h_i\big((\underset{\theta}{R}_{pj} - Q_{pj})F^p_k - (\underset{\theta}{R}_{pk} - Q_{pk} - \underset{1}{T}^q_{pk}\Gamma^s_{qs})F^p_j\big) \\
&+ \underset{1}{T}^h_{ji}\Gamma^s_{ks} + \underset{1}{T}^h_{ki}\Gamma^s_{js} + \underset{1}{T}^h_{pi}F^p_j\Gamma^s_{qs}F^q_k + \underset{1}{T}^h_{pi}F^p_k\Gamma^s_{qs}F^q_j\Big], \quad \theta = 3, 4,
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{ij} = \frac{1}{n+2}\Big(&2\underset{1}{T}^p_{ji}\Gamma^s_{ps} + \underset{1}{T}^r_{pi}F^p_r\Gamma^s_{qs}F^q_j - \underset{1}{T}^r_{pj}F^p_r\Gamma^s_{qs}F^q_i \\
&+ \frac{4}{n-2}\underset{1}{T}^r_{ip}F^p_j\Gamma^s_{qs}F^q_r + \frac{2n}{n-2}\underset{1}{T}^r_{jp}F^p_i\Gamma^s_{qs}F^q_r \\
&+ \frac{n+2}{n-2}\underset{1}{T}^r_{pq}F^p_rF^q_i\Gamma^s_{js} + \frac{n+2}{n-2}\underset{1}{T}^r_{pq}F^p_rF^q_j\Gamma^s_{is}\Big),
\end{aligned}
$$

are invariant with respect to the mapping f.

Proof. The curvature tensors $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ of the generalized hyperbolic Kähler spaces (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$, respectively, satisfy the relation [15]

$$
\begin{aligned}
\underset{3}{\overline{R}}(X,Y)Z = \underset{3}{R}(X,Y)Z &+ \underset{2}{\nabla}_X\underset{1}{P}(Z,Y) - \underset{1}{\nabla}_Y\underset{1}{P}(X,Z) + \underset{1}{P}(X, \underset{1}{P}(Z,Y)) \\
&- \underset{1}{P}(\underset{1}{P}(X,Z),Y) + \underset{1}{T}(\underset{1}{P}(X,Y),Z).
\end{aligned} \tag{3.25}
$$

Let us denote

$$
\underset{\theta}{\psi}(X,Y) = \underset{\theta}{\nabla}_Y\psi(X) + \psi(X)\psi(Y) + \psi(FX)\psi(FY), \ \theta = 1, 2.
$$

Then, by using the definition of the covariant derivative, we conclude that

$$\underset{2}{\psi}(X,Y) = \underset{1}{\psi}(X,Y) + \psi(\underset{1}{T}(X,Y)). \tag{3.26}$$

After substituting (3.10) into (3.25) and by using (3.26) we obtain that

$$\begin{aligned}\overline{\underset{3}{R}}(X,Y)Z = {} & \underset{3}{R}(X,Y)Z + Y\underset{1}{\psi}(Z,X) + Y\psi(\underset{1}{T}(Z,X)) + Z\sum_{CA(X,Y)}\underset{1}{\psi}(Y,X) \\ & + Z\psi(\underset{1}{T}(Y,X)) - X\underset{1}{\psi}(Z,Y) - FX\underset{1}{\psi}(FZ,Y) \\ & + FY\underset{1}{\psi}(FZ,X) + FY\psi(\underset{1}{T}(FZ,X)) - FZ\big(\underset{1}{\psi}(FX,Y) - \underset{1}{\psi}(FY,X)\big) \\ & + FZ\psi(\underset{1}{T}(FY,X)) + \underset{1}{T}(Y,Z)\psi(X) + \underset{1}{T}(X,Z)\psi(Y) + \underset{1}{T}(FY,Z)\psi(FX) \\ & + \underset{1}{T}(FX,Z)\psi(FY).\end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 3.3. □

Theorem 3.5. *Let (M,g,F) and $(\overline{M},\overline{g},\overline{F})$ be two generalized hyperbolic Kähler spaces and $f : M \to \overline{M}$ be an equitorsion holomorphically projective mapping, then the tensor given by*

$$\underset{5}{P}^h_{ijk} = \underset{5}{R}^h_{ijk} + \frac{1}{n+2}\Big[-\underset{5}{R}_{ij}\delta^h_k + \underset{5}{R}_{ik}\delta^h_j - \underset{5}{R}_{pj}F^p_iF^h_k + \underset{5}{R}_{pk}F^p_iF^h_j + 2\underset{5}{R}_{pk}F^p_jF^h_i\Big], \tag{3.27}$$

is invariant with respect to the mapping f.

Proof. The curvature tensors $\underset{5}{R}$ and $\overline{\underset{5}{R}}$ of the generalized hyperbolic Kähler spaces (M,g,F) and $(\overline{M},\overline{g},\overline{F})$, respectively, are related by [15]

$$\begin{aligned}\overline{\underset{5}{R}}(X,Y)Z = {} & \underset{5}{R}(X,Y)Z + \frac{1}{2}\Big(\underset{3}{\nabla}_X\underset{1}{P}(Z,Y) - \underset{4}{\nabla}_Y\underset{1}{P}(Z,X) + \underset{4}{\nabla}_X\underset{1}{P}(Y,Z) - \underset{3}{\nabla}_Y\underset{1}{P}(X,Z) \\ & + \underset{1}{P}(\underset{1}{P}(Z,Y),X) - \underset{1}{P}(Y,\underset{1}{P}(Z,X)) + \underset{1}{P}(X,\underset{1}{P}(Y,Z)) - \underset{1}{P}(\underset{1}{P}(X,Z),Y)\Big).\end{aligned}$$

Since the mapping f is an equitorsion mapping, the bilinear form $\underset{1}{P}$ is symmetric, therefore the last relation can be rewritten as

$$\begin{aligned}\overline{\underset{5}{R}}(X,Y)Z = {} & \underset{5}{R}(X,Y)Z + \frac{1}{2}(\underset{3}{\nabla}_X + \underset{4}{\nabla}_X)\underset{1}{P}(Y,Z) - \frac{1}{2}(\underset{3}{\nabla}_Y + \underset{4}{\nabla}_Y)\underset{1}{P}(X,Z) \\ & + \underset{1}{P}(\underset{1}{P}(Z,Y),X) - \underset{1}{P}(Y,\underset{1}{P}(Z,X)),\end{aligned}$$

i.e.

$$\begin{aligned}\overline{\underset{5}{R}}(X,Y)Z = {} & \underset{5}{R}(X,Y)Z + \nabla_X\underset{1}{P}(Y,Z) - \nabla_Y\underset{1}{P}(X,Z) \\ & + \underset{1}{P}(\underset{1}{P}(Z,Y),X) - \underset{1}{P}(Y,\underset{1}{P}(Z,X)).\end{aligned}$$

After changing (3.10) in the last relation we obtain that

$$\begin{aligned}\overline{\underset{5}{R}}(X,Y)Z = {} & \underset{5}{R}(X,Y)Z - X\psi(Y,Z) + Y\psi(X,Z) - FX\psi(Y,FZ) \\ & + FY\psi(X,FZ) + 2\psi(X,FY)FZ,\end{aligned} \tag{3.28}$$

where we have denoted

$$\psi(X,Y) = \nabla_Y \psi(X) + \psi(X)\psi(Y) + \psi(FX)\psi(FY).$$

In local coordinates, equation (3.28) reads

$$\underset{5}{\overline{R}}{}^h_{ijk} = \underset{5}{R}{}^h_{ijk} - \delta^h_k \psi_{ji} + \delta^h_j \psi_{ki} - F^h_k \psi_{jp} F^p_i + F^h_j \psi_{kp} F^p_i + 2\psi_{kp} F^p_j F^h_i. \tag{3.29}$$

By contracting the last relation with respect to h and k we get

$$\underset{5}{\overline{R}}_{ij} = \underset{5}{R}_{ij} - (n+2)\psi_{ji}. \tag{3.30}$$

Plugging (3.30) in (3.29) we obtain that

$$\underset{5}{P}{}^h_{ijk} = \underset{5}{\overline{P}}{}^h_{ijk},$$

where the tensor $\underset{5}{P}{}^h_{ijk}$ is defined by (3.27) and the tensor $\underset{5}{\overline{P}}{}^h_{ijk}$ is defined in the same manner. This proves the theorem. □

4. Concluding remarks

In the case when the generalized (non-symmetric) Riemannian metric g is symmetric i.e. has vanishing skew-symmetric part $\underset{\vee}{g}$, a generalized hyperbolic Kähler space reduces to a usual hyperbolic Kähler space. Then the curvature tensors $\underset{\theta}{R}$, $\theta = 1, \dots, 5$ reduce to the Riemannian curvature tensor R and the geometric objects $\underset{\theta}{P}{}^h_{ijk}$, $\theta = 1, \dots, 4$ and the tensor $\underset{5}{P}{}^h_{ijk}$ reduce to the paraholomorphic curvature tensor [19]

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\Big(R_{ik}\delta^h_j - R_{ij}\delta^h_k + R_{pk}F^p_i F^h_j - R_{pj}F^p_i F^h_k + 2R_{pk}F^p_j F^h_i\Big).$$

All these invariant geometric objects can be quite interesting for further investigations.

We have already mentioned that there exist twelve curvature tensors which appeared in various Ricci type identities obtained by Minčić in [11,12], and that he showed in [13] that five of them are linearly independent. In this paper we started with the mentioned five linearly independent curvature tensors and obtained one tensor $\underset{5}{P}{}^h_{ijk}$ which is analogous to the paraholomorphic curvature tensor. An open question is: Does there exist any other tensor which would be reduced to the paraholomorphic curvature tensor in the usual hyperbolic Kähler space and, if the answer is affirmative, which is the maximum number of such tensors?

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References

[1] V. Cruceanu, P. Fortuny, P.M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. Math. 26 (1996) 83–115.
[2] V.V. Domashev, J. Mikeš, Theory of holomorphically projective mappings of Kählerian spaces, Mat. Zametki 23 (1978) 297–303.
[3] L.P. Eisenhart, Generalized Riemannian spaces, Proc. Natl. Acad. Sci. USA 37 (1951) 311–315.
[4] L.P. Eisenhart, Generalized Riemannian spaces, II, Proc. Natl. Acad. Sci. USA 38 (1952) 505–508.

[5] I. Hinterleitner, J. Mikeš, On F-planar mappings of spaces with affine connections, Note Mat. 27 (2007) 111–118.
[6] T. Janssen, T. Prokopec, Problems and hopes in nonsymmetric gravity, J. Phys. A: Math. Theor. 40 (2007) 7067–7074.
[7] J. Mikeš, On holomorphically projective mappings of Kählerian spaces, Ukr. Geom. Sb. 23 (1980) 90–98.
[8] J. Mikeš, Holomorphically projective mappings and their generalizations, J. Math. Sci. (N. Y.) 89 (1998) 1334–1353.
[9] J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic Mappings and Some Generalizations, Palacky University, Faculty of Science, Olomouc, 2009.
[10] J. Mikeš, H. Chudá, I. Hinterleitner, Conformal holomorphically projective mappings of almost Hermitian manifolds with a certain initial condition, Int. J. Geom. Methods Mod. Phys. 11 (2014) 1450044.
[11] S.M. Minčić, Ricci identities in the space of non-symmetric affine connexion, Mat. Vesnik 10 (1973) 161–172.
[12] S.M. Minčić, New commutation formulas in the non-symmetric affine connexion space, Publ. Inst. Math., N. S. 22 (1977) 189–199.
[13] S.M. Minčić, Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion, in: Differential Geometry, Budapest, 1979, in: Colloq. Math. Soc. János Bolyai, vol. 31, North-Holland, Amsterdam, 1982, pp. 445–460.
[14] S.M. Minčić, On Ricci type identities in manifolds with non-symmetric affine connection, Publ. Inst. Math., N. S. 94 (2013) 205–217.
[15] S.M. Minčić, M.S. Stanković, Equitorsion geodesic mappings of generalized Riemannian spaces, Publ. Inst. Math., N. S. 61 (1997) 97–104.
[16] S.M. Minčić, M.S. Stanković, Lj.S. Velimirović, Generalized Kählerian spaces, Filomat 15 (2001) 167–174.
[17] S.M. Minčić, M.S. Stanković, Lj.S. Velimirović, Generalized Riemannian Spaces and Spaces of Non-symmetric Affine Connection, University of Niš, Faculty of Sciences and Mathematics, Niš, 2013.
[18] J.W. Moffat, A new nonsymmetric gravitational theory, Phys. Lett. B 355 (1995) 447–452.
[19] M. Prvanović, Holomorphically projective transformations in a locally product space, Math. Balkanica 1 (1971) 195–213.
[20] M. Prvanović, Four curvature tensors of non-symmetric affine connexion, in: Proc. Conf.: "150 Years of Lobachevsky Geometry", Moscow, 1977, pp. 199–205 (in Russian).
[21] M. Prvanović, A note on holomorphically projective transformations of the Kähler spaces, Tensor (N. S.) 35 (1981) 99–104.
[22] M.S. Stanković, S.M. Minčić, Lj.S. Velimirović, On holomorphically projective mappings of generalized Kählerian spaces, Mat. Vesnik 54 (2002) 195–202.
[23] M.S. Stanković, S.M. Minčić, Lj.S. Velimirović, On equitorsion holomorphically projective mappings of generalized Kählerian spaces, Czechoslovak Math. J. 54 (2004) 701–715.
[24] M.S. Stanković, M.Lj. Zlatanović, Lj.S. Velimirović, Equitorsion holomorphically projective mappings of generalized Kählerian space of the first kind, Czechoslovak Math. J. 60 (2010) 635–653.
[25] K. Yano, Differential Geometry of Complex and Almost Complex Spaces, Pergamon Press, New York, 1965.
[26] M.Lj. Zlatanović, New projective tensors for equitorsion geodesic mappings, Appl. Math. Lett. 25 (2012) 890–897.
[27] M.Lj. Zlatanović, Lj.S. Velimirović, M.S. Stanković, Necessary and sufficient conditions for equitorsion geodesic mapping, J. Math. Anal. Appl. 435 (2016) 578–592.