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Generalized Zalcman conjecture for some classes of analytic functions ☆

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A B S T R A C T

For functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in various subclasses of normalized analytic functions, we consider the problem of estimating the generalized Zalcman coefficient functional $\phi(f, n, m; \lambda) := |\lambda a_n a_m - a_{n+m-1}|$. For all real parameters λ and $\beta < 1$, we provide the sharp upper bound of $\phi(f, n, m; \lambda)$ for functions f satisfying $\mathrm{Re} f'(z) > \beta$ and hence settle the open problem of estimating $\phi(f, n, m; \lambda)$ recently proposed by Agrawal and Sahoo (2016) [1]. For all real values of λ, the estimations of $\phi(f, n, m; \lambda)$ are provided for starlike and convex functions of order α ($\alpha < 1$) which are sharp for $\lambda \le 0$ or for certain positive values of λ. Moreover, for certain positive λ, the sharp estimation of $\phi(f, n, m; \lambda)$ is given when f is a typically real function or a univalent function with real coefficients or is in some subclasses of close-to-convex functions.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ defined on the open unit disc $\mathbb{D}$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{S}_{\mathbb{R}}$ be the class of all functions in $\mathcal{S}$ with real coefficients. For $\alpha < 1$, we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, the classes of functions $f \in \mathcal{A}$ satisfying $\mathrm{Re}\big(zf'(z)/f(z)\big) > \alpha$ and $\mathrm{Re}\big(1 + zf''(z)/f'(z)\big) > \alpha$ respectively. For $0 \le \alpha < 1$, these classes are subclasses of $\mathcal{S}$ and were first introduced by Robertson [22] in 1936. Later, for all $\alpha < 1$, these classes were considered in [23,4]. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ represent the classes of starlike and convex functions respectively. We denote the closed convex hulls of $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ by $H\mathcal{S}^*(\alpha)$ and $H\mathcal{K}(\alpha)$ respectively. The class of typically real functions, denoted by T, consists of all functions in $\mathcal{A}$ which have real values on the real axis and non-real values elsewhere. Denote by $\mathcal{P}$, the class of all analytic functions $p(z) = 1 + c_1z + c_2z^2 + \cdots$ defined on $\mathbb{D}$ such that $\mathrm{Re}p(z) > 0$. The class $\mathcal{P}_{\mathbb{R}}$ consists of all functions in $\mathcal{P}$ with real coefficients.

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In 1916, Bieberbach conjectured the inequality $|a_n| \leq n$ for $f \in \mathcal{S}$. Since then, several attempts were made to prove the Bieberbach conjecture which was finally proved by de Branges in 1985. In 1960, as an approach to prove the Bieberbach conjecture, Lawrence Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ $(n \geq 2)$ for $f \in \mathcal{S}$. This led to several works related to Zalcman conjecture and its generalized version $|\lambda a_n^2 - a_{2n-1}| \leq \lambda n^2 - 2n + 1$ $(\lambda \geq 0)$ for various subclasses of $\mathcal{S}$ [18,20,6,16,5,14] but the Zalcman conjecture remained open for many years for the class $\mathcal{S}$. However, for $n \leq 6$, Krushkal [11] proved the conjecture for the class $\mathcal{S}$ by using holomorphic homotopy of univalent functions and with the similar geometric idea, he has recently proved it for all $n \geq 2$ in his unpublished work [12].

In 1999, Ma [19] proposed a generalized Zalcman conjecture for $f \in \mathcal{S}$ that

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \quad (n \geq 2, m \geq 2)$$

which is still an open problem, however he proved it for the classes $\mathcal{S}^*$ and $\mathcal{S}_{\mathbb{R}}$. For $\lambda \in \mathbb{R}$, let $\phi(f, n, m; \lambda) := |\lambda a_n a_m - a_{n+m-1}|$ denote the generalized Zalcman coefficient functional over $\mathcal{A}$. For $\beta < 1$, the class $\mathcal{C}(\beta)$ of close-to-convex functions of order β consists of $f \in \mathcal{A}$ such that $\operatorname{Re}\big(zf'(z)/\big(e^{i\theta}g(z)\big)\big) > \beta$ for some $g \in \mathcal{S}^*$ and $\theta \in \mathbb{R}$. For $0 \leq \beta < 1$, the class $\mathcal{C}(\beta)$ is a subclass of $\mathcal{S}$ and was considered in [17] in a more general form. The class of close-to-convex functions is denoted by $\mathcal{C} := \mathcal{C}(0)$, for details, see [8]. Let $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ be the subclasses of $\mathcal{C}(\beta)$ $(\beta < 1)$ corresponding to $\theta = 0$ and the starlike functions $g(z) = z/(1-z)$ and $g(z) = z/(1-z^2)$ respectively. For $\beta < 1$, let $\mathcal{R}(\beta)$ denote the class of functions $f \in \mathcal{A}$ satisfying $\operatorname{Re} f'(z) > \beta$. For $0 \leq \beta < 1$, $\mathcal{R}(\beta)$ is a subclass of $\mathcal{S}$ and was first introduced in [9]. Here, we are interested in $\mathcal{R}(\beta)$ for all values of β $(\beta < 1)$. Recently, for some positive values of λ and $0 \leq \beta < 1$, Agrawal and Sahoo [1] gave the sharp estimation of $\phi(f, n, m; \lambda)$ for the classes $\mathcal{R}(\beta)$ and $H\mathcal{K}$.

In this paper, for all real values of λ, we give the sharp estimation of $\phi(f, n, m; \lambda)$ for $f \in \mathcal{R}(\beta)$ $(\beta < 1)$. Also, for $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{K}(\alpha)$ $(\alpha < 1)$, the estimations of $\phi(f, n, m; \lambda)$ are given for all real values of λ which are sharp when $\lambda \leq 0$ or when λ is taking certain positive values. Moreover, for certain positive values of λ, the sharp estimations of $\phi(f, n, m; \lambda)$ are provided for the classes T, $\mathcal{S}_{\mathbb{R}}$, $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ $(\beta < 1)$.

We prove our results either by applying the well-known estimation of $|\lambda c_n c_m - c_{n+m}|$ for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ or by applying some characterization of functions in the class $\mathcal{P}$ and that of typically real functions in terms of some positive semi-definite Hermitian form, see [13,21]. Earlier, such characterization of functions with positive real part in terms of some positive semi-definite Hermitian form [13] was used in [3,2,21]. It should be pointed out that in the literature, for various subclasses of $\mathcal{S}$ which are invariant under rotations, the estimation of $\phi(f, n, n; \lambda)$ is usually obtained by using the fact that the expression $\phi(f, n, n; \lambda)$ is invariant under rotations and by an application of the Cauchy–Schwarz inequality which requires λ to be non-negative. However, we are able to give the sharp estimation of $\phi(f, n, m; \lambda)$ for various subclasses of $\mathcal{A}$ when $\lambda \leq 0$. Moreover, for certain positive λ, our technique is giving the estimation of $\phi(f, n, m; \lambda)$ when f is in some subclasses of $\mathcal{A}$ which are not necessarily invariant under rotations. We need the following lemmas to prove our results.

Lemma 1.1. *[21, Lemma 2.3, p. 507] If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then for all $n, m \in \mathbb{N}$,*

$$|\mu c_n c_m - c_{n+m}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \textit{elsewhere}. \end{cases}$$

The result is sharp.

Lemma 1.2. *[13, Theorem 4(b), p. 678] A function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| 2z_j + \sum_{k=1}^{\infty} c_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} c_{k+1} z_{k+j} \right|^2 \right\} \geq 0$$

for every sequence $\{z_k\}$ of complex numbers which satisfy $\limsup_{k\to\infty} |z_k|^{1/k} < 1$.

Lemma 1.3. *[13, Theorem 4(f), p. 678] A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in T$ if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| 2z_j + \sum_{k=1}^{\infty} (a_{k+1} - a_{k-1}) z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} (a_{k+2} - a_k) z_{k+j} \right|^2 \right\} \geq 0$$

for every sequence $\{z_k\}$ of complex numbers which satisfy $\limsup_{k\to\infty} |z_k|^{1/k} < 1$.

Lemma 1.4. *Let $\nu(t)$ be a probability measure on $[0, 2\pi]$. Then for all $n, m \in \mathbb{N}$,*

$$\left| \lambda \int_0^{2\pi} e^{int}\, d\nu(t) \int_0^{2\pi} e^{imt}\, d\nu(t) - \int_0^{2\pi} e^{i(n+m)t}\, d\nu(t) \right| \leq \begin{cases} 1, & 0 \leq \lambda \leq 2; \\ |\lambda - 1|, & \textit{elsewhere}. \end{cases}$$

Proof. The function $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given by the Herglotz representation formula [10, Corollary 3.6, p. 30],

$$p(z) = \int_0^{2\pi} \frac{1 + e^{it} z}{1 - e^{it} z}\, d\nu(t)$$

is clearly in $\mathcal{P}$. On comparing the coefficients on both sides in the above equation, we obtain

$$c_n = 2 \int_0^{2\pi} e^{int}\, d\nu(t) \quad (n \geq 1).$$

An application of Lemma 1.1 to the function p gives

$$\left| 2\mu \int_0^{2\pi} e^{int}\, d\nu(t) \int_0^{2\pi} e^{imt}\, d\nu(t) - \int_0^{2\pi} e^{i(n+m)t}\, d\nu(t) \right| \leq \begin{cases} 1, & 0 \leq \mu \leq 1; \\ |2\mu - 1|, & \text{elsewhere}. \end{cases}$$

On substituting $\lambda = 2\mu$, the desired estimates follow. □

For $\lambda = 2$, the above lemma is proved in [19, Lemma 2.1, p. 330].

2. Generalized Zalcman conjecture for $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$

For $\alpha < 1$, define a function $f_1 : \mathbb{D} \to \mathbb{C}$ by

$$f_1(z) := \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} A_n z^n, \tag{1}$$

where

$$A_n = \frac{1}{(n-1)!}\prod_{j=0}^{n-2}\big(2(1-\alpha)+j\big). \tag{2}$$

It is known that f_1 and its rotations work as extremal functions for the coefficient bounds of functions in the class $\mathcal{S}^*(\alpha)$ [23, Theorem 5.6, p. 324]. Therefore, they could be the expected extremal functions for the upper bound of the generalized Zalcman coefficient functional $\phi(f,n,m;\lambda)$ when $f\in\mathcal{S}^*(\alpha)$. This is shown to be true by the following theorem at least when $\lambda \geq 2A_{n+m-1}/(A_nA_m)$ or $\lambda \leq 0$.

Theorem 2.1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in H\mathcal{S}^*(\alpha)$ $(\alpha<1)$, then for all $n,m=2,3,\dots$,*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} A_{n+m-1}, & 0\leq\lambda\leq \dfrac{2A_{n+m-1}}{A_nA_m};\\ |\lambda A_nA_m - A_{n+m-1}|, & \textit{elsewhere}, \end{cases}$$

where A_n is given by (2). The second inequality is sharp for the function f_1 and its rotations where f_1 is given by the equation (1).

Proof. Since $f\in H\mathcal{S}^*(\alpha)$ $(\alpha<1)$, there exists a probability measure $\nu(t)$ on $[0,2\pi]$ [4, Theorem 3, p. 417] such that

$$f(z)=\int_0^{2\pi}\frac{z}{(1-e^{it}z)^{2(1-\alpha)}}\,d\nu(t).$$

On comparing the coefficients on both sides, we obtain

$$a_n = A_n\int_0^{2\pi} e^{i(n-1)t}\,d\nu(t) \quad (n\geq 2),$$

where A_n is given by the equation (2). This implies

$$\begin{aligned}&|\lambda a_n a_m - a_{n+m-1}|\\ &= A_{n+m-1}\left|\lambda\frac{A_nA_m}{A_{n+m-1}}\int_0^{2\pi}e^{i(n-1)t}\,d\nu(t)\int_0^{2\pi}e^{i(m-1)t}\,d\nu(t)-\int_0^{2\pi}e^{i(n+m-2)t}\,d\nu(t)\right|.\end{aligned}$$

An application of Lemma 1.4 to the above equation yields

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} A_{n+m-1}, & 0\leq\lambda\leq \dfrac{2A_{n+m-1}}{A_nA_m};\\ |\lambda A_nA_m - A_{n+m-1}|, & \text{elsewhere}. \end{cases} \qquad \square$$

For $m=n$, we have the following sharp result.

Corollary 2.2. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in H\mathcal{S}^*(\alpha)$ $(\alpha<1)$, then for all $n=2,3,\dots$,*

$$|\lambda a_n^2 - a_{2n-1}| \leq \begin{cases} A_{2n-1}, & 0\leq\lambda\leq \dfrac{2A_{2n-1}}{A_n^2};\\ |\lambda A_n^2 - A_{2n-1}|, & \textit{elsewhere}, \end{cases}$$

where A_n is given by (2). The second inequality is sharp for the function f_1, given by the equation (1), and its rotations whereas the first inequality is sharp for the function of the form

$$f(z) = \sum_{k=1}^{2(n-1)} m_k g_k(z), \tag{3}$$

where $0 \le m_k \le 1$, $\sum_{k=1}^{n-1} m_{2k} = \sum_{k=1}^{n-1} m_{2k-1} = 1/2$, $g_k(z) = e^{-i\theta_k} f_1(e^{i\theta_k} z)$ and $\theta_k = (2k+1)\pi/(2n-2)$.

For $\alpha = 0$ and $\lambda \ge 0$, the above corollary reduces to the inequalities mentioned in [5, p. 474]. It is a well-known result given by Alexander that a function $f \in \mathcal{A}$ is in $\mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. This implies that for $\alpha < 1$, $f \in H\mathcal{K}(\alpha)$ if and only if $zf'(z) \in H\mathcal{S}^*(\alpha)$ and therefore, we have the following deduction from the Theorem 2.1.

Corollary 2.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H\mathcal{K}(\alpha)$ $(\alpha < 1)$, then for all $n, m = 2, 3, \dots,$*

$$|\lambda a_n a_m - a_{n+m-1}| \le \begin{cases} \dfrac{A_{n+m-1}}{n+m-1}, & 0 \le \lambda \le \dfrac{2nmA_{n+m-1}}{(n+m-1)A_n A_m}; \\ \left|\lambda \dfrac{A_n A_m}{nm} - \dfrac{A_{n+m-1}}{n+m-1}\right|, & \text{elsewhere,} \end{cases}$$

where A_n is given by the equation (2). The second inequality is sharp for the function f_2 and its rotations, where

$$f_2(z) = \begin{cases} \dfrac{(1-z)^{-(1-2\alpha)} - 1}{1-2\alpha}, & \alpha \ne 1/2; \\ -\log(1-z), & \alpha = 1/2. \end{cases} \tag{4}$$

For $\alpha = 0$ and $\lambda \ge 2$, the above corollary reduces to [1, Theorem 2.1, p. 3]. For $m = n$, we have the following sharp result which has been proved in [16] by maximizing the real-valued functional $\operatorname{Re}(\lambda a_n^2 - a_{2n-1})$ for the case $\lambda \ge 0$.

Corollary 2.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H\mathcal{K}(\alpha)$ $(\alpha < 1)$, then for all $n = 2, 3, \dots,$*

$$\left|\lambda a_n^2 - a_{2n-1}\right| \le \begin{cases} \dfrac{A_{2n-1}}{2n-1}, & 0 \le \lambda \le \dfrac{2n^2 A_{2n-1}}{(2n-1)A_n^2}; \\ \left|\lambda \dfrac{A_n^2}{n^2} - \dfrac{A_{2n-1}}{2n-1}\right|, & \text{elsewhere,} \end{cases}$$

where A_n is given by the equation (2). The second inequality is sharp for the function f_2, given by (4), and its rotations whereas the first inequality is sharp for the function given by the equation (3) with $g_k(z) = e^{-i\theta_k} f_2(e^{i\theta_k} z)$.

If $\lambda \ge 0$, the above corollary reduces to [14, Theorem 3.3] and [16, Theorem 4] for $\alpha = -1/2$ and $\alpha = 1/2$ respectively. Also, for $\alpha = 0$ and $0 \le \lambda \le 2$, the above corollary was proved in [6, Theorem 3, p. 3].

3. Generalized Zalcman conjecture for the class $\mathcal{R}(\beta)$ and for typically real functions

For $\lambda \ge nm/((1-\beta)(n+m-1))$ and $0 \le \beta < 1$, the second inequality of the following theorem has been recently proved by Agrawal and Sahoo [1] and they proposed it as an open problem for $0 < \lambda < nm/((1-\beta)(n+m-1))$ which has now been settled in the following theorem by making use of the Hermitian form for functions in the class $\mathcal{P}$.

Theorem 3.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}(\beta)$ $(\beta < 1)$, then for all $n, m = 2, 3, \ldots,$*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} \dfrac{2(1-\beta)}{n+m-1}, & 0 \leq \lambda \leq \dfrac{nm}{(1-\beta)(n+m-1)}; \\ \left|\dfrac{4\lambda(1-\beta)^2}{nm} - \dfrac{2(1-\beta)}{n+m-1}\right|, & \textit{elsewhere.} \end{cases}$$

The result is sharp.

Proof. Since $f \in \mathcal{R}(\beta)$, $\big(f'(z) - \beta\big)/(1-\beta) = 1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}/(1-\beta)z^n \in \mathcal{P}$ which gives

$$|a_n| \leq \frac{2(1-\beta)}{n} \quad (n \geq 2). \tag{5}$$

Clearly, the bounds are sharp for the function $f_0 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = (1-\beta)\int_0^z \frac{1+t}{1-t}\,dt + \beta z. \tag{6}$$

For fixed $n, m = 2, 3, \ldots,$ choose the sequence $\{z_k\}$ of complex numbers by $z_{n-2} = \lambda(1-\beta)a_m$, $z_{n+m-3} = -n(1-\beta)/(n+m-1)$, $z_k = 0$ for all $k \neq n-2, n+m-3$. An application of Lemma 1.2 to the function $(f' - \beta)/(1-\beta) \in \mathcal{P}$ gives

$$\begin{aligned} & n^2\left|\lambda a_n a_m - a_{n+m-1}\right|^2 \\ & \leq \left|\left(2\lambda(1-\beta) - \frac{mn}{n+m-1}\right)a_m\right|^2 - \left|\frac{mn}{n+m-1}a_m\right|^2 + \frac{4n^2(1-\beta)^2}{(n+m-1)^2} \\ & = 4\lambda(1-\beta)\left(\lambda(1-\beta) - \frac{mn}{n+m-1}\right)|a_m|^2 + \frac{4n^2(1-\beta)^2}{(n+m-1)^2}. \end{aligned}$$

By using the bounds given by (5) in the above inequality, we have

$$|\lambda a_n a_m - a_{n+m-1}|^2 \leq \begin{cases} \dfrac{4(1-\beta)^2}{(n+m-1)^2}, & 0 \leq \lambda \leq \dfrac{nm}{(1-\beta)(n+m-1)}; \\ \left(\dfrac{4\lambda(1-\beta)^2}{nm} - \dfrac{2(1-\beta)}{n+m-1}\right)^2, & \text{elsewhere.} \end{cases}$$

For $0 \leq \lambda \leq nm/\big((1-\beta)(n+m-1)\big)$, the inequality is sharp for the function

$$f(z) = (1-\beta)\int_0^z \frac{1+t^{n+m-2}}{1-t^{n+m-2}}\,dt + \beta z.$$

For $\lambda \leq 0$ or $\lambda \geq nm/\big((1-\beta)(n+m-1)\big)$, the inequality is sharp for the function f_0 given by the equation (6). □

For $\beta = 0$ and $0 < \lambda \leq 4/3$, the above theorem was proved in [6] by maximizing the real valued functional $\mathrm{Re}(\lambda a_n^2 - a_{2n-1})$ over $\mathcal{R}(\beta)$. Also, we have the following simple result.

Corollary 3.2. *If the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies Re$(f(z)/z) > \beta$ $(\beta < 1)$ in $\mathbb{D}$, then for all $n, m = 2, 3, \ldots,$*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \begin{cases} 2(1-\beta), & 0 \leq \lambda \leq \dfrac{1}{(1-\beta)}; \\ 2(1-\beta)\,|2\lambda(1-\beta)-1|\,, & \textit{elsewhere.} \end{cases}$$

The result is sharp.

The following theorem generalizes [19, Theorem 3.1, p. 335] which was proved for $\lambda = 1$ by induction on n and m. Although, it can be proved by induction on n and m but here, we are giving it as an application of the Hermitian form for typically real functions.

Theorem 3.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T$ and $\lambda \geq 1$, then*

(i) if $n = 2$ and m is even, the upper bound of $|\lambda a_n a_m - a_{n+m-1}|$ is
   (a) $3 + (2\lambda - 1)(m-2)$ for $1 \leq \lambda \leq 3/2$,
   (b) $2\lambda m - m - 1$ for $\lambda \geq 3/2$;
(ii) if $m = 2$ and n is even, the upper bound of $|\lambda a_n a_m - a_{n+m-1}|$ is
   (a) $3 + (2\lambda - 1)(n-2)$ for $1 \leq \lambda \leq 3/2$,
   (b) $2\lambda n - n - 1$ for $\lambda \geq 3/2$;
(iii) in the other cases, we have

$$|\lambda a_n a_m - a_{n+m-1}| \leq \lambda mn - n - m + 1.$$

The bounds given by (i)(b), (ii)(b) and (iii) are sharp whereas the bounds in (i)(a) and (ii)(a) are sharp for $\lambda = 1$ or the case when $n = 2$ and $m = 2$.

Proof. For fixed $n, m = 2, 3, \ldots$, choose the sequence $\{z_k\}$ of real numbers by $z_{n-2} = \lambda a_m$, $z_{n+m-3} = -1$, $z_k = 0$ for all $k \neq n-2, n+m-3$. Since $f \in T$, $|a_n| \leq n$ $(n \geq 2)$. So, by using Lemma 1.3 to the function $f \in T$, we have

$$\begin{aligned} |(\lambda a_n a_m - a_{n+m-1}) - (\lambda a_{n-2} a_m - a_{n+m-3})|^2 &\leq |(2\lambda - 1)a_m + a_{m-2}|^2 - |a_m - a_{m-2}|^2 + 4 \\ &= 4\lambda(\lambda - 1)a_m^2 + 4\lambda a_m a_{m-2} + 4 \\ &\leq 4(\lambda m - 1)^2. \end{aligned} \tag{7}$$

Since $f \in T$, therefore $f(z) = \big(z/(1-z^2)\big)p(z)$ for some $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}_{\mathbb{R}}$. This gives

$$a_{2k} = c_1 + c_3 + \cdots + c_{2k-1} \quad \text{and} \quad a_{2k+1} = 1 + c_2 + c_4 + \cdots + c_{2k}. \tag{8}$$

By [5, Theorem 1, p. 468], we have $\lambda a_2^2 - a_3 \leq 4\lambda - 3$. Clearly, $\lambda a_2^2 - a_3 \geq -a_3 \geq -3$. Also, we observe that $1 \leq \lambda \leq 3/2$ is equivalent to $1 \leq 4\lambda - 3 \leq 3$. Therefore, we have

$$|\lambda a_2^2 - a_3| \leq \begin{cases} 3, & \text{for } 1 \leq \lambda \leq 3/2, \\ 4\lambda - 3, & \text{for } \lambda \geq 3/2. \end{cases} \tag{9}$$

The first inequality in (9) is sharp for the function $f(z) = z(1+z^2)/(1-z^2)^2$ and the second inequality holds for the Koebe function $k(z) = z/(1-z)^2$. If $n = 2$ and $m = 2k$ $(k \geq 2)$, then

$$\begin{aligned} |\lambda a_2 a_m - a_{m+1}| &= |\lambda a_2 a_{2k} - a_{2k+1}| \\ &= |\lambda c_1(c_1 + c_3 + \cdots + c_{2k-1}) - (1 + c_2 + c_4 + \cdots + c_{2k})| \end{aligned}$$

$$\begin{aligned}&= |(\lambda c_1^2 - c_2 - 1) + (\lambda c_1 c_3 - c_4) + \cdots + (\lambda c_1 c_{2k-1} - c_{2k})|\\ &= |(\lambda a_2^2 - a_3) + (\lambda c_1 c_3 - c_4) + \cdots + (\lambda c_1 c_{2k-1} - c_{2k})|.\end{aligned} \tag{10}$$

An application of Lemma 1.1 and the inequality (9) in the equation (10) gives

$$|\lambda a_2 a_m - a_{m+1}| \leq \begin{cases} 3 + (2\lambda - 1)(m-2), & \text{for } 1 \leq \lambda \leq 3/2; \\ 2\lambda m - m - 1, & \text{for } \lambda \geq 3/2. \end{cases}$$

This proves (i).

When $m = 2$ and n is even, the desired bounds in (ii) follow by interchanging the roles of n and m in the equation (10) and in the above inequality. For $\lambda = 1$, the sharpness in $(i)(a)$ and $(ii)(a)$ follow for the function $f(z) = z(1 + z^2)/(1 - z^2)^2$. Now, it is left to prove the inequality in the case (iii). Since $\lambda a_4^2 - a_7 \leq 16\lambda - 7$ [5, Theorem 1, p. 468] and clearly $\lambda a_4^2 - a_7 \geq -7 \geq -9 \geq -(16\lambda - 7)$, we have

$$|\lambda a_4^2 - a_7| \leq 16\lambda - 7. \tag{11}$$

For $n = 4$ and $m = 2k$ $(k \geq 3)$, by proceeding as in the equation (10), we have

$$\begin{aligned}&|\lambda a_4 a_m - a_{m+3}|\\ &= |\lambda a_4 a_{2k} - a_{2k+3}|\\ &= |\lambda(c_1 + c_3)(c_1 + c_3 + \cdots + c_{2k-1}) - (1 + c_2 + c_4 + \cdots + c_{2(k+1)})|\\ &= |(\lambda a_4^2 - a_7) + \lambda c_1(c_5 + \cdots + c_{2k-1}) + (\lambda c_3 c_5 - c_8) + \cdots + (\lambda c_3 c_{2k-1} - c_{2k+2})|.\end{aligned}$$

An application of Lemma 1.1 and the inequality (11) in the above equation gives

$$|\lambda a_4 a_m - a_{m+3}| \leq 4\lambda m - m - 3. \tag{12}$$

Therefore, if n, m are even and $n > 4$, $m > 2$, then

$$\begin{aligned}|\lambda a_n a_m - a_{n+m-1}| \leq{}& |(\lambda a_n a_m - a_{n+m-1}) - (\lambda a_{n-2} a_m - a_{n+m-3})|\\ &+ |(\lambda a_{n-2} a_m - a_{n+m-3}) - (\lambda a_{n-4} a_m - a_{n+m-5})| + \cdots\\ &+ |(\lambda a_6 a_m - a_{m+5}) - (\lambda a_4 a_m - a_{m+3})| + |\lambda a_4 a_m - a_{m+3}|.\end{aligned} \tag{13}$$

In view of (7), (11) and (12), we have

$$|\lambda a_n a_m - a_{n+m-1}| \leq (\lambda m - 1)(n - 4) + 4m\lambda - m - 3 = \lambda mn - m - n + 1.$$

Next, we consider the case when n is even and m is odd. If $n = 2$ and $m = 2k+1$ $(k \geq 1)$, then by proceeding similarly as in the equation (10) and applying Lemma 1.1, we obtain

$$\begin{aligned}|\lambda a_2 a_m - a_{m+1}| &= |\lambda a_2 a_{2k+1} - a_{2k+2}|\\ &= |\lambda c_1(1 + c_2 + c_4 + \cdots + c_{2k}) - (c_1 + c_3 + \cdots + c_{2k+1})|\\ &\leq m(2\lambda - 1) - 1.\end{aligned} \tag{14}$$

If $n = 2k$ $(k > 1)$ and m is odd, then by proceeding as in the inequality (13) and applying (7) and (14), we have

$$|\lambda a_n a_m - a_{n+m-1}| \leq 2(\lambda m - 1)(k-1) + m(2\lambda - 1) - 1 = \lambda mn - m - n + 1.$$

Finally, we consider the case when n is odd. In this case, we have

$$\begin{aligned} |\lambda a_n a_m - a_{n+m-1}| \leq\, & |(\lambda a_n a_m - a_{n+m-1}) - (\lambda a_{n-2} a_m - a_{n+m-3})| \\ & + |(\lambda a_{n-2} a_m - a_{n+m-3}) - (\lambda a_{n-4} a_m - a_{n+m-5})| + \cdots \\ & + |(\lambda a_3 a_m - a_{m+2}) - (\lambda a_1 a_m - a_m)| + |\lambda a_1 a_m - a_m| \quad (a_1 = 1). \end{aligned}$$

Using inequality (7) and the bound of $|a_m|$ in the above inequality, we obtain

$$|\lambda a_n a_m - a_{n+m-1}| \leq \lambda mn - m - n + 1.$$

The sharpness in the cases $(i)(b)$, $(ii)(b)$ and (iii) follow for the Koebe function $k(z) = z/(1-z)^2$. □

For $\lambda = 1$, the following result is given in [19, Theorem 3.2, p. 338].

Corollary 3.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_{\mathbb{R}}$ and $\lambda \geq 1$, then for $n, m = 2, 3, \ldots,$*

$$|\lambda a_n a_m - a_{n+m-1}| \leq \lambda mn - n - m + 1.$$

The result is sharp.

Proof. Since $\mathcal{S}_{\mathbb{R}} \subset \mathcal{S}$, by using [7, Theorem 2, p. 35], we have

$$|a_2^2 - a_3| \leq 1. \tag{15}$$

Also, $\mathcal{S}_{\mathbb{R}} \subset T$, therefore for $\lambda \geq 1$, by [5, Theorem 1, p. 468], we have $\lambda a_2^2 - a_3 \leq 4\lambda - 3$. For $1 \leq \lambda \leq 3/2$, an application of the inequality (15) gives $\lambda a_2^2 - a_3 \geq a_2^2 - a_3 \geq -1 \geq -(4\lambda - 3)$. Thus, in view of the inequality (9), we must have the sharp inequality

$$|\lambda a_2^2 - a_3| \leq 4\lambda - 3 \tag{16}$$

where the sharpness follows for the Koebe function $k(z) = z/(1-z)^2$. For even $m > 2$, an application of (16) and Lemma 1.1 in the equation (10) gives

$$|\lambda a_2 a_m - a_{m+1}| \leq 2m\lambda - m - 1.$$

When $m = 2$ and $n > 2$ is even, the desired estimate follows by interchanging the roles of m and n in the above inequality. The other cases follow immediately from the Theorem 3.3. The result is sharp for the Koebe function. □

4. Generalized Zalcman conjecture for some subclasses of close-to-convex functions

Recall that the classes $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ $(\beta < 1)$ are defined as follows:

$$\mathcal{F}_1(\beta) := \{f \in \mathcal{A} : \mathrm{Re}\big((1-z)f'(z)\big) > \beta\}$$

and

$$\mathcal{F}_2(\beta) := \{f \in \mathcal{A} : \operatorname{Re}\big((1-z^2)f'(z)\big) > \beta\}.$$

For $0 \leq \beta < 1$, the classes $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ are subclasses of $\mathcal{C}$, the class of close-to-convex functions. Define the functions $f_{1,\beta} : \mathbb{D} \to \mathbb{C}$ and $f_{2,\beta} : \mathbb{D} \to \mathbb{C}$, in $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ respectively, by

$$f_{1,\beta}(z) = \frac{2(1-\beta)z}{1-z} + (1-2\beta)\log(1-z) \tag{17}$$

and

$$f_{2,\beta}(z) = \frac{z(1-\beta)}{1-z^2} + \frac{\beta}{2}\log\left(\frac{1+z}{1-z}\right).$$

Recently, for certain positive values of λ, the sharp estimation of $\phi(f,n,n;\lambda)$ over $\mathcal{C}$ is given in [15] by using the fact that $\mathcal{C}$ and $\phi(f,n,n;\lambda)$ are invariant under rotations. Note that the classes $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ are not necessarily invariant under rotations. For instance, $\mathcal{F}_1(0)$ and $\mathcal{F}_2(0)$ are not invariant under rotations since $\operatorname{Re}\Big((1-z)\big(-if_{1,0}(iz)\big)'\Big) = -2$ at $z = 1/2 - i/2$ and $(1-z^2)\big(-if_{2,0}(iz)\big)' = (1-z^2)^2/(1+z^2)^2$ maps $\mathbb{D}$ to the whole complex plane except the negative real axis. In this section, for certain positive values of λ, we give the sharp estimation of the generalized Zalcman coefficient functional $\phi(f,n,m;\lambda)$ when $f \in \mathcal{F}_1(\beta)$ or $f \in \mathcal{F}_2(\beta)$.

Theorem 4.1. *If $\mu \geq \max\{nm/\big((n+m-1)(1-\beta)\big), nm/(n+m-1)\}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}_1(\beta)$ $(\beta < 1)$, then for all $n, m = 2, 3, \ldots$,*

$$|\mu a_n a_m - a_{n+m-1}| \leq \mu B_n B_m - B_{n+m-1},$$

where

$$B_n = \frac{1 + 2(n-1)(1-\beta)}{n} \quad (n \geq 2). \tag{18}$$

The inequality is sharp.

Proof. Let $g(z) := (1-z)f'(z)$. Since $f \in \mathcal{F}_1(\beta)$, therefore

$$\frac{g(z)-\beta}{1-\beta} = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P},$$

which gives

$$c_n = \frac{(n+1)a_{n+1} - na_n}{1-\beta} \quad (n \geq 1)$$

and

$$a_n = \frac{1 + (1-\beta)(c_1 + c_2 + \cdots + c_{n-1})}{n} \quad (n \geq 2). \tag{19}$$

Since $|c_n| \leq 2$ $(n \geq 1)$, the equation (19) gives

$$|a_n| \leq B_n, \tag{20}$$

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where B_n is given by the equation (18). For fixed $n, m = 2, 3, \ldots$ and $\lambda \in \mathbb{R}$, choose the sequence $\{z_k\}$ of complex numbers by $z_{n-2} = \lambda(1-\beta)a_m$, $z_{n+m-3} = -(1-\beta)$, $z_k = 0$ for all $k \neq n-2, n+m-3$. Then Lemma 1.2 yields

$$\begin{aligned}
&\left|\left(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\right) - \left(\lambda(n-1)a_{n-1}a_m - (n+m-2)a_{n+m-2}\right)\right|^2 \\
&\leq |2\lambda(1-\beta)a_m - m a_m + (m-1)a_{m-1}|^2 - |m a_m - (m-1)a_{m-1}|^2 + 4(1-\beta)^2 \\
&= 4\lambda(1-\beta)\left(\lambda(1-\beta) - m\right)|a_m|^2 + 4(m-1)\lambda(1-\beta)\mathrm{Re}\, a_m\overline{a_{m-1}} + 4(1-\beta)^2.
\end{aligned}$$

If $\lambda \geq \max\{m/(1-\beta), m\}$, then by using equation (20) in the above inequality, we obtain

$$\begin{aligned}
&|\left(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\right) - \left(\lambda(n-1)a_{n-1}a_m - (n+m-2)a_{n+m-2}\right)|^2 \\
&\leq 4(1-\beta)^2\left(\lambda B_m - 1\right)^2.
\end{aligned} \tag{21}$$

For $\lambda \geq \max\{m/(1-\beta), m\}$, consider

$$\begin{aligned}
&|\lambda n a_n a_m - (n+m-1)a_{n+m-1}| \\
&\leq |\left(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\right) - \left(\lambda(n-1)a_{n-1}a_m - (n+m-2)a_{n+m-2}\right)| + \cdots \\
&\quad + |\left(2\lambda a_2 a_m - (m+1)a_{m+1}\right) - \left(\lambda a_1 a_m - m a_m\right)| + |\lambda a_1 a_m - m a_m| \quad (a_1 = 1).
\end{aligned}$$

By applying the inequality (21) and the bounds given by (20) in the above inequality, we have

$$(n+m-1)\left|\frac{\lambda n}{n+m-1}a_n a_m - a_{n+m-1}\right| \leq 2(1-\beta)\left(\lambda B_m - 1\right)(n-1) + (\lambda - m)B_m.$$

On substituting $\mu = \lambda n/(n+m-1)$ in the above inequality and simplifying, we obtain

$$|\mu a_n a_m - a_{n+m-1}| \leq \mu B_n B_m - B_{n+m-1}$$

where $\mu \geq \max\{nm/\left((n+m-1)(1-\beta)\right), nm/(n+m-1)\}$ and B_n is given by (18). The result is sharp for the function $f_{1,\beta}$ given by (17). □

For $\beta = 0$ and $m = n$, we have the following.

Corollary 4.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}_1(0)$ and $\mu \geq n^2/(2n-1)$, then*

$$|\mu a_n^2 - a_{2n-1}| \leq \frac{\mu(2n-1)^2}{n^2} + \frac{3-4n}{2n-1}.$$

The result is sharp.

Theorem 4.3. *If $\mu \geq \max\{nm/\left((n+m-1)(1-\beta)\right), nm/(n+m-1)\}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}_2(\beta)$ $(\beta < 1)$, then for all $n, m = 2, 3, \ldots$ except when both n and m are even,*

$$|\mu a_n a_m - a_{n+m-1}| \leq \mu C_n C_m - C_{n+m-1}$$

where, for $n \geq 2$,

$$C_n = \begin{cases} \dfrac{1 + (n-1)(1-\beta)}{n}, & \text{if } n \text{ is odd;} \\ 1-\beta, & \text{if } n \text{ is even.} \end{cases} \tag{22}$$

The result is sharp.

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Proof. Let $g(z) := (1-z^2)f'(z)$. Since $f \in \mathcal{F}_2(\beta)$, therefore $(g(z)-\beta)/(1-\beta) = p(z)$ for some $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$. This gives

$$c_n = \frac{(n+1)a_{n+1} - (n-1)a_{n-1}}{1-\beta},$$
$$a_{2k} = \frac{(1-\beta)(c_1 + c_3 + \cdots + c_{2k-1})}{2k} \tag{23}$$

and

$$a_{2k+1} = \frac{1 + (1-\beta)(c_2 + c_4 + \cdots + c_{2k})}{2k+1}. \tag{24}$$

Since $|c_n| \le 2$ $(n \ge 1)$, the equations (23) and (24) give

$$|a_n| \le C_n \tag{25}$$

for all $n \ge 2$, where C_n is given by the equation (22). Define a function $f_3 : \mathbb{D} \to \mathbb{C}$ by

$$\begin{aligned} f_3(z) &= \frac{z(1-\beta)}{1-z} + \frac{\beta}{2}\log\frac{1+z}{1-z} \\ &= z + C_2 z^2 + C_3 z^3 + C_4 z^4 + C_5 z^5 + \cdots . \end{aligned} \tag{26}$$

Clearly, the bounds given in (25) are sharp for the function f_3.

For fixed $n, m = 2, 3, \ldots$ and $\lambda \in \mathbb{R}$, choose the sequence $\{z_k\}$ of complex numbers by $z_{n-2} = \lambda(1-\beta)a_m$, $z_{n+m-3} = -(1-\beta)$, $z_k = 0$ for all $k \ne n-2, n+m-3$. Then Lemma 1.2 yields

$$\begin{aligned} &\left|\big(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\big) - \big(\lambda(n-2)a_{n-2}a_m - (n+m-3)a_{n+m-3}\big)\right|^2 \\ &\le |2\lambda(1-\beta)a_m - m a_m + (m-2)a_{m-2}|^2 - |m a_m - (m-2)a_{m-2}|^2 + 4(1-\beta)^2 \\ &= 4\lambda(1-\beta)\big(\lambda(1-\beta) - m\big)|a_m|^2 + 4(m-2)\lambda(1-\beta)\mathrm{Re}\, a_m\overline{a_{m-2}} + 4(1-\beta)^2. \end{aligned}$$

If $\lambda \ge \max\{m/(1-\beta), m\}$, then an application of the equation (25) in the previous inequality gives

$$\begin{aligned} &\left|\big(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\big) - \big(\lambda(n-2)a_{n-2}a_m - (n+m-3)a_{n+m-3}\big)\right| \\ &\le 2(1-\beta)\left(\lambda C_m - 1\right). \end{aligned} \tag{27}$$

If $n = 2$ and $m = 2k+1$ $(k \ge 1)$, then

$$\begin{aligned} |2\lambda a_2 a_m - (m+1)a_{m+1}| &= |2\lambda a_2 a_{2k+1} - (2k+2)a_{2k+2}| \\ &= \left|\frac{\lambda(1-\beta)}{2k+1}c_1\left(1 + (1-\beta)\sum_{j=1}^{k} c_{2j}\right) - (1-\beta)\sum_{j=1}^{k+1} c_{2j-1}\right| \\ &= \left|\left(\frac{\lambda}{m} - 1\right)(1-\beta)c_1 + (1-\beta)\sum_{j=1}^{k}\left(\frac{\lambda(1-\beta)}{m}c_1 c_{2j} - c_{2j+1}\right)\right|. \end{aligned}$$

For $\lambda \ge \max\{m/(1-\beta), m\}$, an application of Lemma 1.1 in the above equation gives

$$|2\lambda a_2 a_m - (m+1)a_{m+1}| \le (1-\beta)\big(2\lambda C_m - (1+m)\big) \tag{28}$$

where C_m is given by the equation (22). If $n > 2$ is even and m is odd, then

$$\begin{aligned}
&|\lambda n a_n a_m - (n+m-1)a_{n+m-1}| \\
&\leq |\big(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\big) - \big(\lambda(n-2)a_{n-2}a_m - (n+m-3)a_{n+m-3}\big)| + \cdots \\
&\quad + |\big(4\lambda a_4 a_m - (m+3)a_{m+3}\big) - \big(2\lambda a_2 a_m - (m+1)a_{m+1}\big)| + |2\lambda a_2 a_m - (m+1)a_{m+1}|.
\end{aligned}$$

For $\lambda \geq \max\{m/(1-\beta), m\}$, in view of (27) and (28), we have

$$(n+m-1)\left|\frac{\lambda n}{n+m-1}a_n a_m - a_{n+m-1}\right| \leq (1-\beta)(\lambda n C_m - n - m + 1).$$

On substituting $\mu = \lambda n/(n+m-1)$ in the above inequality and simplifying, we obtain

$$|\mu a_n a_m - a_{n+m-1}| \leq (1-\beta)(\mu C_m - 1)$$

where $\mu \geq \max\{nm/\big((n+m-1)(1-\beta)\big), nm/(n+m-1)\}$. Next, we consider the case when n is odd. In this case, we have

$$\begin{aligned}
&|\lambda n a_n a_m - (n+m-1)a_{n+m-1}| \\
&\leq |\big(\lambda n a_n a_m - (n+m-1)a_{n+m-1}\big) - \big(\lambda(n-2)a_{n-2}a_m - (n+m-3)a_{n+m-3}\big)| + \cdots \\
&\quad + |\big(3\lambda a_3 a_m - (m+2)a_{m+2}\big) - \big(\lambda a_1 a_m - m a_m\big)| + |\lambda a_1 a_m - m a_m| \quad (a_1 = 1).
\end{aligned}$$

For $\lambda \geq \max\{m/(1-\beta), m\}$, use of the equation (27) and the bound of $|a_m|$ in the above inequality give

$$(n+m-1)\left|\frac{\lambda n}{n+m-1}a_n a_m - a_{n+m-1}\right| \leq (n-1)(1-\beta)\left(\lambda C_m - 1\right) + (\lambda - m)C_m$$

where C_m is given by the equation (22). Substitution of $\mu = \lambda n/(n+m-1)$ in the previous inequality and simplification give

$$|\mu a_n a_m - a_{n+m-1}| \leq \mu C_n C_m - C_{n+m-1},$$

where $\mu \geq \max\{nm/\big((n+m-1)(1-\beta)\big), nm/(n+m-1)\}$. The sharpness follows for the function f_3 given by the equation (26). □

In the case when $\mu \geq \max\{nm/\big((n+m-1)(1-\beta)\big), nm/(n+m-1)\}$ and n, m are simultaneously even, the similar procedure which is used in the above theorem or in the Theorem 3.3 gives an upper bound (probably not sharp) of $|\mu a_n a_m - a_{n+m-1}|$, better than the one by the triangle inequality.

For $\beta = 0$ and $m = n = 2k+1$ $(k \geq 1)$, we have the following.

Corollary 4.4. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}_2(0)$ and $\mu \geq (2k+1)^2/(4k+1)$, then*

$$|\mu a_{2k+1}^2 - a_{4k+1}| \leq \mu - 1 \quad (k \geq 1).$$

The result is sharp.

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