



A note on stability of Mackey–Glass equations with two delays



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ABSTRACT

If the Mackey–Glass equation

$$\dot{x}(t) = r(t) \left[\frac{ax(h(t))}{1 + x^\nu(g(t))} - x(t) \right]$$

with $a > 1$ and $\nu > 0$ incorporates not one but two variable delays, some new phenomena arise: there may exist non-oscillatory about the positive equilibrium unstable solutions, the effect of possible absolute stability for certain a and ν disappears. We obtain sufficient conditions for local and global stability of the positive equilibrium and illustrate the stability tests, as well as new effects of two different delays, with examples.

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1. Introduction

Many models of mathematical biology are described by a delay differential equation

$$\dot{x}(t) = \sum_{k=1}^m F_k(t, x(h_1(t)), \dots, x(h_l(t))) - G(t, x(t)), \quad (1.1)$$

where F_k, G are nonnegative continuous functions. Here the functions F_k describe production incorporating delay, and G corresponds to the instantaneous mortality. Positivity, boundedness and persistence for solutions of (1.1) were investigated in our recent paper [7]. Similar models with some applications were considered in [2,4,7,8,14,17].

For (1.1), usual assumptions are the following: the functions F_k are either monotone or unimodal, $G(t, u)$ is monotone increasing in u , there is only one delay involved in each of F_k , and a positive equilibrium

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is unique. However, it is possible to consider more general models, for example, the modified Nicholson blowflies equation

$$\dot{x}(t) = \sum_{k=1}^m a_k(t)x(h_k(t))e^{-\lambda_k x(g_k(t))} - b(t)x(t), \quad t \geq 0, \quad (1.2)$$

(in the standard model, $h_k \equiv g_k$) and the modified Mackey–Glass type equation

$$\dot{x}(t) = \sum_{k=1}^m \frac{a_k(t)x(h_k(t))}{1 + x^{n_k}(g_k(t))} - \left(b(t) - \frac{c(t)}{1 + x^n(t)} \right) x(t), \quad t \geq 0, \quad (1.3)$$

which in the case $h_k \equiv g_k$ and $c \equiv 0$ coincides with the usual case [9].

There are also many generalizations of Eqs. (1.1)–(1.3) to the case of distributed delays and integro-differential equations [6,11,12,16,18]. However, whatever complicated delay is involved, there is usually exactly one delay incorporated in each nonlinear function. This guarantees that the delay model, at least for small delays, inherits some properties of a nondelay system, for example, all nonoscillatory about the unique positive equilibrium solutions converge to this equilibrium, and for small enough delays the positive equilibrium is globally attractive, see, for example, [12,16,18] and references therein. The purpose of the present paper is to illustrate that the situation may change when two or more delays are involved in the same nonlinear function. This is possible when the function increases in some arguments and decreases in the others. The situation when the same production function involves different delays is quite common in real-world models. Two types of delays occur, for example, in gene regulatory systems where the translation delay and transcription delays involved in the function significantly differ. Such delays can lead to chaotic oscillations, see, for example, [10].

The presence of several delays instead of one delay can create a new type of dynamics: an equation which was stable for coinciding delays can become unstable, once the two delays are different, creating sustainable oscillations. The maximal delay value will influence the oscillation amplitude.

In [7], we presented an example of the modified Mackey–Glass equation with two delays

$$\dot{x}(t) = \frac{2x(h(t))}{1 + x^2(g(t))} - x(t), \quad t \geq 0, \quad (1.4)$$

which has the unique positive equilibrium $x = 1$, the function $f(x) = 2x/(1 + x^2)$ is increasing on $[0, 1]$, so any positive solution of (1.4) with $h \equiv g$ satisfies $\lim_{t \rightarrow \infty} x(t) = 1$, for example, by [9, Theorem 3.13] or [12, Theorem 3.3]. However, the situation changes if the two delays are different. In [7, Example 1.1] we presented an example of h, g in (1.4) such that the solution experiences sustainable oscillations.

Typically, if a nonlinear equation with one delay has two equilibrium solutions, one trivial and one positive, any nonoscillatory about the equilibria solution tends to an equilibrium (or $\pm\infty$). However, the situation changes when two delays are involved in the nonlinear production function, as Example 1.1 illustrates.

Example 1.1. The equation

$$\dot{x}(t) = \frac{3x(h(t))}{1 + \sqrt{x(g(t))}} - 2\sqrt{x}, \quad t \geq 0, \quad x(t) = \varphi(t) \geq 0, \quad t \leq 0 \quad (1.5)$$

has the zero equilibrium and the unique positive equilibrium $K = 4$. Let us note that the initial value problem

$$\dot{x}(t) = 12 - 2\sqrt{x}, \quad (1.6)$$

$x(0) = 9$ satisfies $x(a) = 16$, where $a \approx 1.43279$, while for the initial value problem

$$\dot{x}(t) = \frac{27}{5} - 2\sqrt{x}, \quad (1.7)$$

$x(0) = 16$ we have $x(b) = 9$, where $b \approx 4.95911$. Let $\varphi(-b) = 16$, $\varphi(0) = 9$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and

$$h(t) = \begin{cases} (a+b)n - b, & t \in [(a+b)n, (a+b)n + a), \\ (a+b)n, & t \in [(a+b)n + a, (a+b)(n+1)), \end{cases} \quad n \in \mathbb{N}_0, \quad (1.8)$$

$$g(t) = \begin{cases} (a+b)n, & t \in [(a+b)n, (a+b)n + a), \\ (a+b)n + a, & t \in [(a+b)n + a, (a+b)(n+1)), \end{cases} \quad n \in \mathbb{N}_0. \quad (1.9)$$

Then equation (1.5) on $[(a+b)n, (a+b)n + a]$ has the form of (1.6) with $x((a+b)n) = 9$, while on $[(a+b)n + a, (a+b)(n+1)]$ has the form of (1.7) with $x((a+b)n + a) = 16$, $n \in \mathbb{N}_0$. It is easy to check that the solution of (1.6) with the given initial condition monotonically increases from 9 to 16, while the solution of (1.7) monotonically decreases from 16 to 9. Thus the solution is nonoscillatory about $K = 4$ (exceeds K for any $t \geq -b$) and also does not tend to K .

Similarly, a solution which oscillates between 1 and 2.25 can be designed, since the inequalities $\frac{3 \cdot 2.25}{1 + \sqrt{1}} > 2\sqrt{2.25}$ and $\frac{3 \cdot 1}{1 + \sqrt{2.25}} < 2\sqrt{1}$ hold.

Let us note that, according to [6], all solutions of (1.7) with $h \equiv g$ and $\varphi(t) > 4$ should tend to $+\infty$, while all solutions of (1.5) with $h \equiv g$ and $\varphi(t) < 4$ should converge to zero. Involvement of two different delays breaks this pattern.

2. Nonoscillatory solutions of Mackey–Glass equations

Further, we consider the Mackey–Glass equation with two delays in the production function

$$\dot{x}(t) = r(t) \left[\frac{ax(h(t))}{1 + x^\nu(g(t))} - x(t) \right], \quad t \geq t_0, \quad (2.1)$$

which for $a > 1$ has the unique positive equilibrium

$$K = (a - 1)^{1/\nu}. \quad (2.2)$$

We assume that

- (a1) r, h, g are Lebesgue measurable, $r(t) \geq 0$ is an essentially locally bounded function, $a > 1$, $\nu > 0$;
- (a2) $h(t) \leq t$, $g(t) \leq t$, $\lim_{t \rightarrow \infty} h(t) = \infty$, $\lim_{t \rightarrow \infty} g(t) = \infty$.

Together with equation (2.1) we consider for any $t_0 \geq 0$ the initial value problem for (2.1) with the initial condition

$$x(t) = \varphi(t), \quad t \leq t_0, \quad \varphi(t_0) > 0, \quad (2.3)$$

where

- (a3) $\varphi \in C[(-\infty, t_0], \mathbb{R}^+]$ is bounded.

Here $C[I, D]$ is a set of continuous functions $f: I \rightarrow D$, $\mathbb{R}^+ = [0, \infty)$.

For a positive bounded r , there exists a unique global solution of (2.1)–(2.3), see [9].

Theorem 2.1. Suppose that (a1)–(a2) hold and $\int_0^\infty r(t) dt = \infty$. Let $x(t)$ be a positive global solution of (2.1) satisfying either $x(t) > K$, $t \geq t_0$ or $x(t) \in (0, K)$, $t \geq t_0$. Then $x(t) > K$, $t \geq t_0$, implies

$$K = \liminf_{t \rightarrow \infty} x(t), \quad (2.4)$$

while $x(t) \in (0, K)$, $t \geq t_0$ yields that

$$K = \limsup_{t \rightarrow \infty} x(t). \quad (2.5)$$

Proof. Let $x(t) > K$ for $t \geq t_0$, then there exists t_1 such that $x(h(t)) > K$ and $x(g(t)) > K$ for $t \geq t_1$, and x is continuous on $[t_0, t_1]$.

If (2.4) does not hold, there is $m > K$ such that

$$m := \inf_{t \in [t_0, \infty)} x(t) > K.$$

Denote

$$M_0 = \max_{t \in [t_0, t_1]} x(t).$$

Since $\frac{a}{1+m^\nu} < \frac{a}{1+K^\nu} = 1$, there exists $\varepsilon_0 \in (0, M_0)$ such that

$$a \frac{M_0}{1+m^\nu} = M_0 - \varepsilon_0.$$

Therefore, for any $x(t) \in (M_0 - \varepsilon_0, M_0)$, the solution of (2.1) for $t > t_1$ is decreasing. The definition of M_0 immediately implies $x(t) \leq M_0$ for any $t \geq t_1$.

Thus, due to $\int_0^\infty r(t) dt = \infty$, there is a point $\bar{t} \in [t_1, \infty)$ such that $x(h(t)) < M_0 - \varepsilon_0/2$, $t \geq \bar{t}$, which yields that $\dot{x}(t) < 0$ if $x(t) > M_0 - \varepsilon_0 - \varepsilon_1$, where

$$\varepsilon_1 := M_0 - a \frac{M_0 - \varepsilon_0/2}{1+m^\nu} - \varepsilon_0 = \frac{aM_0}{1+m^\nu} - a \frac{M_0 - \varepsilon_0/2}{1+m^\nu} = \frac{a\varepsilon_0}{2(1+m^\nu)} > 0.$$

Thus there is a point $t^* \geq t_1$ such that $x(t^*) \leq M_0 - \varepsilon_0$, and we can also obtain $x(t) \leq M_0 - \varepsilon_0$ for any $t \geq t^*$.

Let t_2 be such that $h(t) \geq t^*$, $t \geq t_2$. Denote

$$M_1 = \max_{t \in [t^*, t_2]} x(t) \leq M_0 - \varepsilon_0.$$

Continuing this process, we obtain a decreasing sequence M_k , $k \in \mathbb{N}_0$ which has a limit $M > K$.

Since $1 + m^\nu > a$, we can find λ such that

$$0 < \frac{a}{1+m^\nu} < \lambda < 1.$$

Since $\limsup_{t \rightarrow \infty} x(t) = M$, there is a \hat{t} such that $x(t) \leq \frac{M}{\lambda}$, $x(h(t)) \leq \frac{M}{\lambda}$, $t \geq \hat{t}$. Then for $t \geq \hat{t}$,

$$\dot{x}(t) \leq \frac{aM}{\lambda(1+m^\nu)} - x(t) \leq 0$$

for any $x(t) \geq \alpha M < M$, where $\alpha := \frac{a}{\lambda(1+m^\nu)} \in (0, 1)$. Hence $x(t) < \frac{1+\alpha}{2}M$ for t large enough, which contradicts to the assumption $\limsup_{t \rightarrow \infty} x(t) = M$. The contradiction proves (2.4). The proof of (2.5) for $0 < x(t) < K$ is similar. \square

Example 2.2 illustrates Theorem 2.1 in the following sense:

1. the non-oscillatory about the equilibrium $K = 1$ solution of (2.1) does not tend to K ;
2. the conclusion of Theorem 2.1 holds: the equilibrium K is a limit point of the solution.

Example 2.2. The equation

$$\dot{x}(t) = \frac{2x(h(t))}{1+x^2(g(t))} - x(t) \quad (2.6)$$

has the positive equilibrium $K = 1$. Assume that $x(t_0) = 3$, $h(t) \equiv t$, $g(t) \equiv t_0$ on $[t_0, t_1]$, where $x(t_1) = \sqrt{1+0.1^2}$, and $x(t)$ on $[t_0, t_1]$ satisfies the equation

$$\dot{x} = \frac{2x}{1+3^2} - x = -0.8x \Rightarrow x(t) = 3e^{-0.8(t-t_0)} \quad \text{and} \quad x(t_1) = \sqrt{1.1}, \quad t_1 = 1.25 \ln \left(\frac{3}{\sqrt{1.1}} \right) + t_0.$$

We define

$$h(t) \equiv t_0, \quad g(t) \equiv t_1, \quad t \in [t_1, t_2],$$

so on $[t_1, t_2]$ we solve the initial value problem

$$\dot{x}(t) = \frac{2 \cdot 3}{1+1.1} - x(t), \quad x(t_1) = \sqrt{1.1}.$$

The solution increases as long as $x(t) < 3/1.05$, so there is a $t_2 > t_1$ such that

$$x(t_2) = \frac{(2+0.1^1)(2+0.5^1)}{2} = 2.625 < \frac{3}{1.05}.$$

On $[t_2, t_3]$ we assume $h(t) = t$, $g(t) = t_2$. The solution of the equation

$$\dot{x}(t) = \left(\frac{2}{1+2.625^2} - 1 \right) x, \quad x(t_2) = 2.625$$

is decreasing, thus there is a $t_3 > t_2$ such that $x(t_3) = \sqrt{1+0.1^2}$.

Further, denote

$$h(t) = \begin{cases} t, & t \in [t_{2k}, t_{2k+1}], \\ t_{2k}, & t \in [t_{2k+1}, t_{2k+2}] \end{cases} \quad k = 0, 1, 2, \dots, \\ g(t) = \begin{cases} t_{2k+1}, & t \in [t_{2k}, t_{2k+1}], \\ t_{2k+1}, & t \in [t_{2k+1}, t_{2k+2}] \end{cases} \quad k = 0, 1, 2, \dots$$

Let us prove that we can have

$$x(t_{2k+1}) = \sqrt{1+0.1^{k+1}}, \quad x(t_{2k}) = \frac{(2+0.1^k)(2+0.5^k)}{2}, \quad (2.7)$$

with the solution satisfying $x(t) \in (x(t_{2k+1}), x(t_{2k}))$ on (t_{2k}, t_{2k+1}) and $x(t) \in (x(t_{2k+1}), x(t_{2k+2}))$ on (t_{2k+1}, t_{2k+2}) .

First, on $[t_{2k}, t_{2k+1})$ the solution $x(t)$ of the equation

$$\dot{x}(t) = \frac{2x(t)}{2 + 0.1^k} - x(t) = -\alpha_k x(t), \quad \alpha_k = 1 - \frac{2}{2 + 0.1^k} > 0$$

is decreasing:

$$x(t) = x(t_{2k})e^{-\alpha_k(t-t_{2k})}.$$

Thus $x(t_{2k+1}) = \sqrt{1 + 0.1^{k+1}}$ can be attained due to

$$\frac{(2 + 0.1^k)(2 + 0.5^k)}{2} \geq (2 + 0.1^k)(1 + 0.25^k) > 2 > \sqrt{1 + 0.1^{k+1}}$$

for any $k \in \mathbb{N}_0$.

Next, on $[t_{2k+1}, t_{2k+2}]$ we have the initial value problem

$$\dot{x}(t) = \frac{(2 + 0.1^k)(2 + 0.5^k)}{2 + 0.1^{k+1}} - x(t) > 2 + 0.5^k - x(t), \quad x(t_{2k+1}) = \sqrt{1 + 0.1^{k+1}},$$

its solution is increasing. The value $x(t_{2k+2}) = \frac{(2 + 0.1^{k+1})(2 + 0.5^{k+1})}{2}$ can be attained if

$$2(2 + 0.5^k) > (2 + 0.1^{k+1})(2 + 0.5^{k+1}) \Leftrightarrow 2 \cdot 0.5^{k+1} > 2 \cdot 0.1^{k+1} + 0.05^{k+1},$$

the latter inequality can be easily verified by induction.

Equalities (2.7) imply $\lim_{k \rightarrow \infty} x(t_{2k}) = 2$. The derivative $\dot{x}(t)$ is bounded, therefore $\lim_{k \rightarrow \infty} t_k = +\infty$, so $M = \limsup_{t \rightarrow \infty} x(t) = 2 > K = 1$.

Let us note that for $g(t) \equiv h(t)$ all positive solutions of (2.6) tend to K [5], and solutions with $x(t) \in (K, \infty)$ are eventually decreasing. The involvement of two delays can lead to oscillations, however, the lower limit is still $\limsup_{t \rightarrow \infty} x(t) = 1 = K$, as stated in Theorem 2.1.

3. Local stability

In addition to conditions (a1)–(a2) we assume that r is bounded, separated from zero, and the delays are bounded:

(a4) r is essentially bounded, and there exist $r_0 > 0$, $\tau > 0$ and $\sigma > 0$ such that $r(t) \geq r_0$, $t - h(t) \leq \tau$, $t - g(t) \leq \sigma$.

Denote $f(x_1, x_2) = \frac{ax_1}{1 + x_2^\nu}$, then

$$\frac{\partial f}{\partial x_1}(K, K) = 1, \quad \frac{\partial f}{\partial x_2}(K, K) = -\frac{\nu(a-1)}{a}.$$

Hence the linearized equation for (2.1) at the equilibrium K has the form

$$\dot{x}(t) = r(t)[-x(t) + x(h(t)) - \alpha x(g(t))], \quad (3.1)$$

where

$$\alpha = \frac{\nu(a-1)}{a}. \quad (3.2)$$

We will further apply the following lemma, which is based on classical results (see, for example, [13,15]).

Lemma 3.1. *Consider the scalar equation*

$$\dot{x}(t) = -a(t)x(t) - b(t)x(h(t)), \quad (3.3)$$

where a and b are essentially bounded on $[t_0, \infty)$ functions, h is a measurable function satisfying $0 \leq t - h(t) \leq \sigma$.

1) If $a(t) \equiv 0, b(t) \geq b_0 > 0$ and $\limsup_{t \rightarrow \infty} \int_{h(t)}^t b(s)ds < \frac{3}{2}$ then (3.3) is exponentially stable.

2) If $a(t) \equiv 0, b(t) = b > 0, h(t) = t - \tau$ and $b\tau < \frac{\pi}{2}$ then (3.3) is exponentially stable. If $b\tau \geq \frac{\pi}{2}$ then (3.3) is not exponentially stable.

3) If $a(t) \geq a_0 > 0$ and $\limsup_{t \rightarrow \infty} \frac{|b(t)|}{a(t)} < 1$ then (3.3) is exponentially stable.

Definition 3.2. The positive equilibrium K of equation (2.1) is **locally exponentially stable** if there exist constants $\varepsilon > 0, \gamma > 0$ and $M > 0$ such that for any solution x of the problem (2.1), (2.3), where $|\varphi(t) - K| < \varepsilon$, the exponential estimate

$$|x(t) - K| \leq Me^{-\gamma(t-t_0)} \sup_{t \leq t_0} |\varphi(t) - K| \quad (3.4)$$

holds, where M, γ do not depend on t_0 and the solution x .

Equilibrium K is **globally exponentially stable** if inequality (3.4) holds for all positive solutions of problem (2.1), (2.3).

By the well-known linearized principle (see, for example, [3, Theorem 3]), exponential stability of linear equation (3.1) implies local exponential stability of the equilibrium K for equation (2.1).

Lemma 3.1 immediately implies the following result.

Theorem 3.3. *Let (a1)–(a4) hold.*

1. If $h(t) \equiv t, \alpha$ is defined in (3.2) and

$$\limsup_{t \rightarrow \infty} \nu \alpha \int_{g(t)}^t r(s)ds < \frac{3}{2}$$

then the positive equilibrium K of equation (2.1) is locally exponentially stable.

2. If $h(t) \equiv t, g(t) \equiv t - \sigma, r(t) \equiv R$ and $\alpha \nu \sigma R < \frac{\pi}{2}$ then the positive equilibrium of equation (2.1) is locally exponentially stable. If $\alpha \nu \sigma R \geq \frac{\pi}{2}$ then the positive equilibrium of equation (2.1) is not globally exponentially stable.

3. If $g(t) \equiv t$ then the positive equilibrium of equation (2.1) is locally exponentially stable.

To formulate the next lemma, consider a more general than (3.1) equation

$$\dot{x}(t) = -a_1(t)x(t) - a_2(t)x(h(t)) - a_3(t)x(g(t)), \quad (3.5)$$

where $a_i(t)$, $i = 1, 2, 3$ are essentially bounded on $[0, \infty)$ functions, and $t - \tau \leq h(t) \leq t$, $t - \sigma \leq g(t) \leq t$ for some $\tau > 0$, $\sigma > 0$, denote

$$a(t) = \sum_{k=1}^3 a_k(t).$$

Lemma 3.4. [3, Corollary 1.6] Suppose $a_1(t) + a_3(t) \geq b_0 > 0$ and

$$\limsup_{t \rightarrow \infty} \left[\frac{|a_3(t)|}{a_1(t) + a_3(t)} \int_{g(t)}^t \sum_{k=1}^3 |a_k(s)| ds + \frac{|a_2(t)|}{a_1(t) + a_3(t)} \right] < 1.$$

Then equation (3.5) is exponentially stable.

If we denote

$$a_1(t) = r(t), a_2(t) = -r(t), a_3(t) = \alpha r(t), a(t) = \alpha r(t), \sum_{k=1}^3 |a_k(t)| = (2 + \alpha)r(t)$$

then as a direct corollary of Lemma 3.4 we obtain the following result.

Theorem 3.5. Suppose (a1)–(a4) hold and

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t r(s) ds < \frac{1}{2 + \alpha}.$$

Then equation (2.1) is locally exponentially stable.

To obtain different stability conditions consider together with equation (3.1) the following initial value problem

$$\dot{x}(t) = r(t)[-x(t) + x(h(t)) - \alpha x(g(t))] + f(t), \quad x(t) = 0, t \leq t_0. \quad (3.6)$$

Lemma 3.6. [1, Theorem 4.3.1] If for any essentially bounded on $[t_0, \infty)$ function f the solution of (3.6) is bounded on $[t_0, \infty)$ then equation (3.1) is exponentially stable.

Choose an arbitrary $t_1 > t_0$. Denote $T = [t_0, t_1]$, $|u|_T = \max_{t \in T} |u(t)|$ for a continuous function u and $\|f\| = \text{ess sup}_{t \in [t_0, \infty)} |f(t)|$ for any essentially bounded on $[t_0, \infty)$ function f .

Lemma 3.7. Let (a1)–(a4) hold. We have the following estimates for the solution x of problem (3.6) and its derivative \dot{x} .

1. If $\alpha < 2$ then

$$|x|_T \leq \frac{\|h - g\|}{2 - \alpha} |\dot{x}|_T + M_1. \quad (3.7)$$

2.

$$|x|_T \leq \sigma |\dot{x}|_T + M_2. \quad (3.8)$$

3.

$$|\dot{x}|_T \leq (2 + \alpha)\|r\||x|_T + M_3. \quad (3.9)$$

4. If $\tau\|r\| < 1$ then

$$|\dot{x}|_T \leq \frac{\alpha\|r\|}{1 - \tau\|r\|}|x|_T + M_4 \quad (3.10)$$

where M_i are some positive numbers which do not depend on the interval T .

Proof. 1. We have the following transformations for the equation in (3.6):

$$\begin{aligned} \dot{x}(t) + r(t)x(t) &= r(t)[x(h(t)) - \alpha x(g(t))] + f(t), \\ \dot{x}(t) + r(t)x(t) &= r(t) \left[\int_{g(t)}^{h(t)} \dot{x}(s) ds - (\alpha - 1)x(g(t)) \right] + f(t), \\ x(t) &= \int_{t_0}^t e^{-\int_s^t r(\zeta) d\zeta} r(s) \left[\int_{g(s)}^{h(s)} \dot{x}(\zeta) d\zeta - (\alpha - 1)x(g(s)) \right] ds + f_1(t), \end{aligned}$$

where $f_1(t) = \int_{t_0}^t e^{-\int_s^t r(\zeta) d\zeta} f(s) ds$ is an essentially bounded on $[t_0, \infty)$ function.

Hence

$$|x|_T \leq \|h - g\||\dot{x}|_T + (\alpha - 1)|x|_T + \|f_1\|,$$

which implies inequality (3.7) with $M_1 = \frac{\|f_1\|}{2 - \alpha}$.

2. We have the following transformations for the equation in (3.6):

$$\begin{aligned} \dot{x}(t) + (1 + \alpha)r(t)x(t) &= r(t)[x(h(t)) + \alpha(x(t) - x(g(t)))] + f(t), \\ \dot{x}(t) + (1 + \alpha)r(t)x(t) &= r(t) \left[x(h(t)) + \alpha \int_{g(t)}^t \dot{x}(s) ds \right] + f(t), \\ x(t) &= \int_{t_0}^t e^{-\int_s^t (1 + \alpha)r(\zeta) d\zeta} (1 + \alpha)r(s) \frac{1}{1 + \alpha} \left[x(h(s)) + \alpha \int_{g(s)}^t \dot{x}(\zeta) d\zeta \right] ds + f_2(t), \end{aligned}$$

where $f_2(t) = \int_{t_0}^t e^{-\int_s^t (1 + \alpha)r(\zeta) d\zeta} f(s) ds$ is an essentially bounded on $[t_0, \infty)$ function. Hence

$$|x|_T \leq \frac{1}{1 + \alpha}|x|_T + \frac{\alpha\sigma}{1 + \alpha}|\dot{x}|_T + \|f_2\|,$$

which implies inequality (3.8) with $M_2 = \frac{(1 + \alpha)\|f_2\|}{\alpha}$.

3. Inequality (3.9) is evident, where $M_3 = \|f\|$.

4. From the equation in (3.6) we have

$$\dot{x}(t) = r(t) \left[- \int_{h(t)}^t \dot{x}(s) ds - \alpha x(g(t)) \right] + f(t).$$

Hence

$$|\dot{x}|_T \leq \|r\|(\tau|\dot{x}|_T + \alpha|x|_T) + \|f\|,$$

which implies inequality (3.10) with $M_4 = \frac{\|f\|}{1-\tau\|r\|}$. \square

Theorem 3.8. Suppose that (a1)–(a4) are satisfied, and at least one of the following conditions (a)–(d) holds:
(a) $\alpha < 2$ and

$$\frac{(2+\alpha)\|h-g\|\|r\|}{2-\alpha} < 1; \quad (3.11)$$

(b) $\alpha < 2, \tau\|r\| < 1$ and

$$\frac{\alpha\|h-g\|\|r\|}{(2-\alpha)(1-\tau\|r\|)} < 1; \quad (3.12)$$

(c)

$$(2+\alpha)\sigma\|r\| < 1; \quad (3.13)$$

(d) $\tau\|r\| < 1$ and

$$\frac{\alpha\sigma\|r\|}{1-\tau\|r\|} < 1. \quad (3.14)$$

Then equation (2.1) is locally exponentially stable.

Proof. By Lemma 3.6 it is sufficient to prove that the solution of (3.6) is bounded on $[t_0, \infty)$ for any essentially bounded on $[t_0, \infty)$ function f .

(a) From inequalities (3.7) and (3.9) we have

$$|x|_T \leq \frac{\|h-g\|}{2-\alpha}(2+\alpha)\|r\||x|_T + M_5, \quad \text{where} \quad M_5 = \frac{M_3\|h-g\|}{2-\alpha} + M_1.$$

Condition (3.11) implies $|x|_T \leq M$, where

$$M = M_5 \left(1 - \frac{(2+\alpha)\|h-g\|\|r\|}{2-\alpha} \right)^{-1}$$

is a positive number which does not depend on the solution x and the interval T . Hence the solution x is a bounded on $[t_0, \infty)$ function. By Lemma 3.6 equation (3.1) is exponentially stable, hence (2.1) is locally exponentially stable.

(b) follows from inequalities (3.7) and (3.10).

(c) follows from inequalities (3.8) and (3.9).

(d) follows from inequalities (3.8) and (3.10). \square

Remark 3.9. Part (c) of Theorem 3.8 follows from Theorem 3.5. All other conditions of Theorems 3.5 and 3.8 are independent.

4. Global stability and attractivity

We start with the proof of global stability in the case when delays tend to zero as $t \rightarrow \infty$.

Applied to (2.1), the result of [7, Theorem 5.6] can be formulated as follows.

Lemma 4.1. [7, Theorem 5.6] *Let (a1)–(a4) hold, a positive $A > 0$ be an essential upper bound for r : $r(t) \leq A$ for almost all $t \geq t_0$, and denote $\tau_{\max} = \max\{\tau, \sigma\}$.*

Then any solution of (2.1) satisfies

$$\liminf_{t \rightarrow \infty} x(t) \geq K e^{-2A\tau_{\max}}, \quad \limsup_{t \rightarrow \infty} x(t) \leq K e^{4A\tau_{\max}}. \quad (4.1)$$

Theorem 4.2. *Suppose that (a1)–(a4) hold and*

$$\lim_{t \rightarrow \infty} (t - h(t)) = 0, \quad \lim_{t \rightarrow \infty} (t - g(t)) = 0. \quad (4.2)$$

Then any positive solution $x(t)$ of (2.1) satisfies

$$\lim_{t \rightarrow \infty} x(t) = K. \quad (4.3)$$

Proof. By the assumptions of the theorem, there is $A > 0$ such that $0 < r_0 \leq r(t) \leq A$ for almost all $t \geq t_0$. For any $\varepsilon > 0$, where $\varepsilon < K$, we can define

$$\tau_0 < \min \left\{ \frac{1}{2A} \ln \left(\frac{2K}{2K - \varepsilon} \right), \frac{1}{4A} \ln \left(1 + \frac{\varepsilon}{2K} \right) \right\}.$$

According to (4.2), we can choose $t_0 \geq 0$ such that $t - \tau_0 \leq h(t) \leq t$, $t - \tau_0 \leq g(t) \leq t$. By Lemma 4.1,

$$\begin{aligned} \liminf_{t \rightarrow \infty} x(t) &\geq K e^{-2A\tau_{\max}} > K \frac{2K - \varepsilon}{2K} = K - \frac{\varepsilon}{2}, \\ \limsup_{t \rightarrow \infty} x(t) &\leq K e^{4A\tau_{\max}} < K \left(1 + \frac{\varepsilon}{2K} \right) = K + \frac{\varepsilon}{2}. \end{aligned}$$

Thus there exists $t_1 \geq t_0$ such that

$$K - \varepsilon < x(t) < K + \varepsilon,$$

which concludes the proof of (4.3). \square

Similarly to the local stability analysis, we consider the cases when either $h(t) \equiv t$ or $g(t) \equiv t$.

For the case $g(t) \equiv t$ we also have a global stability result.

Theorem 4.3. [4, Theorem 2, p. 2619] *If (a1)–(a2) hold, r is essentially bounded, $t - g(t) \leq \sigma$ for some $\sigma > 0$, $\int_0^\infty r(s)ds = \infty$, $h(t) \equiv t$ and*

$$\limsup_{t \rightarrow \infty} \frac{a\nu}{4} \int_{g(t)}^t r(s)ds < 1 + \frac{1}{e},$$

then equation (2.1) is globally exponentially stable.

Theorem 4.4. *Let (a1)–(a4) hold. If $g(t) \equiv t$ then the equilibrium K of (2.1) is globally attractive.*

Proof. Let $g(t) \equiv t$. Under (a1)–(a3), with a nonnegative $\varphi(t)$ satisfying $\varphi(t_0) > 0$, (2.1) has a positive global solution [7,9] which is also bounded. Denote

$$r_1(t) = \frac{r(t)}{1 + x^\nu(t)},$$

where r_1 is positive, bounded and separated from zero. Thus, $x(t)$ is a solution of the equation

$$\dot{x}(t) = r_1(t) [ax(h(t)) - x(t)(1 + x^\nu(t))], \quad (4.4)$$

where both $F(x) = ax$ and $G(x) = x(1 + x^\nu)$ are increasing in x , $F(x) > G(x)$ for $x \in (0, K)$ and $F(x) < G(x)$ for $x > K$.

By [5, Theorem 3.1], all solutions of (4.4) converge to K . Thus, the solution $x(t)$ of (2.1) also converges to K , which concludes the proof. \square

Remark 4.5. Let us note that for $h \equiv g$ the condition $a < 1/(1 - \nu)$ guarantees global asymptotic stability for (2.1) with any delay satisfying (a2), see [9,12], while there are no delay-independent conditions for (2.1).

The following result claims that, whatever small the delays are, there may still be sustainable oscillations about the positive equilibrium, and the positive equilibrium $K = (a/b - 1)^{1/\nu}$ is unstable.

Theorem 4.6. Let (a1)–(a2) hold and the function r satisfy $0 < r_0 \leq r(t) \leq A$ for any $t \geq 0$, $a > 1$. If for fixed $a > 1$ and $\nu > 0$ there exist $\varepsilon > 0$ and $\delta \in (0, K)$ such that

$$r_0 \varepsilon \geq \max \left\{ \frac{2\delta}{K} \left(\frac{K^\nu + 1}{(K - \delta/2)^\nu + 1} - 1 \right)^{-1}, \frac{2\delta}{K} \left(1 - \frac{K^\nu + 1}{(K + \delta/2)^\nu + 1} \right)^{-1} \right\}, \quad (4.5)$$

then there is an equation of the form (2.1) with bounded delay functions

$$0 \leq t - h(t) \leq \varepsilon, \quad 0 \leq t - g(t) \leq \varepsilon \quad (4.6)$$

and a solution x of this equation satisfying

$$\limsup_{t \rightarrow \infty} x(t) - \liminf_{t \rightarrow \infty} x(t) = \delta. \quad (4.7)$$

Proof. Let us note that (4.5) implies

$$\delta \leq r_0 \frac{\varepsilon}{2} K \left(\frac{K^\nu + 1}{(K - \delta/2)^\nu + 1} - 1 \right) \quad (4.8)$$

and

$$\delta \leq r_0 \frac{\varepsilon}{2} \frac{K}{2} \left(1 - \frac{K^\nu + 1}{(K + \delta/2)^\nu + 1} \right). \quad (4.9)$$

We construct a solution of (2.1) with the delays satisfying (4.6) such that (4.7) holds. Introducing

$$m = K - \frac{\delta}{2}, \quad M = K + \frac{\delta}{2}, \quad (4.10)$$

we notice that, as $\delta \in (0, K)$,

$$\frac{K}{2} < m = K - \frac{\delta}{2} < K, \quad K < M = K + \frac{\delta}{2} < \frac{3K}{2}. \quad (4.11)$$

We choose the initial function such that $x(t_{-1}) = M$ for some $t_{-1} \in (-\varepsilon/2, 0)$ and $x(0) = m$. Next, assume $t_0 = 0$ and prove that there exists an increasing sequence of t_k such that

$$t_{n+1} - t_n < \frac{\varepsilon}{2}, \quad n \in \mathbb{N}$$

and

$$x(t_{2k}) = m, \quad x(t_{2k+1}) = M, \quad k \in \mathbb{N}, \quad (4.12)$$

where on the intervals $[t_{2k}, t_{2k+1}]$ the solution increases, while on $[t_{2k+1}, t_{2k+2}]$ it decreases. Then, we obtain (4.7). Suppose the sequence t_n is constructed. Then we denote

$$h(t) = \begin{cases} t_{2k-1}, & t \in [t_{2k}, t_{2k+1}), \\ t_{2k}, & t \in [t_{2k+1}, t_{2k+2}), \end{cases} \quad k \in \mathbb{N},$$

$$g(t) = \begin{cases} t_{2k}, & t \in [t_{2k}, t_{2k+1}), \\ t_{2k+1}, & t \in [t_{2k+1}, t_{2k+2}), \end{cases} \quad k \in \mathbb{N}.$$

Therefore, on $[t_0, t_1)$, where $t_0 = 0$, $x(t_0) = m$ and $t_1 > t_0$ is still to be determined, $x(t)$ satisfies the equation

$$\dot{x}(t) = r(t) \left[\frac{a(K + \delta/2)}{1 + (K - \delta/2)^\nu} - x(t) \right]. \quad (4.13)$$

Assuming that $x(t) \leq M$ on $[t_0, t_1)$ and taking into account $r(t) \geq r_0$ and $a = 1 + K^\nu$, we obtain the differential inequality

$$\dot{x}(t) \geq B_1 := r_0 \left[\frac{K^\nu + 1}{1 + (K - \delta/2)^\nu} - 1 \right] \left(K + \frac{\delta}{2} \right) > 0.$$

From (4.8), see also (4.11), we obtain

$$r_0 \left[\frac{K^\nu + 1}{1 + (K - \delta/2)^\nu} - 1 \right] \left(K + \frac{\delta}{2} \right) \frac{\varepsilon}{2} \geq r_0 \frac{\varepsilon}{2} \left[\frac{K^\nu + 1}{1 + (K - \delta/2)^\nu} - 1 \right] K \geq \delta.$$

If we have an initial value problem satisfying

$$\dot{x}(t) \geq B_1, \quad t \in \left[t_0, t_0 + \frac{\varepsilon}{2} \right], \quad x(t_0) = m = K - \frac{\delta}{2},$$

where $B_1 \frac{\varepsilon}{2} \geq \delta$, we get

$$x\left(t_0 + \frac{\varepsilon}{2}\right) \geq K - \frac{\delta}{2} + \delta = K + \frac{\delta}{2} = M.$$

Hence there exists t_1 satisfying $t_1 - t_0 \leq \varepsilon/2$ and $x(t_1) = M$.

Similarly, on $[t_1, t_2)$, where t_2 is still to be determined and $x(t_1) = M$, we have the equation

$$\dot{x}(t) = r(t) \left[\frac{a(K - \delta/2)}{1 + (K + \delta/2)^\nu} - x(t) \right]. \quad (4.14)$$

Assuming that $x(t) \geq m$ on $[t_0, t_1]$ and taking into account $r(t) \geq r_0$ and $a = 1 + K^\nu$, we get the differential inequality

$$\dot{x}(t) \leq B_2 := r_0 \left[\frac{K^\nu + 1}{1 + (K + \delta/2)^\nu} - 1 \right] \left(K - \frac{\delta}{2} \right) < 0.$$

From (4.9), see also (4.11), we obtain

$$-r_0 \left[1 - \frac{K^\nu + 1}{1 + (K + \delta/2)^\nu} \right] \left(K - \frac{\delta}{2} \right) \frac{\varepsilon}{2} \leq -r_0 \frac{\varepsilon}{2} \left[1 - \frac{K^\nu + 1}{1 + (K + \delta/2)^\nu} \right] \frac{K}{2} \leq -\delta.$$

For an initial value problem satisfying

$$\dot{x}(t) \leq -B_2, \quad t \in \left[t_1, t_1 + \frac{\varepsilon}{2} \right], \quad x(t_1) = M = K + \frac{\delta}{2},$$

where $-B_2 \geq -\delta$, we have

$$x \left(t_1 + \frac{\varepsilon}{2} \right) \leq K + \frac{\delta}{2} - \delta = K - \frac{\delta}{2} = m.$$

Therefore there exists t_2 satisfying $t_2 - t_1 \leq \varepsilon/2$ and $x(t_2) = m$. We repeat this construction on $[t_{2k}, t_{2k+1}]$ and $[t_{2k+1}, t_{2k+2}]$ and obtain (4.12).

Finally, let us prove that $t_n \rightarrow \infty$ as $n \rightarrow \infty$.

By the construction of the solution of (2.1),

$$m \leq x(t) \leq M, \quad m = \inf_{t \geq 0} x(t), \quad M = \sup_{t \geq 0} x(t),$$

so for any $t \geq 0$,

$$\dot{x}(t) \leq \sup_{t \geq 0} r(t) \left[\sup_{t, s \geq 0} \frac{ax(t)}{1 + x^\nu(s)} - \inf_{t \geq 0} x(t) \right] \leq A \left(\frac{aM}{1 + m^\nu} - m \right),$$

since $\frac{am}{1 + M^\nu} < M$, due to $m < K < M$.

Also, as the expression in the brackets below is negative by $\frac{am}{1 + M^\nu} < M$, we have

$$\dot{x}(t) \geq \sup_{t \geq 0} r(t) \left[\inf_{t, s \geq 0} \frac{ax(t)}{1 + x^\nu(s)} - \sup_{t \geq 0} x(t) \right] \geq A \left(\frac{am}{1 + M^\nu} - M \right).$$

Finally,

$$A \left(\frac{am}{1 + M^\nu} - M \right) \leq \dot{x}(t) \leq A \left(\frac{aM}{1 + m^\nu} - m \right).$$

Denoting

$$P := A \max \left\{ M - \frac{am}{1 + M^\nu}, \frac{aM}{1 + m^\nu} - m \right\},$$

we obtain $|\dot{x}(t)| \leq P$ and $t_{n+1} - t_n \geq (M - m)/P$.

Thus $\lim_{n \rightarrow \infty} t_n = \infty$, which, taking into account (4.12), implies (4.7). \square

Remark 4.7. Let us note that for all positive δ , the product $r_0\varepsilon$ in (4.5) cannot be chosen arbitrarily small. Consider the equation

$$\dot{x}(t) = \frac{2x(h(t))}{1 + x^\nu(g(t))} - x(t), \quad t \geq 0 \quad (4.15)$$

which has a positive equilibrium $K = 1$. Fig. 1 estimates numerically the maximal values of the right-hand side in (4.5) for $\nu = y \in [1, 20]$.

However, sustainable oscillations about the positive equilibrium can also occur when (4.5) does not hold, as the following example illustrates.

Example 4.8. Consider the equation

$$\dot{x}(t) = \frac{2x(h(t))}{1 + \sqrt{x(g(t))}} - x(t), \quad t \geq 0. \quad (4.16)$$

It is well known, see, for example, [12], that (4.16) is asymptotically stable for any argument deviations $h \equiv g$ satisfying (a2). Denote

$$\begin{aligned} a &= \ln \left(\frac{2.42}{1.9} - 0.81 \right) - \ln \left(\frac{2.42}{1.9} - 1.21 \right) \approx 1.98527, \\ b &= \ln \left(1.21 - \frac{1.62}{2.1} \right) - \ln \left(0.81 - \frac{1.62}{2.1} \right) \approx 2.43101. \end{aligned}$$

Assume

$$x(0) = 0.81, \quad x(-b) = 1.21$$

and

$$\begin{aligned} h(t) &= \begin{cases} \left\lfloor \frac{t}{a+b} \right\rfloor - b, & t \in [n(a+b), n(a+b) + a), \\ \left\lfloor \frac{t}{a+b} \right\rfloor - a - b, & t \in [n(a+b) + a, (n+1)(a+b)), \end{cases} \\ g(t) &= \begin{cases} \left\lfloor \frac{t}{a+b} \right\rfloor - a - b, & t \in [n(a+b), n(a+b) + a), \\ \left\lfloor \frac{t}{a+b} \right\rfloor - b, & t \in [n(a+b) + a, (n+1)(a+b)). \end{cases} \end{aligned}$$

Then on $[0, a]$ the equation is $\dot{x}(t) + x(t) = \frac{2 \cdot 1.21}{1 + \sqrt{0.81}}$, $x(0) = 0.81$, thus $x(a) = 1.21$. Further, on $[a, a+b]$ the equation is $\dot{x}(t) + x(t) = \frac{2 \cdot 0.81}{1 + \sqrt{1.21}}$, $x(a) = 1.21$, so $x(a+b) = 0.81$, and the solution is periodic. Fig. 2 illustrates both the delays and the solution of (4.16).

As stated in Theorem 4.6, involvement of two different delays in (4.16) can lead to sustainable oscillations whenever, according to (4.5) with $K = 1$, $r_0 = 1$, for some $\delta > 0$,

$$\varepsilon \geq \max \left\{ 2\delta \left(\frac{2}{\sqrt{1 - \delta/2} + 1} - 1 \right)^{-1}, 2\delta \left(1 - \frac{2}{\sqrt{1 + \delta/2} + 1} \right)^{-1} \right\},$$

or $\varepsilon > 16$. However, in the example, the maximal delay of $2(a+b) \approx 8.832556 < 16$. Thus, assumption (4.5) in Theorem 4.6 is sufficient for possible instability but not necessary. Also, we once again illustrated the fact that absolutely stable for identical delays equations lose the stability property, once the argument deviations are different.

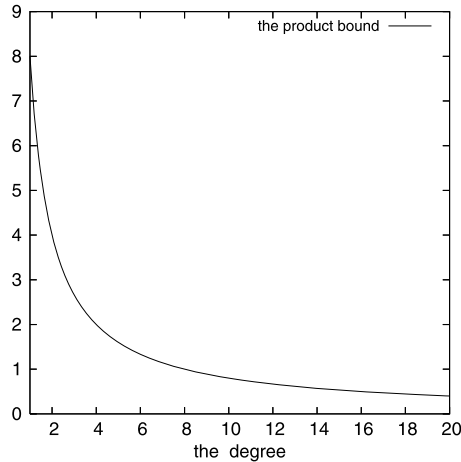


Fig. 1. The numerically evaluated threshold of the product of the maximal delay by the minimal growth rate guaranteeing possible instability of (4.15) for the values of the degree ν changing from 1 to 20.

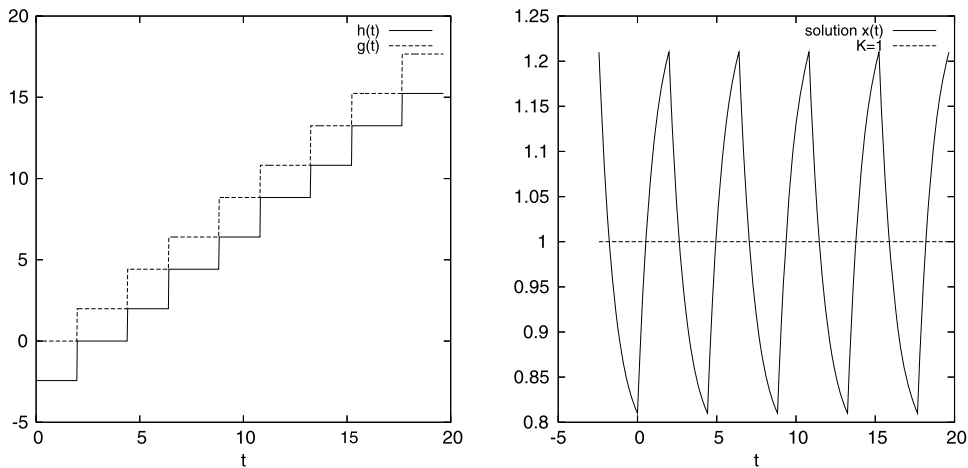


Fig. 2. The left graphs represent delayed arguments $h(t)$ and $g(t)$, while the right figure illustrates the solution $x(t)$ of (4.16) which is oscillatory about $K = 1$.

Remark 4.9. Let us note that instability condition (4.5) assumes that the product of r_0 and the maximal delay is large enough. In Example 4.8 it includes the condition that the maximal delay exceeds 16, while in Example 4.8 it does not exceed $a + b \approx 4.416278$. Thus, (4.5) is a sufficient instability condition only.

Finally, we present sufficient conditions for the global attractivity of K .

Theorem 4.10. Assume that (a1)–(a4) hold, a positive $A > 0$ is an essential upper bound for r : $r(t) \leq A$ for almost all $t \geq t_0$, $\tau_{\max} = \max\{\tau, \sigma\}$, where

$$\tau_{\max} < \frac{1}{4A} \ln 2, \quad (4.17)$$

and there exist $\lambda_1, \lambda_2 \in (0, 1)$ such that

$$(1 - e^{-A\tau_{\max}}) \frac{2a}{1 + (K/\sqrt{2})^\nu} \leq \lambda_1 \quad (4.18)$$

and

$$\sqrt{2} \left[1 - \frac{a}{\sqrt{2}(1 + (2K)^\nu)} \right] \sup_{t \geq t_0 + \tau_{\max}} \int_{t - \tau_{\max}}^t r(s) \, ds \leq \lambda_2. \quad (4.19)$$

Then any solution of (2.1) satisfies $\lim_{t \rightarrow \infty} x(t) = K$.

Proof. We will apply Lemma 4.1, in particular, (4.1) which in this case has the form

$$\liminf_{t \rightarrow \infty} x(t) \geq K e^{-2A\tau_{\max}}, \quad \limsup_{t \rightarrow \infty} x(t) \leq K e^{4A\tau_{\max}}. \quad (4.20)$$

Let us note that estimate (4.17) yields that there exists t_1 for which

$$\frac{1}{\sqrt{2}}K \leq x(t) \leq 2K, \quad t \geq t_1 - \tau_{\max}. \quad (4.21)$$

Further, we consider (2.1) for $t \geq t_1$ only.

By Theorem 2.1, for any $c_n \rightarrow 0$ there exists an increasing sequence $t_n^* \rightarrow \infty$ such that

$$|x(t_n^*) - K| < c_n.$$

Let us assume first that $x(t^*) = K + \varepsilon$, where $t^* = t_{n_0}^*$ for some n_0 , and estimate the maximal possible value of the solution x on an interval $[t^*, \infty)$. Without loss of generality, we can assume $t^* > t_1$ large enough such that

$$\varepsilon < \frac{1 - \lambda_1}{2}K. \quad (4.22)$$

Denote $\lambda_3 := \frac{1}{2}(\lambda_1 + 1) \in (\lambda_1, 1)$. For $t \in [t^*, t^* + \tau_{\max}]$, as $x(h(s))$ and $x(g(s))$ satisfy (4.21), (2.1), we have by (4.18) and (4.22),

$$\begin{aligned} x(t) &= (K + \varepsilon)e^{-\int_{t^*}^t r(\zeta) \, d\zeta} + \int_{t^*}^t r(s)e^{-\int_s^t r(\zeta) \, d\zeta} \frac{ax(h(s))}{1 + x^\nu(g(s))} \, ds \\ &\leq K + \varepsilon + K \frac{2a}{1 + (K/\sqrt{2})^\nu} (1 - e^{-A\tau_{\max}}) \\ &\leq K + \frac{1 - \lambda_1}{2}K + \lambda_1 K = K + \frac{1 + \lambda_1}{2}K \\ &= (1 + \lambda_3)K. \end{aligned}$$

Thus $x(t) \leq (1 + \lambda_3)K$ for $t \in [t^*, t^* + \tau_{\max}]$. To prove the global upper estimate $x(t) \leq (1 + \lambda_3)K$, it remains to verify that if $x(t) \geq K + \varepsilon$, $t \in [t^* + \tau_{\max}, t_2]$, we also have $x(t) \leq (1 + \lambda_3)K$. For $t \in [t^* + \tau_{\max}, t_2]$, the argument $g(t)$ satisfies $g(t) \geq t^*$ and $1 + x^\nu(g(t)) \geq 1 + (K + \varepsilon)^\nu > a$. Denote

$$\varepsilon_1 := \frac{a}{1 + (K + \varepsilon)^\nu} \in (0, 1).$$

Assume the contrary that $x(t_2) > (1 + \lambda_3)K$. Without loss of generality, we can consider $x(t_2) > x(s)$, $s \in [t^* + \tau_{\max}, t_2]$ and $x(t_2) \leq \varepsilon_1^{-1}(1 + \lambda_3)K$: otherwise, we pick t_3 where the inequality $x(t_3) \leq \varepsilon_1^{-1}(1 + \lambda_3)K$ holds, as x takes all the intermediate values, and choose the minimal $t_4 \in [t^* + \tau_{\max}, t_2]$ where this value is attained. However, for $t \in [t^* + \tau_{\max}, t_2]$, we have $x(h(t)) \leq x(t_2)$ and, by (2.1),

$$\begin{aligned}\dot{x}(t) &\leq r(t) \left[\frac{ax(h(t))}{1 + (K + \varepsilon)^\nu} - x(t) \right] \\ &\leq r(t) [\varepsilon_1 x(h(t)) - x(t)] \leq r(t) [(1 + \lambda_3)K - x(t)] < 0\end{aligned}$$

for any $x(t) > (1 + \lambda_3)K$. Since $\dot{x}(t) < 0$ on any segment where $x(t)$ increases from $(1 + \lambda_3)K$ to a larger value of $x(t_2)$, we obtain a contradiction, thus $x(t_2) \leq (1 + \lambda_3)K$ for any $t_2 > t^* + \tau_{\max}$.

We choose the next point t^{**} such that

$$x(t^{**}) = K + \varepsilon_1, \quad \varepsilon_1 < \frac{1 - \lambda_1}{2} \lambda_3 K,$$

and, repeating the same argument with the maximum estimate of x as $(1 + \lambda_3)K$ in the numerator, we obtain

$$x(t) \leq (1 + \lambda_3^2)K, \quad t \geq t^{**} + \tau_{\max}.$$

Proceeding by induction, we get

$$\limsup_{t \rightarrow \infty} x(t) = K. \quad (4.23)$$

Next, let us evaluate the lower limit of the solution. We have from (2.1) and (4.21), for t large enough,

$$\dot{x}(t) \geq r(t) \left[\frac{a(K/\sqrt{2})}{1 + (2K)^\nu} - x(t) \right].$$

Similarly to the proof of the upper-limit case, we take t^* such that $x(t^*) = K - \varepsilon$, where

$$\varepsilon < K \frac{1 - \lambda_2}{2\sqrt{2}}$$

and consider only the time segment $[t^*, t^* + \tau_{\max}]$, since for a larger t we get $\frac{a}{1 + x^\nu(g(t))} > 1$, and the function increases as soon as it reaches the previously achieved minimum. On $[t^*, t^* + \tau_{\max}]$ we have

$$\sqrt{2} \left[1 - \frac{a}{\sqrt{2}(1 + (2K)^\nu)} \right] \sup_{t \geq t_0 + \tau_{\max}} \int_{t - \tau_{\max}}^t r(s) \, ds \leq \lambda_2,$$

and, by (4.19) and our choice of ε ,

$$\begin{aligned}x(t) &= K - \varepsilon + \int_{t^*}^t r(s) \left[\frac{ax(h(s))}{1 + x^\nu(g(s))} - x(s) \right] \, ds \\ &\geq K - \varepsilon - \left[\sqrt{2}K - \frac{aK/\sqrt{2}}{1 + (2K)^\nu} \right] \sup_{t \geq t_0 + \tau_{\max}} \int_{t - \tau_{\max}}^t r(s) \, ds \\ &\geq K - \frac{1 - \lambda_2}{2\sqrt{2}}K + \lambda_2 \frac{K}{\sqrt{2}} = \\ &= K - \lambda_4 \frac{K}{\sqrt{2}},\end{aligned}$$

where $\lambda_4 := (\lambda_2 + 1)/2 \in (0, 1)$.

As previously, continuing the process by induction, we obtain

$$\liminf_{t \rightarrow \infty} x(t) = K, \quad (4.24)$$

where (4.23) and (4.24) imply attractivity of K , which concludes the proof. \square

Corollary 4.11. Let $\alpha > \beta > 0$, $\tau > 0$, $\sigma > 0$, $\tau_0 = \max\{\tau, \sigma\}$, $\beta\tau_0 < \frac{1}{4} \ln 2$, and

$$(1 - e^{-\beta\tau_0}) \frac{\alpha}{\beta(1 + (K/\sqrt{2})^\nu)} < \frac{1}{2}, \quad \beta\tau_0 \left[1 - \frac{\alpha}{\sqrt{2}\beta(1 + (2K)^\nu)} \right] < \frac{1}{\sqrt{2}}, \quad (4.25)$$

where $K = \left(\frac{\alpha}{\beta} - 1 \right)^{1/\nu}$. Then any solution of

$$\dot{x}(t) = \frac{\alpha x(t - \tau)}{1 + x^\nu(t - \sigma)} - \beta x(t), \quad t \geq t_0, \quad \alpha > \beta > 0, \quad \tau > 0, \quad \sigma > 0 \quad (4.26)$$

with a nonnegative initial function and $x(t_0) > 0$ satisfies $\lim_{t \rightarrow \infty} x(t) = K$.

Proof. Substituting $r(t) \equiv \beta$, $a = \frac{\alpha}{\beta}$ in (2.1) and $A = \beta$ in the assumptions of Theorem 4.10, we obtain the statement of the corollary. \square

Example 4.12. Consider the equation

$$\dot{x}(t) = \frac{2x(t - 0.1)}{1 + x^2(t - 0.15)} - x(t) \quad (4.27)$$

which is a particular case of (2.6). For (2.6), estimate (4.17) of the maximal delay is $\tau_{\max} < \tau \approx 0.17329$. If, as in (4.27), $A = 1$, $a = 2$, $K = 1$, $\tau_{\max} = 0.15$, (4.18) and (4.19) are satisfied. Numerical runs with Matlab illustrate convergence of positive solutions to K .

Remark 4.13. Let us note that, under (4.17), the two other conditions can be considered separately: (4.18) yields (4.23), while (4.19) implies (4.24). We also note that the form of (4.18) and (4.19) is significantly influenced by (4.17): if we choose other a priori estimates than (4.21), we obtain a different set of asymptotic stability conditions.

5. Discussion and open problems

It is well known that for a linear delay differential equation (DDE), under natural assumptions, exponential stability is preserved under small perturbations of its parameters. For nonlinear DDE and global exponential stability it is not so. For example, the equation $\dot{x} = -x$ is exponentially stable. However, the zero equilibrium of the equation $\dot{x} = -x + \alpha x^2$ is not globally exponentially stable for any parameter $\alpha > 0$. On the other hand, small perturbations may preserve global exponential stability for special classes of equations and their parameters.

We have obtained for equation (2.1) both local stability and global attractivity conditions, as well as instability conditions. However, there is still a gap between stability and instability tests. For example, (2.6) is unstable when the maximal delay exceeds 4 (see Fig. 1) while global stability is obtained for the maximal delay not exceeding 0.17 (see Example 4.12). However, the maximal delay bound is a part of a

priori estimate and presumably can be improved. Filling the gap between stability and instability conditions is a suggested area for further investigations.

Let us describe some other open problems and topics for future research.

1. [Theorem 4.10](#) presents stability conditions when the maximal delay is reduced. Is it possible to get global exponential stability for (2.1) without the restriction on $\max\{\tau, \sigma\}$ but the difference $|\sigma - \tau|$ being sufficiently small?
2. In [Theorems 4.2–4.4](#) we presented global stability conditions for several special cases of equation (2.1). Consider the following classes of delays for (2.1):
 - $\limsup_{t \rightarrow \infty} (t - h(t)) = 0$, g is arbitrary;
 - $\limsup_{t \rightarrow \infty} (t - g(t)) = 0$, h is arbitrary.
3. For (2.1) with $g(t) = t - \sigma$, $h(t) = t - \tau$, obtain sharper global stability results than in [Theorem 4.10](#).
4. Extend the results of the paper to integro-differential equations and equations with distributed delays.
5. Consider other models of population dynamics that in classical version involved two coinciding delays, when the two delays may vary, for example, Nicholson's blowflies equation with two different delays

$$\dot{x}(t) = r(t) \left[Px(h(t))e^{-\alpha x(g(t))} - \delta x(t) \right],$$

as well as modified Nicholson's blowflies equation (1.2) and modified Mackey–Glass equation (1.3).

6. Generalize the results of the present paper to equation (1.1), where f is decreasing in some of the x -arguments and increasing in the others.

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