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Soliton solutions of an integrable nonlocal modified Korteweg–de Vries equation through inverse scattering transform

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A B S T R A C T

It is well known that the nonlinear Schrödinger (NLS) equation is a very important integrable equation. Ablowitz and Musslimani introduced and investigated an integrable nonlocal NLS equation through inverse scattering transform. Very recently, we proposed an integrable nonlocal modified Korteweg–de Vries equation (mKdV) which can also be found in the papers of Ablowitz and Musslimani. We have constructed the Darboux transformation and soliton solutions for the nonlocal mKdV equation. In this paper, we will investigate further the nonlocal mKdV equation. We will give its exact solutions including soliton and breather through inverse scattering transformation. These solutions have some new properties, which are different from the ones of the mKdV equation.

1. Introduction

As is well known, the nonlinear Schrödinger (NLS) equation

$$iq_t(x,t) = q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t) \tag{1}$$

has been investigated deeply since the important work of Zakharov and Shabat [31]. In physics, the NLS equation can characterize plenty of models in varies aspects, such as nonlinear optics [6], plasma physics [18], deep water waves [8] and in pure mathematics like motion of curves in differential geometry [25]. The NLS equation can be derived from the theory of deep water wave, and also from the Maxwell equations.

It should be noted that the NLS equation is parity-time-symmetric (PT-symmetric), which has become an interesting topic in quantum mechanics [7], optics [24,26], Bose–Einstein condensates [9] and quantum chromodynamics [23], etc.

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A nonlocal NLS equation has been introduced by Ablowitz and Musslimani in [2]:

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(-x,t). \tag{2}$$

It can be yielded from the famous AKNS system. Ablowitz and Musslimani gave its infinitely many conservation laws and solved it through the inverse scattering transformation [2]. Eq. (2) has different properties from Eq. (1), e.g., Eq. (2) contains both bright and dark soliton [27] and solutions with periodic singularities [2]. But, like the NLS equation (1), the nonlocal NLS equation (2) is also PT-symmetric. This is a very important property of the nonlocal NLS equation (2). This implies possible physical application of the nonlocal NLS equation (2) [10,20,21]. It has been demonstrated in [7] that in the spectrum of the Hamiltonian, the PT-symmetry has considerable influences. Recent progress on nonlinear wave dynamics in PT-symmetric systems is comprehensively reviewed in [16]. It has been shown that PT-symmetric systems possess new properties and phenomena apart from traditional conservative or dissipative systems. Ref. [22] shows applications of PT-symmetry to invisibility. The constant-intensive wave solutions to a PT-symmetric system due to the advantage of the balance of gain and loss potentials and their modulation instability are discussed [22].

Very recently, motivated by the work of nonlocal NLS equation due to Ablowitz and Musslimani, we proposed and investigated a nonlocal modified Korteweg–de Vries (mKdV) equation in [14],

$$q_t(x,t) + 6q(x,t)q(-x,-t)q_x(x,t) + q_{xxx}(x,t) = 0. \tag{3}$$

Its Lax integrability, Darboux transformation, and soliton solution have been discussed in our paper [14]. We should remark here that the nonlocal mKdV equation (3) and its soliton solutions also appeared in the papers of Ablowitz and Musslimani [3,4]. The mKdV equation can be derived from Euler equation and has applications in varies physical fields [15,19]. Wadati used inverse scattering transformation to study mKdV equation and obtained explicit solutions, including N-solitons, multiple-pole solutions and solutions derived from PT-symmetric potentials [28–30]. Hirota also obtained N-solitons by bilinear technique and investigated multiple collisions of solitons [12]. The possible physical application of the nonlocal mKdV equation (3) is discussed in [20].

In this paper, we will investigate further the new integrable nonlocal mKdV equation (3). We will construct exact solutions of the nonlocal mKdV equation (3) including soliton and breather through inverse scattering transformation. These solutions have some new properties, which are different from the ones of the mKdV equation. We remark here that the main content of this paper appeared in [13].

2. Inverse scattering transformation on nonlocal mKdV equation

The invention of inverse scattering transformation (IST) is due to the pioneering work of Gardner, Greene, Kruskal, and Miura for the Cauchy problem of KdV equation [11]. IST has been developed into a systematic method to construct exact solutions for integrable nonlinear systems [1,5,17]. In this section, we will give the IST for the nonlocal mKdV equation (3). Start with the following linear problem,

$$\varphi_x = \mathbf{U}\varphi = (-ik\sigma_3 + \mathbf{Q})\varphi, \tag{4}$$

$$\varphi_t = \mathbf{V}\varphi = (-4ik^3\sigma_3 + 4k^2\mathbf{Q} - 2ik\mathbf{V}_1 + \mathbf{V}_2)\varphi, \tag{5}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix},$$

$$\mathbf{V}_1 = (\mathbf{Q}^2 + \mathbf{Q}_x)\sigma_3, \qquad \mathbf{V}_2 = -\mathbf{Q}_{xx} + 2\mathbf{Q}^3 + \mathbf{Q}_x\mathbf{Q} - \mathbf{Q}\mathbf{Q}_x,$$

where $\varphi = (\varphi_1(x,t), \varphi_2(x,t))^{\mathrm{T}}$, and k is the spectral parameter. The compatibility condition of system (4) and (5) $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$ leads to

$$\begin{aligned} &q_t(x,t) + q_{xxx}(x,t) - 6q(x,t)r(x,t)q_x(x,t) = 0, \\ &r_t(x,t) + r_{xxx}(x,t) - 6q(x,t)r(x,t)r_x(x,t) = 0. \end{aligned} \tag{6}$$

Nonlocal mKdV equation (3) is obtained from system (6) under the reduction

$$r(x,t) = -q(-x,-t). \tag{7}$$

Next, following the standard procedure of inverse scattering transformation (e.g. see [1,3,5]), we will give the inverse scattering for nonlocal mKdV equation. Assume $q(x,t)$ and its derivatives with respect to x vanish rapidly at infinity. So does $r(x,t)$. Fix time $t = 0$. Define $\phi(x,k)$ and $\bar{\phi}(x,k)$ as a pair of eigenfunctions of Eq. (4), which satisfy the following boundary conditions,

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad x \to -\infty. \tag{8}$$

Similarly, $\psi(x,k)$ and $\bar{\psi}(x,k)$ are defined as another pair of eigenfunctions of Eq. (4) satisfying a different boundary conditions,

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad x \to +\infty. \tag{9}$$

Note that, in this paper, we denote the complex conjugation of ϕ by ϕ^* instead of $\bar{\phi}$. Furthermore, ϕ and ψ are required to be analytic in upper half k-plane, while $\bar{\phi}$ and $\bar{\psi}$ are required to be analytic in lower half k-plane. For a solution $u(x,k)$ and $v(x,k)$ to Eq. (4), their Wronskian $W[u,v] = u_1v_2 - u_2v_1$ is independent of x. Since $\{\phi, \bar{\phi}\}$ and $\{\psi, \bar{\psi}\}$ are linearly dependent, we set

$$\begin{aligned} \phi(x,k) &= a(k)\bar{\psi}(x,k) + b(k)\psi(x,k), \\ \bar{\phi}(x,k) &= \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k). \end{aligned} \tag{10}$$

The scattering data therefore can be expressed as

$$\begin{aligned} a(k) &= W[\phi(x,k), \psi(x,k)], b(k) = W[\bar{\psi}(x,k), \phi(x,k)], \\ \bar{a}(k) &= W[\bar{\psi}(x,k), \bar{\phi}(x,k)], \bar{b}(k) = W[\bar{\phi}(x,k), \psi(x,k)]. \end{aligned} \tag{11}$$

One can prove that ϕe^{ikx}, ψe^{-ikx} and $a(k)$ are analytic functions in upper half k-plane; $\bar{\phi}e^{-ikx}$, $\bar{\psi}e^{ikx}$ and $\bar{a}(k)$ are analytic functions in lower half k-plane [5]. Define $\rho(k) = b(k)/a(k)$ and $\bar{\rho}(k) = \bar{b}(k)/\bar{a}(k)$ as reflection coefficients. Assume k_m $(m = 1, 2, \cdots, N)$, the zeros of $a(k)$ in upper half k-plane, are single, as well as $\bar{k}_n$ $(n = 1, 2, \cdots, \bar{N})$ denoted as the zeros of $\bar{a}(k)$ in lower half k-plane. When $a(k_m) = 0$, by Eq. (11), it yields that $\phi(x, k_m)$ and $\psi(x, k_m)$ are linearly dependent, i.e. there exist constants γ_m such that $\phi(x, k_m) = \gamma_m \psi(x, k_m)$. Similarly, denote $\bar{\gamma}_n$ such that $\bar{\phi}(x, \bar{k}_n) = \bar{\gamma}_n \bar{\psi}(x, \bar{k}_n)$. The normalizing coefficients $\{c_m, \bar{c}_n\}$ are defined by

$$c_m^2 = \frac{i\gamma_m}{\dot{a}(k_m)} \ \ (m = 1, 2, \cdots, N); \qquad \bar{c}_n^2 = \frac{i\bar{\gamma}_n}{\dot{\bar{a}}(\bar{k}_n)} \ \ (n = 1, 2, \cdots, \bar{N}). \tag{12}$$

We should note here, under the reduction (7), the scattering data obeys $b(k) = -\bar{b}(-k^*)$, $a(k) = a^*(-k^*)$ and $\bar{a}(k) = \bar{a}^*(-k^*)$, when $q(x)$ is a real function. That means the eigenvalues are pure imaginary or appear in pairs $\{k_m, -k_m^*\}$ and $\{\bar{k}_n, -\bar{k}_n^*\}$.

Suppose eigenfunctions ψ and $\bar{\psi}$ hold for the following form:

$$\begin{aligned} \psi(x,k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} + \int_x^\infty K(x,s) e^{iks} \mathrm{d}s, \\ \bar{\psi}(x,k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} + \int_x^\infty \bar{K}(x,s) e^{-iks} \mathrm{d}s, \end{aligned} \tag{13}$$

where $K(x,s) = (K_1(x,s), K_2(x,s))^{\mathrm{T}}$ and $\bar{K}(x,s) = (\bar{K}_1(x,s), \bar{K}_2(x,s))^{\mathrm{T}}$, $x < s$. Substituting Eq. (13) into Eq. (4) yields that $K_1(x,s)$ and $K_2(x,s)$ satisfy a Goursat problem, which means the solution exists and is unique. Moreover, one can get the relations between potentials and $K(x,y)$ and $\bar{K}(x,y)$:

$$q(x) = -2K_1(x,x), \qquad r(x) = -2\bar{K}_2(x,x). \tag{14}$$

Let

$$\begin{aligned} F_c(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \rho(k) e^{ikx} \mathrm{d}k, \quad F_d(x) = \sum_{m=1}^{N} c_m^2 e^{ik_m x}, \quad F(x) = F_c(x) - F_d(x), \\ \bar{F}_c(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{\rho}(k) e^{-ikx} \mathrm{d}k, \quad \bar{F}_d(x) = \sum_{n=1}^{\bar{N}} \bar{c}_n^2 e^{-i\bar{k}_n x}, \quad \bar{F}(x) = \bar{F}_c(x) - \bar{F}_d(x). \end{aligned} \tag{15}$$

Through Eq. (10), one arrives at Gel'fand–Levitan–Marchenko integral equation (GLM):

$$\begin{aligned} \bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^\infty K(x,s) F(s+y) \mathrm{d}s = 0, \\ K(x,y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) + \int_x^\infty \bar{K}(x,s) \bar{F}(s+y) \mathrm{d}s = 0. \end{aligned} \tag{16}$$

The time evolution of scattering data $\{\rho(k,t), \bar{\rho}(k,t)\}$ and normalizing coefficients $\{c_m^2, \bar{c}_n^2\}$ are given by

$$\begin{aligned} \rho(k,t) &= \rho(k,0) e^{8ik^3 t}, \quad c_m^2(t) = c_m^2(0) e^{8ik_m^3 t} \quad (m = 1, 2, \cdots, N), \\ \bar{\rho}(k,t) &= \bar{\rho}(k,0) e^{-8i\bar{k}^3 t}, \quad \bar{c}_n^2(t) = \bar{c}_n^2(0) e^{-8i\bar{k}_n^3 t} \quad (n = 1, 2, \cdots, \bar{N}). \end{aligned} \tag{17}$$

Then, putting Eq. (17) into Eq. (15) and solving GLM Eq. (16) yields $K(x,y)$ and $\bar{K}(x,y)$. Finally, $q(x,t)$ and $r(x,t)$ are constructed.

Next, we consider the reflectionless inverse scattering problem for Eq. (3). Assume $\rho(k,t) = \bar{\rho}(k,t) \equiv 0$. Denote I_N or $I_{\bar{N}}$ is an N-dimensional or $\bar{N}$-dimensional unit matrix. Introduce $N \times 1$ column vectors

$$g(x,t) = (g_1(x,t), \cdots, g_m(x,t), \cdots, g_N(x,t))^{\mathrm{T}}, \qquad h(x,t) = (h_1(x,t), \cdots, h_m(x,t), \cdots, h_N(x,t))^{\mathrm{T}},$$

$\bar{N} \times 1$ column vectors

$$\bar{g}(x,t) = (\bar{g}_1(x,t), \cdots, \bar{g}_n(x,t), \cdots, \bar{g}_{\bar{N}}(x,t))^{\mathrm{T}}, \qquad \bar{h}(x,t) = (\bar{h}_1(x,t), \cdots, \bar{h}_n(x,t), \cdots, \bar{h}_{\bar{N}}(x,t))^{\mathrm{T}},$$

and matrix $E(x,t) = (e_{pj})_{\bar{N} \times N}$, where

$$h_m(x,t)=c_m(t)e^{ik_mx},\qquad \bar{h}_n(x,t)=\bar{c}_n(t)e^{-i\bar{k}_nx},\qquad e_{nm}(x,t)=\frac{h_m(x,t)\bar{h}_n(x,t)}{k_m-\bar{k}_n}.$$

Suppose $K_1(x,y)$, $\bar{K}_2(x,y)$ have the following expressions:

$$K_1(x,y,t)=\bar{h}(y,t)^{\mathrm{T}}\bar{g}(x,t),\qquad \bar{K}_2(x,y,t)=h(y,t)^{\mathrm{T}}g(x,t). \tag{18}$$

Start from the reflectionless GLM Eq. (16), we have

$$0=K(x,y,t)-\begin{pmatrix}1\\0\end{pmatrix}\bar{F}_d(x+y,t)-\int_x^{\infty}\left[\begin{pmatrix}0\\1\end{pmatrix}F_d(x+s,t)+\int_x^{\infty}K(x,z,t)F_d(z+s,t)\mathrm{d}z\right]\bar{F}_d(s+y,t)\mathrm{d}s,$$

and

$$0=K_1(x,y,t)-\bar{F}_d(x+y,t)-\int_x^{\infty}K_1(x,z,t)\left(\int_x^{\infty}F_d(z+s,t)\bar{F}_d(s+y,t)\mathrm{d}s\right)\mathrm{d}z.$$

Note that

$$\int_x^{\infty}F_d(z+s,t)\bar{F}_d(s+y,t)\mathrm{d}s=\sum_{j=1}^{N}\sum_{p=1}^{\bar{N}}\frac{ic_j^2(t)c_p^2(t)}{k_j-\bar{k}_p}e^{ik_j(x+z)-i\bar{k}_p(x+y)}.$$

We thus obtain

$$\begin{aligned}&\sum_{n=1}^{\bar{N}}\bar{c}_n(t)\bar{g}_n(x,t)e^{-i\bar{k}_ny}-\sum_{n=1}^{\bar{N}}\bar{c}_n^2(t)e^{-i\bar{k}_n(x+y)}\\&-\sum_{p=1}^{\bar{N}}\sum_{j=1}^{N}\sum_{n=1}^{\bar{N}}\int_x^{\infty}\bar{c}_p(t)\bar{g}_p(x,t)e^{-i\bar{k}_pz}\frac{ic_j^2(t)c_n^2(t)}{k_j-\bar{k}_n}e^{ik_j(x+z)-i\bar{k}_n(x+y)}\mathrm{d}z=0.\end{aligned}$$

This gives

$$\bar{g}_n(x,t)-\bar{c}_n(t)e^{-i\bar{k}_nx}+\sum_{p=1}^{\bar{N}}\sum_{j=1}^{N}\bar{g}_p(x,t)e_{pj}(x,t)e_{nj}(x,t)=0,\qquad n=1,2,\cdots,\bar{N},$$

which can be written in a vector form,

$$\bar{g}(x,t)-\bar{h}(x,t)+E(x,t)E(x,t)^{\mathrm{T}}\bar{g}(x,t)=0.$$

So, we obtain

$$\begin{aligned}K_1(x,y,t)&=\bar{h}(y,t)^{\mathrm{T}}\bar{g}(x,t)\\&=\bar{h}(y,t)^{\mathrm{T}}(I_{\bar{N}}+E(x,t)E(x,t)^{\mathrm{T}})^{-1}\bar{h}(x,t)\\&=\mathrm{tr}\left[(I_{\bar{N}}+E(x,t)E(x,t)^{\mathrm{T}})^{-1}\bar{h}(x,t)\bar{h}(y,t)^{\mathrm{T}}\right].\end{aligned}$$

Do the similar work on $\bar{K}_2(x,y,t)$ for $r(x,t)$. Finally, $q(x,t)$ and $r(x,t)$ can be written

$$\begin{aligned}q(x,t)&=-2\mathrm{tr}\left[(I_{\bar{N}}+E(x,t)E(x,t)^{\mathrm{T}})^{-1}\bar{h}(x,t)\bar{h}(x,t)^{\mathrm{T}}\right],\\r(x,t)&=-2\mathrm{tr}\left[(I_N+E(x,t)^{\mathrm{T}}E(x,t))^{-1}h(x,t)h(x,t)^{\mathrm{T}}\right].\end{aligned} \tag{19}$$

When eigenvalues $\{k_m, \bar{k}_n\}$ are suitably selected and Eq. (19) satisfies the constraint (7), $q(x,t)$ becomes the solution of Eq. (3) with initial data $\{c_m(0), \bar{c}_n(0)\}$.

We should emphasize here that the scattering problem for nonlocal mKdV equation is different from the one of the classical mKdV equation. Note that the spatial part of the Lax pair in the two equations are distinct. The boundary conditions of the eigenfunctions are derived from the spatial part of the Lax pair. Therefore, different eigenfunctions in the two equations leads to the different scattering data. So, the scattering data in the nonlocal case perform different properties with ones in the classical problem. The scattering coefficients $a(k)$ and $\bar{a}(k)$ for the nonlocal case have no relations, while ones of classical problems have. This leads to that eigenvalues $k_j, \bar{k}_j$ are not related, either. The normalizing coefficients $c_j, \bar{c}_j$ depend on the eigenvalues $k_j, \bar{k}_j$ in the nonlocal case, which will be mentioned in the next section, rather than being free parameters in the classical case. In the classical case, eigenfunctions, which are analytic in the upper k-plane, are related to those being analytic in the lower k-plane. But, this property does not hold anymore in the nonlocal case. This is the most important difference between these two cases, which is also mentioned in [3]. Again, We emphasize that though the procedure described above of solving nonlocal mKdV equation seems same as the one for the classical mKdV equation, there exist important differences between these two cases.

At the last of this section, we present the conservation laws of Eq. (3). From the Lax pair of nonlocal mKdV equation (3), one can derive infinite number of conservation quantities $\{H_k\}$. Here, we list first few conservation laws as follows:

$$H_1 = \int_{-\infty}^{+\infty} q(x,t)q(-x,-t)\mathrm{d}x, \qquad H_2 = \int_{-\infty}^{+\infty} q(x,t)q_x(-x,-t)\mathrm{d}x,$$

$$H_3 = \int_{-\infty}^{+\infty} [q(x,t)q_{xx}(-x,-t) - q^2(x,t)q^2(-x,-t)]\mathrm{d}x.$$

3. Some solutions and their analysis

In this section, we will derive soliton solutions of integrable nonlocal mKdV equation (3) from the explicit formula (19).

Case 1. One-soliton solutions

Let $N = \bar{N} = 1$ and the eigenvalues be pure imaginary. From formula (19) and the symmetry reduction (7), it can be derived that $c_1(0), \bar{c}_1(0)$ and $k_1, \bar{k}_1$ has the following constraints:

$$(k_1 - \bar{k}_1)^2 + [c_1(0)]^4 = 0, \qquad (k_1 - \bar{k}_1)^2 + [\bar{c}_1(0)]^4 = 0. \tag{20}$$

Denote $k_1 = i\alpha$ and $\bar{k}_1 = -i\beta$, where $\alpha, \beta > 0$. Substituting the above constraints into Eq. (19) yields the one-soliton solution

$$q(x,t) = \frac{2(\alpha+\beta)}{e^{-2\alpha(x-4\alpha^2 t)} + \sigma e^{2\beta(x-4\beta^2 t)}}, \tag{21}$$

where $\sigma = \pm 1$. If $\sigma = 1$, q can be written as

$$q(x,t) = (\alpha+\beta)e^{(\alpha-\beta)x-4(\alpha^3-\beta^3)t}\,\mathrm{sech}((\alpha+\beta)x - 4(\alpha^3+\beta^3)t). \tag{22}$$

It is obvious that for arbitrary fixed t, $q(x,t) \to 0$ as $|x| \to \infty$. This solution $q(x,t)$ of nonlocal mKdV equation is a soliton solution, but it has different property from the one of classical mKdV equation. We

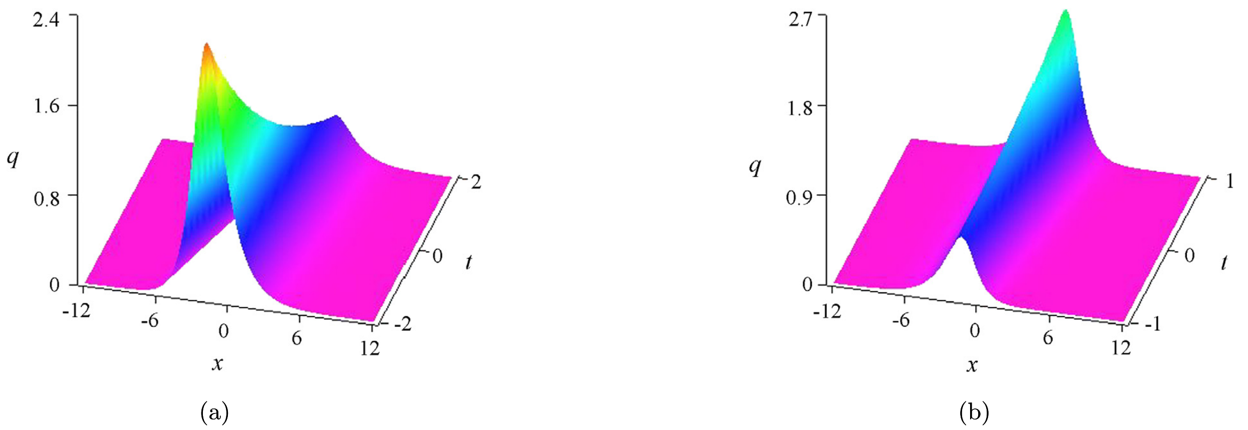

Fig. 1. (a) One-soliton-like solution given by Eq. (22) with $\alpha = 3/5$ and $\beta = 1/3$. The amplitude decays exponentially as t increases; (b) one-soliton-like solution given by Eq. (22) with $\alpha = 1/3$ and $\beta = 3/5$. The amplitude increases exponentially as t increases.

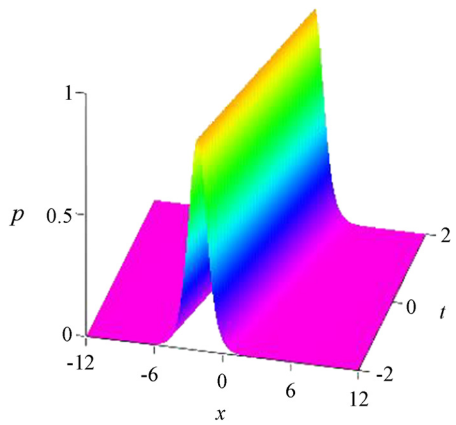

Fig. 2. $p(x,t) \triangleq q(x,t)q(-x,-t)$ given by Eq. (23) with $\alpha = 3/5$ and $\beta = 1/3$.

note that, when x and t satisfy $x/t = k + o(t^{-1})$ $(t \to \infty)$, where k is a constant between $4\alpha^2$ and $4\beta^2$, $q(x,t)$ goes to infinity along these directions as $t \to +\infty$ for $\alpha < \beta$, or $t \to -\infty$ for $\alpha > \beta$. It indicates $q(x,t)$ evolves like a solitary wave with its amplitude increasing or decaying exponentially. Fig. 1 depicts this property. We can see that $q(x,t)$ is a usual soliton in the case of $\alpha = \beta$. Notice that in this case $q(x,t) = q(-x,-t)$, and $\bar{k}_1 = k_1^*$. This means that $q(x,t)$ is also a soliton solution to mKdV equation. It is interesting to note that $q(x,t)q(-x,-t)$ is exactly the mode of a classical one-soliton solution,

$$q(x,t)q(-x,-t) = (\alpha+\beta)^2 \operatorname{sech}^2((\alpha+\beta)x - 4(\alpha^3+\beta^3)t), \tag{23}$$

which is shown in Fig. 2. It follows from Eq. (23) that the behavior of exponential increase or decrease in the amplitude of $q(x,t)$ is counteracted by the one of $q(-x,-t)$. If $\sigma = -1$,

$$q(x,t) = -(\alpha+\beta)e^{(\alpha-\beta)x - 4(\alpha^3-\beta^3)t} \operatorname{csch}((\alpha+\beta)x - 4(\alpha^3+\beta^3)t). \tag{24}$$

So, $q(x,t)$ possesses singularity at the line $\{(x,t)|x = 4(\alpha^2 - \alpha\beta + \beta^2)t\}$.

Case 2. Two-soliton solutions

Set $N = \bar{N} = 2$. First, we obtain the constraints between normalizing coefficients and eigenvalues via Eq. (19) and Eq. (7) by direct calculations:

$$
\begin{aligned}
&[c_1(0)]^4+\frac{(\bar{k}_1-k_1)^2(\bar{k}_2-k_1)^2}{(k_1-k_2)^2}=0, \quad [c_2(0)]^4+\frac{(\bar{k}_1-k_2)^2(\bar{k}_2-k_2)^2}{(k_1-k_2)^2}=0,\\
&[\bar{c}_1(0)]^4+\frac{(k_1-\bar{k}_1)^2(k_2-\bar{k}_1)^2}{(\bar{k}_1-\bar{k}_2)^2}=0, \quad [\bar{c}_2(0)]^4+\frac{(k_1-\bar{k}_2)^2(k_2-\bar{k}_2)^2}{(\bar{k}_1-\bar{k}_2)^2}=0.
\end{aligned} \tag{25}
$$

Then, the general expression of a two-soliton solutions is

$$
\begin{aligned}
q(x,t)&=-2i\frac{F(x,t)}{G(x,t)},\\
F(x,t)&=\frac{\bar{\sigma}_1(k_1-\bar{k}_1)(k_2-\bar{k}_1)}{\bar{k}_1-\bar{k}_2}e^{\bar{\xi}_1}+\frac{\bar{\sigma}_2(k_1-\bar{k}_2)(k_2-\bar{k}_2)}{\bar{k}_1-\bar{k}_2}e^{\bar{\xi}_2}\\
&\quad-\frac{\sigma_1\bar{\sigma}_1\bar{\sigma}_2(\bar{k}_1-k_2)(\bar{k}_2-k_2)}{k_1-k_2}e^{\xi_1+\bar{\xi}_1+\bar{\xi}_2}-\frac{\sigma_2\bar{\sigma}_1\bar{\sigma}_2(\bar{k}_1-k_1)(\bar{k}_2-k_1)}{k_1-k_2}e^{\xi_2+\bar{\xi}_1+\bar{\xi}_2},\\
G(x,t)&=1-\frac{(k_1-\bar{k}_2)(k_2-\bar{k}_1)}{(k_1-k_2)(\bar{k}_1-\bar{k}_2)}\left(\sigma_1\bar{\sigma}_1e^{\xi_1+\bar{\xi}_1}+\sigma_2\bar{\sigma}_2e^{\xi_2+\bar{\xi}_2}\right)\\
&\quad-\frac{(k_1-\bar{k}_1)(k_2-\bar{k}_2)}{(k_1-k_2)(\bar{k}_1-\bar{k}_2)}\left(\sigma_1\bar{\sigma}_2e^{\xi_1+\bar{\xi}_2}+\sigma_2\bar{\sigma}_1e^{\xi_2+\bar{\xi}_1}\right)+\sigma_1\sigma_2\bar{\sigma}_1\bar{\sigma}_2e^{\xi_1+\xi_2+\bar{\xi}_1+\bar{\xi}_2},
\end{aligned} \tag{26}
$$

where $\sigma_j,\bar{\sigma}_j=\pm 1$ $(j=1,2)$ and

$$
\xi_j=2ik_j(x+4k_j^2t), \qquad \bar{\xi}_j=-2i\bar{k}_j(x+4\bar{k}_j^2t), \quad (j=1,2).
$$

Here, we focus on the case of $\{k_j,\bar{k}_j\}_{j=1}^2$ being pure imaginary. Set $k_j=i\alpha_j$, $\bar{k}_j=-i\beta_j$, where $\alpha_j,\beta_j>0$, and $\sigma_j=\bar{\sigma}_j=1$ $(j=1,2)$. For $(\alpha_1-\alpha_2)(\beta_1-\beta_2)>0$, Eq. (26) is simplified to

$$
\begin{aligned}
q(x,t)&=\frac{2F_1(x,t)}{G_1(x,t)},\\
F_1(x,t)&=A[(\alpha_1+\beta_1)e^{u_{2-}}\cosh(u_{2+}+\theta_2)+(\alpha_2+\beta_2)e^{u_{1-}}\cosh(u_{1+}+\theta_1)],\\
G_1(x,t)&=e^{u_{1-}+u_{2-}}[(\alpha_2-\alpha_1)(\beta_2-\beta_1)\cosh(u_{1+}+u_{2+})\\
&\quad+(\alpha_1+\beta_1)(\alpha_2+\beta_2)\cosh(u_{1-}-u_{2-})+(\alpha_1+\beta_2)(\alpha_2+\beta_1)\cosh(u_{1+}-u_{2+})],
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
u_{j\pm}&=\frac{1}{2}(\xi_j\pm\bar{\xi}_j), \qquad A=\sqrt{(\alpha_2-\alpha_1)(\beta_2-\beta_1)(\alpha_1+\beta_2)(\alpha_2+\beta_1)},\\
e^{\theta_1}&=\frac{A}{|\alpha_2-\alpha_1|(\alpha_2+\beta_1)}, \qquad e^{\theta_2}=\frac{A}{|\alpha_2-\alpha_1|(\alpha_1+\beta_2)}.
\end{aligned}
$$

This is a two-soliton solution. In Fig. 3, we describe such a solution with $\alpha_1<\beta_1$ and $\alpha_2=\beta_2$. In this case, we see that the amplitude of one solitary wave in the solution shown has exponential increase as $t\to+\infty$, and the other amplitude is stable but has a change during the collisions of two solitary waves. Furthermore, after the interaction of two solitary waves, there is a shift of phase and no change in the velocity of them. Fig. 4 gives the case of $\alpha_1<\beta_1$ and $\alpha_2>\beta_2$, i.e., the amplitude of a solitary wave increases exponentially, and the one of another solitary wave decreases exponentially. The all solutions above belong to the interactions of bright–bright solitons. Interactions of bright–dark solitons can be found by setting $\sigma_2=-1$ and $\bar{\sigma}_2=-1$. The results are similar with the bright–bright case. In Fig. 5, we give an example of the increase–increase case, i.e. the amplitudes of both two solitary waves have exponential increase as $t\to+\infty$, and the amplitude below zero increases faster than the one above zero. During the

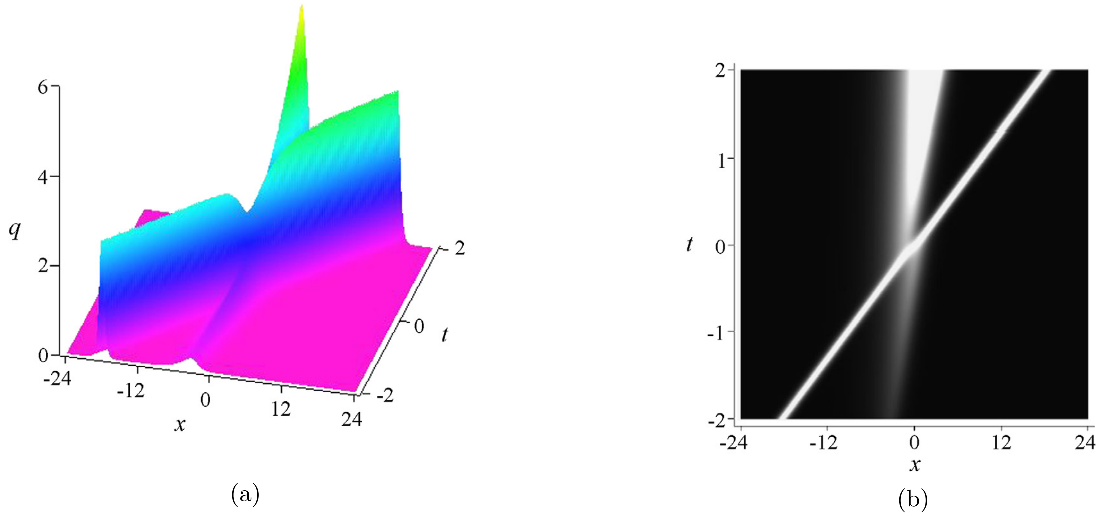

Fig. 3. Two-soliton-like solution of bright–bright kind given by Eq. (27) with $\alpha_1 = 1/4$, $\beta_1 = 3/4$ and $\alpha_2 = \beta_2 = 3/2$. Only one of the amplitudes increases exponentially as t increases.

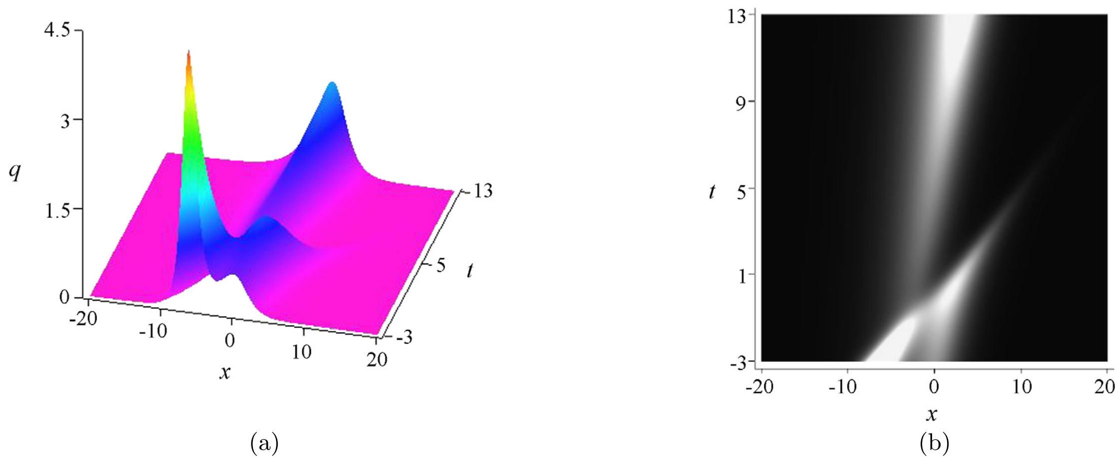

Fig. 4. Two-soliton-like solution of bright–bright kind given by Eq. (27) with $\alpha_1 = 3/16$, $\beta_1 = 3/8$, $\alpha_2 = 3/4$ and $\beta_2 = 9/16$. One of the amplitudes increases exponentially and the other decrease exponentially as t increases.

interaction, both two solitary waves have a shift of phase respectively and no changes in velocity. In the case of $\alpha_j = \beta_j$, i.e., $\bar{k}_j = k_j^*$, $(j = 1, 2)$, the solution is a usual two-soliton solution to nonlocal mKdV equation (3) as well as to mKdV equation. Similar with the case of one-soliton solutions, $q(x,t)q(-x,-t)$ is exactly the mode of a typical two-soliton solution. In fact, we have

$$
\begin{aligned}
q(x,t)q(-x,-t) &= \frac{4F_1(x,t)F_1(-x,-t)}{G_1(x,t)G_1(-x,-t)}, \\
F_1(x,t)F_1(-x,-t) &= A^2\big[(\alpha_1+\beta_1)^2\cosh(u_{2+}+\theta_2)\cosh(u_{2+}-\theta_2) \\
&\quad + (\alpha_2+\beta_2)^2\cosh(u_{1+}+\theta_1)\cosh(u_{1+}-\theta_1) \\
&\quad + (\alpha_1+\beta_1)(\alpha_2+\beta_2)(e^{u_{1-}-u_{2-}}\cosh(u_{1+}+\theta_1)\cosh(u_{2+}-\theta_2) \\
&\quad + e^{u_{2-}-u_{1-}}\cosh(u_{1+}-\theta_1)\cosh(u_{2+}+\theta_2))\big], \\
G_1(x,t)G_1(-x,-t) &= [(\alpha_2-\alpha_1)(\beta_2-\beta_1)\cosh(u_{1+}+u_{2+})
\end{aligned}
\tag{28}
$$

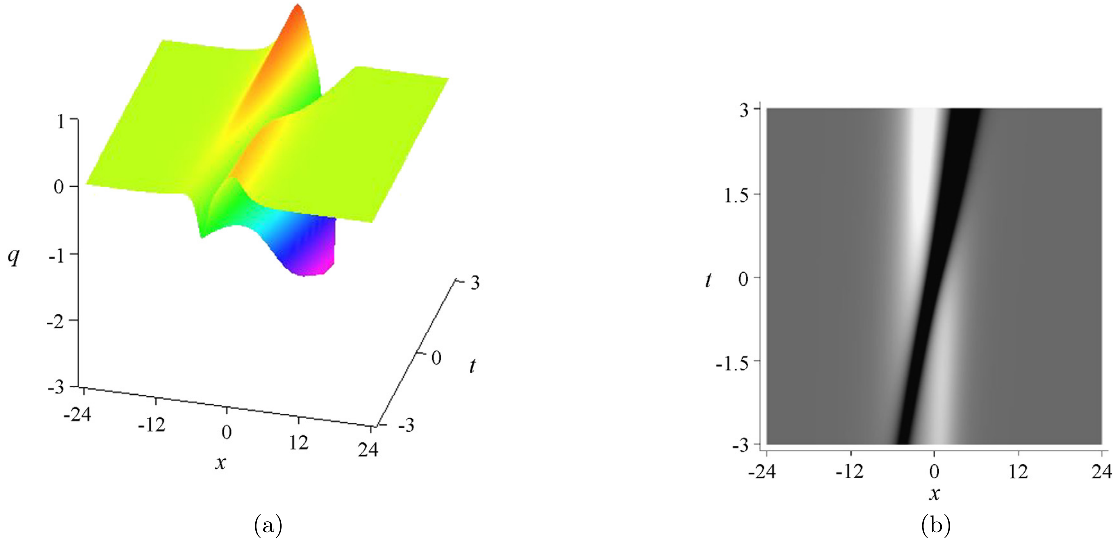

Fig. 5. Two-soliton solution of bright–dark kind given by Eq. (26) with $\sigma_1 = 1$, $\sigma_2 = -1$, $\bar{\sigma}_1 = 1$ and $\bar{\sigma}_2 = -1$, $k_1 = i/2$, $\bar{k}_1 = -i/3$, $k_2 = i/4$ and $\bar{k}_2 = -3i/5$.

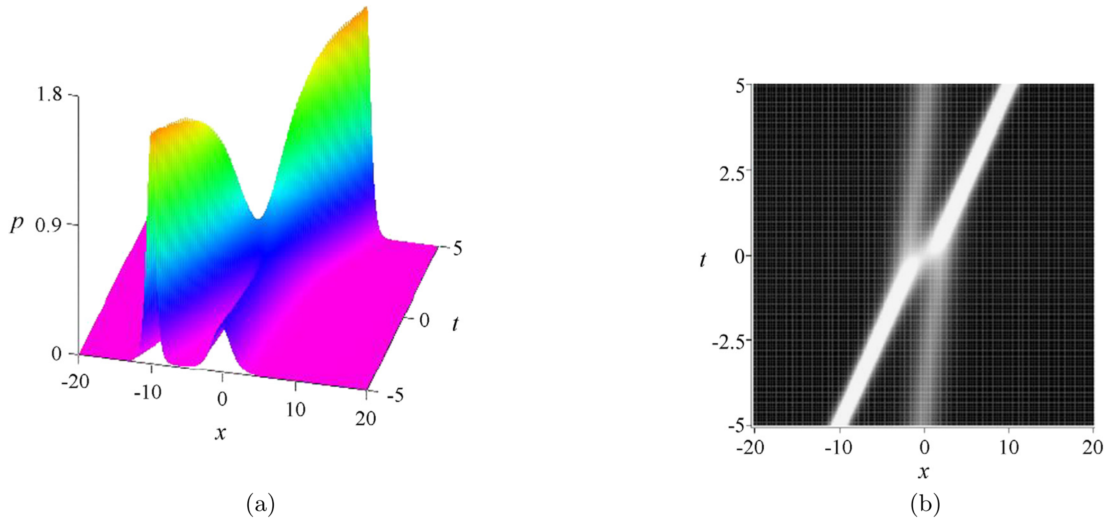

Fig. 6. $p(x,t) \triangleq q(x,t)q(-x,-t)$ defined by Eq. (28) with $\alpha_1 = 3/16$, $\beta_1 = 3/8$, $\alpha_2 = 3/4$ and $\beta_2 = 9/16$.

$$
\begin{aligned}
&+ (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)\cosh(u_{1-} - u_{2-}) \\
&+ (\alpha_1 + \beta_2)(\alpha_2 + \beta_1)\cosh(u_{1+} - u_{2+})]^2.
\end{aligned}
$$

Both the amplitudes of two waves in $q(x,t)q(-x,-t)$ do no longer change exponentially as in $q(x,t)$. We present an example in Fig. 6. For the case of $(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) < 0$, the solution always contains singularity at some sites.

Case 3. Breather solution

Let us consider the case of $k_1 = -k_2^*$ and $\bar{k}_1 = -\bar{k}_2^*$, where k_1 and $\bar{k}_1$ are denoted by $k_1 = \eta_1 + i\zeta_1$ and $\bar{k}_1 = \eta_2 - i\zeta_2$, $(\eta_j, \zeta_j > 0\ j = 1, 2)$, and $\sigma_1\sigma_2 = -1$ and $\bar{\sigma}_1\bar{\sigma}_2 = -1$. In this case, the solution has the expression,

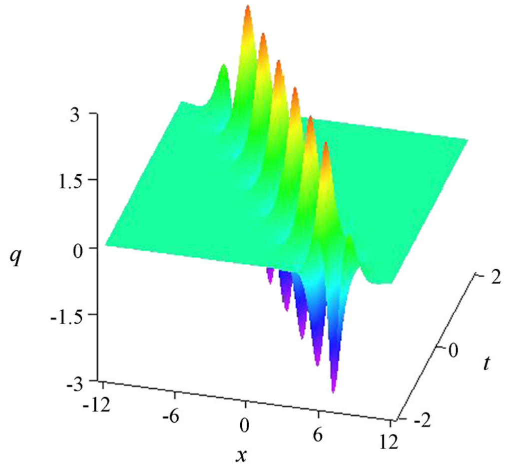

Fig. 7. Breather solution given by Eq. (30) with $\mu = 2/3$.

$$
\begin{aligned}
q(x,t) &= \frac{2F_2(x,t)}{G_2(x,t)},\\
F_2(x,t) &= \eta_1[(\eta_1^2-\eta_2^2+(\zeta_1+\zeta_2)^2)\sin v_{2+} - 2\eta_2(\zeta_1+\zeta_2)\cos v_{2+}]e^{-v_{1-}}\\
&\quad + \eta_2[(\eta_1^2-\eta_2^2-(\zeta_1+\zeta_2)^2)\sin v_{1+} - 2\eta_1(\zeta_1+\zeta_2)\cos v_{1+}]e^{v_{2-}},\\
G_2(x,t) &= 2\eta_1\eta_2\cosh(v_{1-}+v_{2-}) + 2\eta_1\eta_2\cos v_{1+}\cos v_{2+}\\
&\quad + [\eta_1^2+\eta_2^2+(\zeta_1+\zeta_2)^2]\sin v_{1+}\sin v_{2+},
\end{aligned}
\tag{29}
$$

where

$$
v_{j+} = 2\eta_j[x+4(\eta_j^2-3\zeta_j^2)t], \quad v_{j-} = -2\zeta_j[x+4(3\eta_j^2-\zeta_j^2)t], \quad (j=1,2).
$$

The solution possesses singularity if $\eta_1 \neq \eta_2$ and $\zeta_1 \neq \zeta_2$. But, selecting $\eta_1 = \eta_2 = \zeta_1 = \zeta_2$ in Eq. (29) yields an interesting solution,

$$
q(x,t) = 4\mu\frac{\sinh(\xi_+)\sin(\xi_-) - \cosh(\xi_+)\cos(\xi_-)}{\cosh^2(\xi_+) + \sin^2(\xi_-)},
\tag{30}
$$

where $\xi_\pm = -2\mu(x \pm 8\mu^2 t)$ with $\mu > 0$. We view $q(x,t)$ as a function with respect to new variables $\xi_\pm$. Apparently, $q(\xi_+,\xi_-)$ has a period of 2π with respect to ξ_-, i.e. $q(\xi_+,\xi_-) = q(\xi_+,\xi_- + 2\pi)$. Thus, q is a breather solution along ξ_-, as is shown in Fig. 7.

4. Conclusions and discussions

In this paper, we have investigated a new integrable equation—nonlocal mKdV equation through inverse scattering method. We have obtained its solutions in the general form. We have presented its one-soliton, two-soliton and breather solutions. The analysis of the properties of these solutions has been given. We have demonstrated that these solutions for the nonlocal mKdV equation have some different properties from ones of mKdV equation. Very recently, in Ref. [4], Ablowitz and Musslimani introduced new reverse space–time and reverse time nonlocal nonlinear integrable equations. For these nonlocal nonlinear integrable equations, Lax pairs, conservation laws, inverse scattering transformation and one soliton solutions were discussed [4]. We hope to construct richer solutions for these nonlocal nonlinear integrable equations in the future work.

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