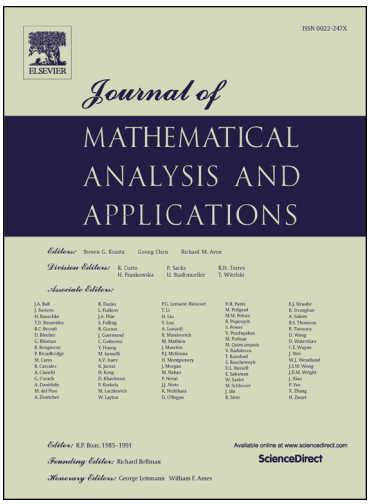
Accepted Manuscript



Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition

Yanxiang Tan, Chuangxia Huang, Bo Sun, Tao Wang

PII: S0022-247X(17)30884-3
DOI: https://doi.org/10.1016/j.jmaa.2017.09.045
Reference: YJMAA 21713

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 20 June 2017

Please cite this article in press as: Y. Tan et al., Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition, *J. Math. Anal. Appl.* (2018), https://doi.org/10.1016/j.jmaa.2017.09.045

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

ACCEPTED MANUSCRIPT

Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition *

Yanxiang Tan[1,2], Chuangxia Huang[2]†, Bo Sun[2], Tao Wang[3]

[1]College of Mathematics and Econometrics, Hunan University
Changsha, Hunan 410082, P. R. China

[2]School of Mathematics and Statistics,
Changsha University of Science and Technology, Changsha, Hunan 410114, China

[3]College of Mathematics and Computing Science,
Hunan University of Science and Technology
Xiangtan, Hunan 411201, P. R. China

Abstract. This paper considers a class of delayed reaction-diffusion systems under the Neumann boundary condition which arise in epidemiology and can describe the temporal and spatial evolutionary phenomena for the bacteria population and the human infective population. With the help of the iterative properties of interval mapping and dynamical system approaches, some positively invariant sets and attractive basins of the considered systems are analyzed detailedly. In addition, combining the global attractivity of interval mapping, we provide some sufficient conditions to ensure local or global attractivity of steady states of the systems. Finally, we apply these theoretical results to some models with different nonlinearity demonstrating "force of infection", and then obtain some sufficient conditions about "force of infection" to describe the survival and extinction of bacteria and infective populations.

Key words: Attraction basin; bacteria-infective population model; delayed reaction-diffusion equation

*Partially supported by the National Natural Science Foundation of P. R. China (Nos. 11571371, 71471020), Hunan Provincial Natural Science Foundation (No. 2016JJ1001), Scientific Research Fund of Hunan Provincial Education Department (No. 15A003), China Postdoctoral Science Foundation (Nos. 2014M550097, 2015T80144)

†Corresponding author. *E-mail address:* cxiahuang@amss.ac.cn (C. Huang); *telephone/fax number:* +86 731 85258293 /+86 731 85258787.

ACCEPTED MANUSCRIPT

1 Introduction

As is known to us, epidemic diseases always bring great harm to human beings. In order to better predict and control the spread of infectious diseases, mathematical modeling and the analysis of dynamics of these models have been subsequently done. In [3], Capasso and Maddalena proposed the following reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= d_1\Delta u(t,x) - a_{11}u(t,x) + a_{12}v(t,x), \\ \frac{\partial v}{\partial t}(t,x) &= d_2\Delta v(t,x) - a_{22}v(t,x) + g(u(t,x)) \end{cases} \tag{1.1}$$

to model the spatial spread of epidemics due to the environmental pollution produced by the infective human population. Here $u(t,x)$ and $v(t,x)$, respectively, represent the spatial densities of bacteria and infective population at a point x in the habitat $\Omega \subseteq \mathbb{R}^m$ and time $t \geq 0$; Δ is Laplacian operator; $d_1 > 0$ and $d_2 > 0$ are diffusion coefficients; a_{11} is the natural death rate of bacteria and $a_{12}v$ is the contribution of infectious population to the density of bacteria; a_{22} is the natural diminishing rate of infected individuals; the nonlinearity $g(u)$ gives the "force of infection" on human due to the concentration of bacteria. For (1.1), one may face an unbounded or a bounded spatial domain depending on the practical situation. When an unbounded domain is considered, travelling wavefront solutions are an important topic, see [14, 36, 40] and references cited therein. When a bounded domain Ω is considered, boundary conditions need to be posed on the boundary $\partial\Omega$ of the domain, depending on the practical scenarios. Taking population dynamics as an example, the homogeneous Dirichlet boundary condition represents the situation where the boundary is hostile, while the homogeneous Neumann boundary condition accounts for the case when the domain is isolated from outside. Capasso and Maddalena [3] studied system (1.1) under homogeneous Neumann boundary conditions by means of the contracting rectangle technique. Later, Capasso and Maddalena [5] investigated the global dynamics of (1.1) with mixed boundary conditions. In [6], a detailed analysis of the steady-state bifurcation pattern was performed for the case of homogeneous Dirichlet boundary conditions applied at the endpoints of a one-dimensional interval. In addition, based on monotone techniques, a partial analysis of the dynamical behavior of (1.1) was carried out. For more related results, one can see [3, 2] and their references therein.

When taking the latent period of pathogen into account, Thieme and Zhao [31] introduced the following modified version of (1.1)

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= d_1\Delta u(t,x) - a_{11}u(t,x) + a_{12}v(t,x), \\ \frac{\partial v}{\partial t}(t,x) &= d_2\Delta v(t,x) - a_{22}v(t,x) + \int_0^+ \infty g(u(t-s,x))P(\mathrm{ds}), \end{cases} \tag{1.2}$$

where P is a probability measure on $\mathbb{R}_+$ that describes the distribution of the latent period. For the case $\Omega = \mathbb{R}^N$ and g is monotone increasing, Thieme and Zhao [31] proved that the spreading speed for the solutions of (1.2) with $d_2 = 0$ having compact support coincides with the minimal wave speed for traveling wave front of (1.2). When g is non-monotone, system (1.2) with $d_2 = 0$ does not satisfy the so-called quasi-monotone condition and the comparison

ACCEPTED MANUSCRIPT

theorems are not applicable. In order to overcome the difficulty, Wu and Liu [34] obtained the existence of traveling waves and the minimal wave speed by the construction of two associated auxiliary monotone integral equations and a profile set in a suitable Banach space. In [33], the authors considered the following nonlocal reaction-diffusion model with distributed delay

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= d_1\Delta u(t,x) - a_{11}u(t,x) + \int_\Omega K(x-y)v(t,y)dy, \\ \frac{\partial v}{\partial t}(t,x) &= d_2\Delta v(t,x) - a_{22}v(t,x) + \int_0^+ \infty g(u(t-s,x))P(\mathrm{ds}), \end{cases} \tag{1.3}$$

Compared with (1.2), the main feature of (1.3) is the presence of the nonlocal term, which is important when dealing with indirect transmission diseases, typical of infectious diseases transmitted via the pollution of the environment due to the infective population. When g is monotone increasing, the authors established the existence of the minimal wave speed of monotone traveling waves of (1.3) with $d_2 = 0$ under proper assumptions on K. For the case that g is non-increasing, some nice results about global attractivity, spreading speeds and traveling waves of (1.3) with $d_2 \neq 0$ are obtained in [35].

If the latent period is a fixed positive constant denoted by τ, (1.2) becomes

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= d_1\Delta u(t,x) - a_{11}u(t,x) + a_{12}v(t,x), \\ \frac{\partial v}{\partial t}(t,x) &= d_2\Delta v(t,x) - a_{22}v(t,x) + g(u(t-\tau,x)). \end{cases} \tag{1.4}$$

Compared with (1.1), (1.4) was cooperated into time delay which is applicable to more realistic models. In this paper, we are interested in the Neumann boundary condition and constant time delay, that is, we consider the following boundary initial value problem (BIVP):

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d_1\Delta u(t,x) - a_{11}u(t,x) + a_{12}v(t,x) \quad \text{for} \quad (t,x) \in (0,+\infty)\times\partial\Omega, \\ \frac{\partial v}{\partial t}(t,x) = d_2\Delta v(t,x) - a_{22}v(t,x) + g(u(t-\tau,x)) \quad \text{for} \quad (t,x) \in (0,+\infty)\times\partial\Omega, \\ (u(\theta,x), v(0,x)) = (\eta(\theta,x), \xi(x)) \quad \text{for} \quad (\theta,x) \in [-\tau,0]\times\bar{\Omega}, \\ \frac{\partial u}{\partial \nu}(\theta,x) = \frac{\partial v}{\partial \nu}(\theta,x) = 0 \quad \text{for} \quad (\theta,x) \in [-\tau,0]\times\partial\Omega, \end{cases} \tag{1.5}$$

which concerns the evolution of the bacteria population with fixed latent period and of the human infective population in a region where the two populations are both completely isolated from outside. Here, $\Omega \subseteq \mathbb{R}^m$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ represents the differentiation in the direction of the outward normal to $\partial\Omega$, parameters $d_i, a_{ij} > 0$, and $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function with $g(0) = 0$ and $g((0,+\infty)) \subseteq (0,+\infty)$. We also assume initial functions $\eta \in C([-\tau,0]\times\bar{\Omega}, \mathbb{R}_+)$ and $\xi \in C(\bar{\Omega}, \mathbb{R}_+)$. Equations (1.5) and their broader forms have been indeed studied by references and reviewers, one will find that the methods and the constraint conditions used in our paper are completely different from those published papers. Just taking the Pao's work as an example [42], they restrict the nonlinear reaction functions to possess some quasimonotone property and adopt the classical method of upper and lower solutions. Our paper consider the bistable, non-monotonic nonlinear reactions, and we mainly use the iterative properties of interval mapping and dynamical system approaches. It is obvious that Pao's work can not cover our results.

ACCEPTED MANUSCRIPT

In the following, we restrict our attention to the existence of positively invariant set and attractive basins, and the sufficient conditions to guarantee local or global attractivity of a steady state of system (1.5). We will address these questions by using dynamical system approaches and the iterative properties of interval mapping. To explain our motivations and methods, let us first recapitulate the achievements for the scalar equation related to system (1.5). We point out that the idea of relating the dynamics of a map to the dynamics of some scalar delay differential equation has been used by some other researchers, among which are Ivanov and Sharkovsky [16], Hale and Verduyn Lunel [[13], Section 12.7] and Huang et al. [15], [22, 18, 4, 7]. Recently, this idea has also been successfully employed to study some scalar delay reaction diffusion equations in Yi and Zou [39, 37].

Motivated by the above discussion, in this paper, by employing the iterative feature of attracting intervals and dynamical system approach, combining the special comparative properties of non-monotone system (1.5), we can obtain some very simple conditions that can assure local or global attractivity of a steady state of the systems.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2 where we present some basic definitions and notations. In particular, we give some basic properties of the compactness and positivity on the semiflow of system (1.5), which can also be prepared for the following argument. In Section 3, we identify some positively invariant set attractive basins, and some sufficient conditions to ensure local or global attractivity of steady states of model (1.5). Section 4, we apply the results obtained in Section 3 to three model systems arising from population biology with different nonlinearity of "force of infection". The paper is concluded by a discussion on some related topics, raising some interesting open problems in Section 5.

2 Preliminary results

We first introduce some notation and basic results. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all nonnegative integers, reals and nonnegative reals, respectively, and let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$, Δ be the Laplacian operator and $\frac{\partial}{\partial\nu}$ be the derivative in the outward normal direction of $\partial\Omega$.

Let $Y = C(\bar{\Omega}, \mathbb{R})$, $Z = C([-\tau, 0] \times \bar{\Omega}, \mathbb{R})$, $Y^2 = Y \times Y$, and $X = Y \times Z$ be equipped with the corresponding usual supremum norms $\|\cdot\|$. Also, let $Y_+ = \{\phi \in Y : \phi(\overline{\Omega}) \subseteq \mathbb{R}_+\}$, $Z_+ = \{\phi \in Z : \phi([-\tau, 0] \times \overline{\Omega}) \subseteq \mathbb{R}_+\}$, $X_+ = Z_+ \times Y_+$, and $Y_+^2 = Y_+ \times Y_+$. It follows that X_+, Y_+, Z_+, Y_+^2 are closed cones in X, Y, Z, Y^2, respectively. Let $Y_I = \{\phi \in Y : \phi(\bar{\Omega}) \subseteq I\}$, $Z_I = \{\phi \in Z : \phi([-\tau, 0] \times \bar{\Omega}) \subseteq I\}$, and $X_I = Z_I \times Y_{\frac{a_{11}}{a_{12}}I}$, for any inteval $I \subseteq \mathbb{R}_+$. For any $a, b \in \mathbb{R}$, we also use $a, b, \binom{a}{b}$ to denote the constant functions in Y, Z, X taking constant value $a, b, \binom{a}{b}$, respectively, when no confusion arises.

For any ξ, $\eta \in *$, we write $\xi \geq_* \eta$ if $\xi - \eta \in *_+$, $\xi >_* \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$, $\xi \gg_* \eta$ if $\xi - \eta \in Int(*_+)$, where $*$ stands for X, Y, Z or Y^2. For simplicity of notations, when there is no confusion about the spaces, we just write $\geq$, $>$, $\gg$ and $||\cdot||$ for $\geq_*$, $>_*$, $\gg_*$ and $||\cdot||_*$,

ACCEPTED MANUSCRIPT

respectively.

For a real interval I, let $I+[-\tau,0]=\{t+\theta : t\in I \quad \text{and } \theta\in[-\tau,0]\}$. For $u:(I+[-\tau,0])\times\mathbb{R}_+\to\mathbb{R}$, $v: I\times\bar{\Omega}\to\mathbb{R}$, $t\in I$, and $w=\binom{u}{v}$, we define $u_t(\cdot,\cdot), w_t(\cdot,\cdot)$ by $u_t(\theta,x)=u(t+\theta,x)$, $w_t(\theta,x)=\binom{u(t+\theta,x)}{v(t,x)}$, respectively, for all $\theta\in[-\tau,0], x\in\bar{\Omega}$.

Let $S(t)$ $(t\geq 0)$ be an analytic, compact, positive and strongly continuous semigroup on Y generated by the Y-realization Δ_Y of $\Delta=\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2}$. Define $T:\mathbb{R}_+\times Y^2\to Y^2$ by

$$T(t)\Big[\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big]=\begin{pmatrix}e^{-a_{11}t}S(d_1t)[\phi]\\ e^{-a_{22}t}S(d_2t)[\psi]\end{pmatrix}\triangleq\begin{pmatrix}T_1(t)[\phi]\\ T_2(t)[\psi]\end{pmatrix},$$

where $t\in\mathbb{R}$ and $\phi,\psi\in Y$. Then, $T(t)$ is the semigroup generated by the following linear system

$$\left\{\begin{array}{rcl}\frac{\partial u}{\partial t}(t,x) &=& d_1\Delta u(t,x)-a_{11}u(t,x) \quad \text{for} \quad (t,x)\in(0,+\infty)\times\partial\Omega,\\ \frac{\partial v}{\partial t}(t,x) &=& d_2\Delta v(t,x)-a_{22}v(t,x) \quad \text{for} \quad (t,x)\in(0,+\infty)\times\partial\Omega,\\ u(0,x) &=& \phi(x), \quad v(0,x)=\psi(x) \quad \text{for} \quad x\in\bar{\Omega},\\ \frac{\partial u}{\partial\nu}(t,x) &=& \frac{\partial v}{\partial\nu}(t,x)=0 \quad \text{for} \quad (t,x)\in\mathbb{R}_+\times\partial\Omega,\end{array}\right. \tag{2.1}$$

where $\phi,\psi\in Y$.

We now give some properties of $T(t)$ directly due to Lemma 3.1 in [39].

Lemma 2.1. *Let $T(t)$ be defined as above. Then we have following:*

(i) $T(t)[Y_+^2]\subseteq Y_+^2$ for all $t\in\mathbb{R}_+$;

(ii) $T(t)[(Y_+\backslash\{0\})\times(Y_+\backslash\{0\})]\subseteq Int(Y_+^2)$,
$T(t)[(Y_+\times(Y_+\backslash\{0\})]\subseteq Y_+\times Int(Y_+)$,
$T(t)[(Y_+\backslash\{0\})\times Y_+]\subseteq Int(Y_+)\times Y_+$ for all $t\in\mathbb{R}_+$;

(iii) $T(t)[\binom{a}{b}]=\binom{ae^{-a_{11}t}}{be^{-a_{22}t}}$ for all $a,b\in\mathbb{R}$, and $t\in\mathbb{R}_+$.

Proof. By Lemma 3.1 in [39], we have $S(t)[Y_+]\subseteq Y_+$, $S(t)[Y_+\backslash\{0\}]\subseteq Int(Y_+)$ and $S(t)[a]=a$ for all $t\in\mathbb{R}_+$ and $a\in\mathbb{R}$. These together with the definition of $T(t)$, imply (i), (ii) and (iii), completing the proof.

We will consider the mild solution of system (1.5) with the initial value $\varphi\equiv\binom{\phi}{\psi}\in X_+$ in the sense of Martin and Smith [23, 24], which solves the following integral equation with the given initial function,

$$\left\{\begin{array}{rcl}w(t,\cdot) &=& T(t)[\varphi(0,\cdot)]+\int_0^t T(t-s)[F(w_s)]\mathrm{d}s, \qquad t\in\mathbb{R}^+,\\ w_0 &=& \varphi\in X_+,\end{array}\right. \tag{2.2}$$

where $F:X\to Y^2$ is defined by

$$F\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}a_{12}\eta\\ g(\xi(-\tau,\cdot))\end{pmatrix}\triangleq\begin{pmatrix}F_1(\eta)\\ F_2(\xi)\end{pmatrix} \text{ for all } \begin{pmatrix}\xi\\ \eta\end{pmatrix}\in X,$$

ACCEPTED MANUSCRIPT

By the step argument and the definition of F, it is easy to see that, for any $\varphi \in X_+$, (2.2) has a unique solution which exists for all $t \geq 0$. Denote this solution by $(w^\varphi)_t \triangleq \binom{(u^\varphi)_t}{(v^\varphi)_t}$. Moreover, $(w^\varphi)_t = \binom{(u^\varphi)_t}{v^\varphi(t,\cdot)} \in X_+$ for all $t \in \mathbb{R}_+$. Since the semigroup $T(t)$ is analytic, by Corollary 2.2.5 [32] (see also [23, 24, 8, 30]), we know that a mild solution of (1.5) is also a classical solution of (1.5) for $t > \tau$ if g is *hölder* continuous.

Define the map $\Phi : \mathbb{R}_+ \times X_+ \to X_+$ by $\Phi(t, \varphi) = (w^\varphi)_t$ for all $(t, \varphi) \in \mathbb{R}_+ \times X_+$.

The following result is a direct consequence of the abstract results in [32, 8, 30].

Lemma 2.2. *The map Φ is a semiflow defined on X_+ such that at any given $t > \tau$, $\Phi(t, \cdot) : X_+ \to X_+$ is completely continuous, more precisely, if $B \subset X_+$ is bounded, then $\Phi(t, B)$ is precompact for $t > \tau$.*

We first introduce some terminologies from dynamical systems theory.

Definition 2.1. *An element $\varphi \in X_+$ is called an equilibrium of Φ if $\Phi(t, \varphi) = \varphi$ for all $t \geq 0$. A subset M of X is said to be positively invariant under Φ if $\Phi(t, \varphi) \in M$ for every $\varphi \in M$ and $t \geq 0$.*

For given $\varphi \in X_+$ and a bounded subset A of X_+, we define $O(\varphi) = \{\Phi(t, \varphi) : t \geq 0\}$, $O(A) = \bigcup_{t \geq 0} \Phi(t, A)$, $\omega(\varphi) = \bigcap_{t \geq 0} \overline{O(\Phi(t, \varphi))}$, and $\omega(A) = \bigcap_{t \geq 0} \overline{O(\Phi(t, A))}$.

Definition 2.2. *Let w^* be an equilibrium and M be a positively invariant set of the semiflow Φ.*

(i) We say that w^* is stable in M if for every neighborhood V of w^* in M, there exists a neighborhood W of w^* in M such that $\Phi(t, \varphi) \in V$ for all $(t, \varphi) \in \mathbb{R}_+ \times W$.

(ii) We say that w^* is globally attractive in M if $\omega(\varphi) = w^*$ for all $\varphi \in M$.

(iii) We say w^* is globally asymptotically stable in M if w^* is stable as well globally attractive in M.

As usual, we will omit M in Definition 2.1 if $M = X_+$.

Remark 2.1. *If w^* is a globally asymptotically stable equilibrium in M and $w^* \in M' \subseteq M$ is another positively invariant set of the semiflow Φ, then w^* is also globally asymptotically stable in M'.*

Remark 2.2. *If φ belongs to X_+ and A is a bounded and positively invariant set, then by Lemma 2.2, we know that $\overline{O(\varphi)}$ and $\overline{O(A)}$ are compact and hence $\omega(\varphi)$ and $\omega(A)$ are nonempty, compact, and invariant. Moreover, $\omega(A)$ is a global attractor of $\Phi|_{\mathbb{R}_+ \times A}$ according to Theorem 2.4.1 in [12].*

ACCEPTED MANUSCRIPT

3 Main results

In this section, by employing the iterative feature of attracting intervals and dynamical system approach, combining the special comparative properties of non-monotone systems (1.5), we obtain some very simple conditions that can assure the existences of some positively invariant subsets and global attractor, as well as the local or global attractivity of steady states of the systems.

To state our results, let us define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(u) = \frac{a_{12}}{a_{11}a_{22}}g(u)$ for all $u \in \mathbb{R}_+$. Next proposition gives the existences of some positively invariant ordered intervals in X_+, which is very useful for our main conclusions.

Proposition 3.1. *If $I = [a, b] \subseteq \mathbb{R}_+$ and $f(I) \subseteq I$, then $\Phi(t, X_I) \subseteq X_I$, that is, X_I is a positively invariant set.*

Proof. Let $\varphi = \binom{\phi}{\psi} \in X_I$ and $w(t, x) = \Phi(t, \varphi)(0, x) \triangleq \binom{u(t,x)}{v(t,x)}$ for all $(t, x) \in \mathbb{R}_+ \times \bar{\Omega}$. It follows from (2.2) that for any $t \in [0, \tau]$, we have

$$\begin{aligned} v(t,\cdot) &= T_2(t)[\psi] + \int_0^t T_2(t-s)[g(u(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}f(u(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}a]\mathrm{d}s \\ &= \frac{a_{11}a}{a_{12}}e^{-a_{22}t} + \frac{a_{11}a_{22}a}{a_{12}}\int_0^t e^{-a_{22}(t-s)}\mathrm{d}s \\ &= \frac{a_{11}}{a_{12}}a. \end{aligned}$$

Similarly, $v(t, \cdot) \leq \frac{a_{11}}{a_{12}}b$, for any $t \in [0, \tau]$. Again, from (2.2) , we may obtain that for any $t \in [0, \tau]$,

$$\begin{aligned} u(t,\cdot) &= T_1(t)[\phi(0,\cdot)] + \int_0^t T_1(t-s)[a_{12}v(s,\cdot)]\mathrm{d}s \\ &\geq T_1(t)[a] + \int_0^t T_1(t-s)[a_{11}a]\mathrm{d}s \\ &= a. \end{aligned}$$

A similar argument gives $u(t, \cdot) \leq b$ for all $t \in [0, \tau]$. Thus, $\Phi(t, \varphi) \in X_I$ for all $t \in [0, \tau]$. By the inductive method and the semigroup property of Φ, we may get $\Phi(t, \varphi) \in X_I$ for all $t \geq 0$. The proof is complete.

Lemma 3.1. *Let $I = [a, b] \subseteq \mathbb{R}_+$ be a bounded and closed interval with $f(I) \subseteq I$. Then we have following:*

ACCEPTED MANUSCRIPT

(i) If $f(x) > a$ for all $x \in (a,b)$, then $\Phi([3\tau,+\infty)\times(X_I \setminus \{\binom{a}{\frac{a_{11}}{a_{12}}a}\}) \subseteq X_{(a,b]}$;

(ii) If $f(x) < b$ for all $x \in (a,b)$, then $\Phi([3\tau,+\infty)\times(X_I \setminus \{\binom{b}{\frac{a_{11}}{a_{12}}b}\}) \subseteq X_{[a,b)}$.

Proof. We only prove (ii), since (i) can be proved by a similar argument.

Note that X_I is a positive invariant set according to Proposition 3.1.

Let $\varphi = \binom{\phi}{\psi} \in X_I \setminus \{\binom{b}{\frac{a_{11}}{a_{12}}b}\}$ and $w(t+\theta,x) = \Phi(t,\varphi)(\theta,x) \triangleq \binom{u(t+\theta,x)}{v(t,x)}$ for all $(\theta,t,x) \in [-\tau,0]\times\mathbb{R}_+\times\bar{\Omega}$.

If $u(t,\cdot) = a$ for all $t \in [-\tau,\frac{\tau}{3}]$, then by (2.2), we know that for any $t \in [0,\frac{\tau}{3}]$,

$$\begin{aligned} u(t,\cdot) &= T_1(t)[\phi(0,\cdot)] + \int_0^t T_1(t-s)[a_{12}v(s,\cdot)]\mathrm{d}s \\ &= ae^{-a_{11}t} + \int_0^t T_1(t-s)[a_{12}v(s,\cdot)]\mathrm{d}s. \end{aligned}$$

This implies $\int_0^t T_1(t-s)[a_{12}v(s,\cdot)]\mathrm{d}s = a(1-e^{-a_{11}t})$ for all $t \in [0,\frac{\tau}{3}]$. By combining with the positive invariance of X_I, we have $v(s,\cdot) = \frac{a_{11}}{a_{12}}a$ for all $t \in [0,\frac{\tau}{3}]$. Thus, by the semi-group property Φ, we may obtain $w(t,\cdot) = \binom{a}{\frac{a_{11}}{a_{12}}a}$ for all $t \geq -\tau$. In particular, statement (ii) holds.

Suppose that $u(t_1,\cdot) > a$ for some $t_1 \in [-\tau,\frac{\tau}{3}]$. We first claim that $a < u(\theta_0,x_0) < b$ for some $(\theta_0,x_0) \in [-\tau,\tau/3]\times\Omega$. Otherwise, $u(\theta,x) = b$ for all $(\theta,x) \in [-\tau,\tau/3]\times\Omega$. Particularly, $\phi = b$, and hence there exist $t_0 \in (0,\tau/3)$ such that $v(t,\cdot) < \frac{a_{11}}{a_{12}}b$ for all $t \in [0,t_0]$. It follows from the definition of $T(t)$ and Lemma 2.1 that $T_1(t_0-s)[v(s,\cdot)] < b\frac{a_{11}}{a_{12}}e^{-a_{11}(t_0-s)}$ for all $s \in [0,t_0)$. From (2.2), we can derive that

$$\begin{aligned} u(t_0,\cdot) &= T_1(t_0)[\phi(0,\cdot)] + \int_0^{t_0} T_1(t_0-s)[a_{12}v(s,\cdot)]\mathrm{d}s \\ &= be^{-a_{11}t_0} + \int_0^{t_0} T_1(t_0-s)[a_{12}v(s,\cdot)]\mathrm{d}s \\ &< be^{-a_{11}t_0} + \int_0^{t_0} ba_{11}e^{-a_{11}(t_0-s)}\mathrm{d}s \\ &= be^{-a_{11}t_0} + b[1-e^{-a_{11}t_0}] \\ &= b, \end{aligned}$$

which yields a contradiction, and thus the claim holds.

Next, we prove that $v(t,x) < \frac{a_{11}}{a_{12}}b$ for all $(t,x) \in [\frac{4\tau}{3},+\infty)\times\bar{\Omega}$.

In fact, for any $(s,t,x) \in [-\tau,+\infty)\times(0,+\infty)\times\bar{\Omega}$, we know that $T_2(t)[g(u(s,\cdot))](x) \leq \frac{a_{11}a_{22}}{a_{12}}be^{-a_{22}t}$ and $T_2(t)[g(u(\theta_0,\cdot))](x) < \frac{a_{11}a_{22}}{a_{12}}be^{-a_{22}t}$ due to $g((a,b)) \subseteq (-\infty,\frac{a_{11}a_{22}}{a_{12}}b)$, the defini-

ACCEPTED MANUSCRIPT

tion of $T(t)$, and Lemma 2.1. These combined with (2.2), imply that for any $(t,x) \in [\frac{4\tau}{3}, +\infty)\times\bar{\Omega}$,

$$\begin{aligned} v(t,x) &= T_2(t)[\psi](x) + \int_0^t T_2(t-s)[g(u(s-\tau,\cdot))](x)\mathrm{d}s \\ &\leq T_2(t)[\frac{a_{11}}{a_{12}}b](x) + \int_0^t T_2(t-s)[g(u(s-\tau,\cdot))](x)\mathrm{d}s \\ &< \frac{a_{11}}{a_{12}}be^{-a_{22}t} + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}b]\mathrm{d}s \\ &= \frac{a_{11}b}{a_{12}}e^{-a_{22}t} + \frac{a_{11}b}{a_{12}}[1-e^{-a_{22}t}] \\ &= \frac{a_{11}b}{a_{12}}. \end{aligned}$$

Again, by (2.2), for any $(t,x) \in [\frac{4\tau}{3}, +\infty)$, we have

$$\begin{aligned} u(t,x) &= T_1(t)[\phi(0,\cdot)](x) + \int_0^t T_1(t-s)[a_{12}v(s,\cdot)](x)\mathrm{d}s \\ &\leq T_1(t)[b](x) + \int_0^t T_1(t-s)[a_{12}v(s,\cdot)](x)\mathrm{d}s \\ &< be^{-a_{11}t} + \int_0^t T_1(t-s)[a_{12}\frac{a_{11}b}{a_{12}}](x)\mathrm{d}s \\ &= be^{-a_{11}t} + b(1-e^{-a_{11}t}). \end{aligned}$$

Therefore, $w^{\varphi}(t,\cdot) \in X_{[a,b)}$, for any $t \geq \frac{7}{3}\tau$, and hence the proof is complete.

The following result gives the existence of the global attractor, whose element values belong to the minimum order intervals determined by f.

Theorem 3.1. *Let $I = [a,b]$ be a bounded and closed interval in $\mathbb{R}_+$ with $f(I) \subseteq I$. Then $\omega(X_I) \subseteq X_{I_f}$, where $I_f = \bigcap_{n\geq 0} f^n(I)$, moreover, $\omega(X_I)$ is the global attractor in X_I.*

Proof. Note that X_I ia a positively invariant set and $\omega(X_I)$ is a global attractor in X_I due to Proposition 3.1 and Remark 2.2.

Let $I^* = \bigcup_{\binom{\phi}{\psi}\in\omega(X_I)} \phi(0,\bar{\Omega})$. It follows from the positive invariance of X_I that $I^* \subseteq I$ is a bounded and closed interval.

Let $n^* = \sup\{n \in \mathbb{N} : I^* \subseteq \bigcap_{k=0}^{n} f^k(I)\}$. We now prove that $n^* = +\infty$; otherwise, $n^* < +\infty$. Thus, there exist a^* and $b^* \in I$ such that $f^{n^*}(I) = [a^*, b^*]$ and $f([a^*,b^*]) \subsetneqq [a^*,b^*]$.

Claim that $\omega(X_I) \subseteq X_{[a^*,b^*]}$. By the definition of I^*, n^* and $[a^*,b^*]$, it is easy to see that $\phi \in Z_{[}a^*,b^*]$ for any $\binom{\phi}{\psi} \in \omega(X_I)$. So, it suffices to prove that $\psi \in Y_{[\frac{a_{11}}{a_{12}}a^*, \frac{a_{11}}{a_{12}}b^*]}$ for any

ACCEPTED MANUSCRIPT

$\binom{\phi}{\psi} \in \omega(X_I)$. In fact, from (2.2), for any $\varphi = \binom{\phi}{\psi} \in \omega(X_I), t \geq 0$, we obtain

$$\begin{aligned} v^{\varphi}(t,\cdot) &= T_2(t)[\psi] + \int_0^t T_2(t-s)[g(u^{\varphi}(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}f(u^{\varphi}(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}a^*]\mathrm{d}s \\ &= \frac{a_{11}}{a_{12}}ae^{-a_{22}t} + \frac{a_{11}a^*}{a_{12}}[1-e^{-a_{22}t}] \\ &= \frac{a_{11}a^*}{a_{12}} + \frac{a_{11}}{a_{12}}[a-a^*]e^{-a_{22}t}. \end{aligned}$$

By the invariance of $\omega(X_I)$, we get that $\psi \geq \frac{a_{11}a^*}{a_{12}} + \frac{a_{11}}{a_{12}}[a-a^*]e^{-a_{22}t}$ for any $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$ and $t \geq 0$. Therefore, letting $t \to +\infty$, we have $\psi \geq \frac{a_{11}}{a_{12}}a^*$ for any $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$. Similarly, we can prove that $\psi \leq \frac{a_{11}}{a_{12}}b^*$ for any $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$. Hence, the claim holds.

In view of $f([a^*,b^*]) \subsetneqq [a^*,b^*]$, there exist $a^{**}, b^{**} \in [a,b]$ such that $[a^{**},b^{**}] = f([a^*,b^*]) \subsetneqq [a^*,b^*]$. It follows from (2.2) that for any $t \geq 0$ and $\varphi = \binom{\phi}{\psi} \in \omega(x_I)$, we deduce

$$\begin{aligned} v^{\varphi}(t,\cdot) &= T_2(t)[\psi] + \int_0^t T_2(t-s)[g(u^{\varphi}(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a^*] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}a^{**}]\mathrm{d}s \\ &= \frac{a_{11}}{a_{12}}a^*e^{-a_{22}t} + \frac{a_{11}a^{**}}{a_{12}}(1-e^{-a_{22}t}) \\ &= \frac{a_{11}a^{**}}{a_{12}} + \frac{a_{11}(a^*-a^{**})}{a_{12}}e^{-a_{22}t}. \end{aligned}$$

By the invariance of $\omega(X_I)$, and letting $t \to +\infty$, we have $\psi \geq \frac{a_{11}a^{**}}{a_{12}}$ for all $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$. Similarly, $\psi \leq \frac{a_{11}b^{**}}{a_{12}}$ for all $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$. On the other hand, we deduce from (2.2) that for any $t \geq 0$ and $\varphi = \binom{\phi}{\psi} \in \omega(X_I)$, we have

$$\begin{aligned} u^{\varphi}(t,\cdot) &= T_1(t)[\phi(0,\cdot)] + \int_0^t T_1(t-s)[a_{12}v^{\varphi}(s,\cdot)]\mathrm{d}s \\ &\leq T_1(t)[b^*] + \int_0^t T_1(t-s)[a_{12}\frac{a_{11}b^{**}}{a_{12}}]\mathrm{d}s \\ &= b^*e^{-a_{11}t} + b^{**}(1-e^{-a_{11}t}) \\ &= b^{**} + (b^*-b^{**})e^{-a_{11}t}. \end{aligned}$$

By the invariance of $\omega(X_I)$, and letting $t \to +\infty$, for any $\binom{\phi}{\psi} \in \omega(X_I)$, we have, $\phi \leq b^{**}$. Similarly, $\phi \geq a^{**}$ for any $\binom{\phi}{\psi} \in \omega(X_I)$. So, $\omega(X_I) \subseteq X_{[a^{**},b^{**}]} = X_{f^{n^*+1}}$, which contradicts with the choice of n^*. This, together with the definition of I_f, gives that $\phi \in Z_{I_f}$ for all $\binom{\phi}{\psi} \in \omega(X_I)$. It

ACCEPTED MANUSCRIPT

suffices to prove $\psi \in Y_{\frac{a_{11}}{a_{12}}I_f}$ for all $\binom{\phi}{\psi} \in \omega(X_I)$. In fact, by (2.2), for any $t \geq 0, \varphi = \binom{\phi}{\psi} \in \omega(X_I)$, we have

$$\begin{aligned} v^{\varphi}(t,\cdot) &= T_2(t)[\psi] + \int_0^t T_2(t-s)[g(u^{\varphi}(s-\tau,\cdot))]\mathrm{d}s \\ &\geq T_2(t)[\frac{a_{11}}{a_{12}}a] + \int_0^t T_2(t-s)[\frac{a_{11}a_{22}}{a_{12}}\inf I_f]\mathrm{d}s \\ &= \frac{a_{11}}{a_{12}}ae^{-a_{22}t} + \frac{a_{11}a_{22}}{a_{12}}\inf I_f \int_0^t e^{-a_{22}(t-s)}\mathrm{d}s \\ &= \frac{a_{11}}{a_{12}}\inf I_f + \frac{a_{11}}{a_{12}}\Big[a - \inf I_f\Big]e^{-a_{22}t}. \end{aligned}$$

Thus, combining this with the invariance of $\omega(X_I)$, we have

$$\psi \geq \frac{a_{11}}{a_{12}}\inf I_f + \frac{a_{11}}{a_{12}}\Big[a - \inf I_f\Big]e^{-a_{22}t},$$

where $\binom{\phi}{\psi} \in \omega(X_I), t \geq 0$. By letting $t \to +\infty$, we have

$$\psi \geq \frac{a_{11}}{a_{12}}\inf I_f,$$

for all $\binom{\phi}{\psi} \in \omega(X_I)$. Similarly, we have

$$\psi \leq \frac{a_{11}}{a_{12}}\sup I_f,$$

for all $\binom{\phi}{\psi} \in \omega(X_I)$. In other words,

$$\psi \in Y_{\frac{a_{11}}{a_{12}}I_f}, \quad \text{for} \quad \text{all} \quad \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in \omega(X_I),$$

and

$$\varphi \in X_{I_f}, \quad \text{for} \quad \text{all} \quad \varphi \in \omega(X_I).$$

That is, $\omega(X_I) \subseteq X_{I_f}$. This completes the proof.

The following results directly follow from Theorem 3.1.

Theorem 3.2. *Let $I = [a,b]$ with $f(I) \subseteq I$ and $I_f^\sharp = 1$, where $I_f^\sharp$ represents the cardinality of I_f. Then there exists $u^* \in I$ such that $I_f = \{u^*\}$ and $\binom{u^*}{\frac{a_{11}u^*}{a_{12}}}$ is globally asymptotically stable in X_I and attracts X_I.*

By Propositions 2.1 and 2.2 in [39] and Theorem 3.2, we easily obtain the following theorem.

Theorem 3.3. *Let $I = [a,b]$ with $f(I) \subseteq I$. Assume that there exists $u^* \in I$ such that f satisfies one of the following assumptions:*

ACCEPTED MANUSCRIPT

(H1) $\{u \in I : f^2(u) = u\} = \{u^*\}$;

(H2) $|f(u) - u^*| < |u - u^*|$ *for all* $u \in I \setminus \{u^*\}$.

Then $\binom{u^*}{\frac{a_{11}u^*}{a_{12}}}$ *is globally asymptotically stable in* X_I *and attracts* X_I.

The above results are all for $I = [a, b]$, a compact interval. In the case that $I = \mathbb{R}_+$, to get the desired results, we need some additional assumptions.

Assumption (I): $f(x) < x$ for all $x \in (0, +\infty)$.

Theorem 3.4. *If* **(I)** *holds, then* 0 *is a globally asymptotically stable in* X_+ *and attracts any bounded set in* X_+.

Proof. Take $I = [0, 1]$. Then $f(I) \subseteq I$ and $I_f = \{0\}$. It follows from Theorem 3.2 that 0 is a global attractor in $X_{[0,1]}$. Thus, it suffices to prove that 0 attracts any bounded set in X_+.

Suppose that $A \subseteq X_+$ is bounded. It follows from **(I)** that there exists $a > 0$ such that $A \subseteq X_{[0,a]}$, $f([0, a]) \subseteq [0, a]$, and $[0, a]_f = 0$.

In view of Theorem 3.2, we know that $\omega(X_{[0,a]}) = \{0\}$, and thus 0 attracts A. The proof is complete.

Assumption (II): $\{u > 0 : f^2(u) = u\} = \{u^*\}$ for some $u^* > 0$.

Theorem 3.5. *Assume that* **(II)** *holds. If* $f'(0) > 1$ *and* $f(u^{**}) < u^{**}$ *for some* $u^{**} > u^*$, *then* $\binom{u^*}{\frac{a_{11}u^*}{a_{12}}}$ *is globally asymptotically stable in* $X_+ \setminus \{0\}$ *and attracts any bounded and closed set in* $IntX_+$.

Proof. In view of (II), $f'(0) > 1$, and $f(u^{**}) < u^{**}$ for some $u^{**} > u^*$, we obtain that $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u > u^*$.

Claim 1: For any bounded and closed inteval $I \subseteq (0, +\infty)$, there exists a bounded and closed interval $J \subseteq (0, +\infty)$ such that $I \subseteq J$, $u^* \in IntJ$, and $f(J) \subseteq J$.

Indeed, set $c = \max f([0, u^*])$ and $I = [a, b]$ with $b > a > 0$. Let $M = \max\{c, 1 + u^*, b\}$, $\varepsilon = \min f([\min\{a, \frac{u^*}{3}\}, M])$, and $J = [\varepsilon, M]$. Then $I \subseteq J$, $u^* \in IntJ$, and $f(J) \subseteq J$, that is, **Claim 1** holds.

Claim 2: $\binom{u^*}{\frac{a_{11}u^*}{a_{12}}}$ is stable and attracts any bounded and closed subset in $Int(X_+)$.

Suppose that A is a given bounded and closed subset in $Int(X_+)$. Then by **Claim 1**, there exists a bounded and closed interval $J \subseteq (0, +\infty)$ such that $A \subseteq X_J$, $u^* \in IntJ$, and $f(J) \subseteq J$. This combined with (II) and Theorem 3.3 shows that $\binom{u^*}{\frac{a_{11}u^*}{a_{12}}}$ is stable and attracts X_J, and thus **Claim 2** holds.

By **Claim 2**, it suffices to prove that $(u^\varphi)_t \in IntX_+$ for all $\varphi \in X_+ \setminus \{0\}$ and $t > 3\tau$.

In fact, suppose that $\varphi = \binom{\phi}{\psi} \in X_+ \setminus \{0\}$. Let $a = 0$, $b = \max\{c, ||\phi||, |||\psi|\}$, and $I = [a, b]$. Then $f(I) \subseteq I$, $f((0, b]) \subseteq (0, +\infty)$, and $\varphi \in X_I$. It follows from Lemma 3.1-(i) that $(u^\varphi)_t \in Int(X_+)$ for all $t > 3\tau$. The proof is complete.

Assumption (III): There exists $u^* > 0$ such that $f(u^*) = u^*$ and $f(x) < x$ for all $x \in (0, +\infty) \setminus \{u^*\}$.

ACCEPTED MANUSCRIPT

Theorem 3.6. *Under the assumptions of* **(III)**, *the following statements hold:*

(i) $X_{[0,b]}$ *is a positively invariant set, for any* $b > 0$;

(ii) $\omega(X_{[0,u^*]}) \subseteq X_{[0,u^*]}$ *is the global attractor and attracts any bounded set in* X_+;

(iii) 0 *is globally asymptotically stable in* $X_{[0,u^*]} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$, *and attracts any closed set contained in* $X_{[0,u^*)}$;

(iv) $\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)$ *is unstable and attracts* $X_{[u^*,(f|_{[u^*,+\infty)})^{-1}(u^*)]}$, *where* $(f|_{[u^*,+\infty)})^{-1}(u^*) \triangleq \sup\{u \geq u^* : f(v) \geq u^* \ \text{ for all } \ v \in [u^*, u]$.

Proof. (i) For any $b > 0$, we know from **(III)** that $f([0, b]) \subseteq [0, b]$. Thus by Proposition 3.1, we easily see that $X_{[0,b]}$ is a positively invariant set.

(ii) Let A be a bounded set in X_+, then there exists $b > 0$ such that $A \subseteq X_{[0,b]}$ and $f([0, b]) \subseteq [0, b] \triangleq I$. By **(III)**, we know that $I_f = [0, u^*]_f = [0, u^*]$. This, together with Theorem 3.1, implies $\omega(X_{[0,u^*]}) = \omega(X_I) \subseteq X_{[0,u^*]}$, which is the global attractor and attracts X_I and A.

(iii) Clearly, $f(u) < u$, $f([0, u]) \subseteq [0, u]$, and $[0, u]_f = \{0\}$ for all $u \in (0, u^*)$. By applying Lemma 3.1-(ii), we know that $(w^\varphi)_t \in X_{[0,u^*)}$ for any $\varphi \in X_{[0,u^*]} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$ and $t > 3\tau$. Thus, it suffices to prove that 0 attracts any closed subset $A \subseteq X_{[0,u^*)}$.

Let A be a closed subset in X and $A \subseteq X_{[0,u^*)}$. Then, there exists $u_1^* \in (0, u^*)$ such that $A \subseteq X_{[0,u_1^*]}$ and $\{0\} = [0, u_1^*]_f \subseteq f([0, u_1^*]) \subseteq [0, u_1^*]$. These and Theorem 3.2 show that 0 attracts $X_{[0,u_1^*]}$ and A.

(iv) Clearly, the unstability of $\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)$ follows from (iii). Let $I = [u^*, (f|_{[u^*,+\infty)})^{-1}(u^*)]$. Then by **(III)** and the choice of $(f\ |_{[u^*,+\infty)})^{-1}(u^*)$, we have $\{u^*\} = I_f \subseteq f(I) \subseteq I$, which, combined with Theorem 3.2 , yields that $\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)$ attracts $X_{[u^*,(f|_{[u^*,+\infty)})^{-1}(u^*)]}$,

This completes the proof.

Assumption (IV): There exist $u^{**} > u^* > 0$ satisfy the following statements:

(i) $Fixf \triangleq \{u \in \mathbb{R}_+ : f(u) = u\} = \{0, u^*, u^{**}\}$;

(ii) $f(u) < u$ for all $u \in (0, u^*) \bigcup (u^{**}, +\infty)$ and $f(u) > u$ for all $u \in (u^*, u^{**})$;

(iii) There exists $c > 0$ such that $f'(u) > 0$ for all $u \in (0, c)$ and $f'(u) < 0$ for all $u \in (c, +\infty)$;

(iv) There exists a unique $c_1 > 0$ such that $f''(c_1) = 0$.

Theorem 3.7. *Let (IV) hold and let* $u_1^* \triangleq (f|_{[c,+\infty)})^{-1}(u^*)$ *and* $(Sf)(u) = \frac{f''(u)}{f'(u)} - \frac{3}{2}(\frac{f''(u)}{f'(u)})^2$ *for all* $u \in \mathbb{R}_+$. *Then the following statements hold:*

(i) 0 *is globally asymptotically stable in* $X_{[0,u^*]} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$, *and attracts any closed set contained in* $X_{[0,u^*)}$;

ACCEPTED MANUSCRIPT

(ii) $\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)$ *is unstable;*

(iii) If $c \geq u^{**}$, *then* $\left(\frac{u^{**}}{\frac{a_{11}u^{**}}{a_{12}}}\right)$ *is globally asymptotically stable in* $X_{[u^*,u_1^*]} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$, *and attracts any closed set contained in* $X_{(u^*,u_1^*)}$;

(iv) If $c < u^{**}$ *and* $f^2(c) > u^*$, *then* $\omega(A) \subseteq X_{[f^2(c),f(c)]}$ *for any closed set* $A \subseteq X_{(u^*,u_1^*)}$;

(v) If $f \in C^3$ *with* $f^2(c) > u^*$, $f'(u^{**}) \geq -1$ *and* $(Sf)(u) < 0$ *for all* $u \in (0,c)\bigcup(c,+\infty)$, *then* $\left(\frac{u^{**}}{\frac{a_{11}u^{**}}{a_{12}}}\right)$ *is globally asymptotically stable in* $X_{[u^*,u_1^*]} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$, *and attracts any closed set contained in* $X_{(u^*,u_1^*)}$.

Proof.

(i) follows from Theorem 3.6-(iii), while (ii) follows from (i).

(iii) Given $\varphi \in X_{\overline{I}} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$ and any closed set contained A in $X_{(u^*,u_1^*)}$. If $\varphi = \left(\frac{u_1^*}{\frac{a_{11}u_1^*}{a_{12}}}\right)$, then by $f(u_1^*) = u^* < u_1^*$, we have $\Phi(\epsilon,\varphi) \in X_{\overline{I}} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right), \left(\frac{u_1^*}{\frac{a_{11}u_1^*}{a_{12}}}\right)\}$ for some $\epsilon \in [0, \frac{\tau}{7})$. So, we may assume that $\varphi \in X_{\overline{I}} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right), \left(\frac{u_1^*}{\frac{a_{11}u_1^*}{a_{12}}}\right)\}$. By applying Lemma 3.1, we know that $(u^\varphi)_t \in X_I$ for $t > 3\tau$. We may choose $a \in (u^*, u^{**})$ and $b = (f|_{[c,+\infty)})^{-1}(f(a))$ such that $(u^\varphi)_{5\tau} \in X_{[a,b]}$ and $A \subseteq [a,b]$. Using the proof of Theorem 3.3-(ii) in [15] with $g = id$, we obtain

$$\begin{cases} f(I) \subseteq I, \\ \{u \in I : f^2(u) = u\} = \{u^{**}\}, \\ f([a,b]) \subseteq [a,b], \end{cases}$$

where $I = (u^*, u_1^*)$. Hence, Theorem 3.3 implies that $\left(\frac{u^{**}}{\frac{a_{11}u^{**}}{a_{12}}}\right)$ is globally asymptotically stable in $X_{[a,b]}$ and attracts $X_{[a,b]}$. Therefore, by the choices of a, b, we know that $\left(\frac{u^{**}}{\frac{a_{11}u^{**}}{a_{12}}}\right)$ attracts φ and A. This proves statement (iii).

(iv) It follows from the proof of Theorem 4.3-(iv) in [15] that we obtain $f(I) \subseteq I$ with $I = [f^2(c), f(c)]$. For any closed subset $A \subseteq X_{(u^*,u_1^*)}$, there exists $a \in (u^*, (f|_{[0,c]})^{-1}(f^2(c))) \triangleq J_1$ such that $A \subseteq X_{[a,b]}$ with $b = (f|_{[c,+\infty)})^{-1}(f(a))$. By the proof of Theorem 4.3-(v) in [15], we easily see that

$$\begin{cases} f([a,b]) \subseteq [a,b], \\ [a,b]_f = I_f, \end{cases}$$

Therefore, $\omega(A) \subseteq X_{I_f} \subseteq X_I$ due to Theorem 3.1, and hence statement (v) holds.

(v) Suppose that $\varphi \in X_{\overline{I}} \setminus \{\left(\frac{u^*}{\frac{a_{11}u^*}{a_{12}}}\right)\}$ and A is a closed set in $X_{(u^*,u_1^*)}$. By the discussions in statement (iii), we know that $(u^\varphi)_{5\tau} \in X_I$. Hence, there exist $a \in (u^*, (f|_{[0,c]})^{-1}(f^2(c))) \triangleq J_1$ and $b = (f|_{[c,+\infty)})^{-1}(f(a))$ such that $\{(u^\varphi)_{5\tau}\} \cup A \subseteq X_{[a,b]}$. It follows from Proposition 1-(3d) in [19] the proof of statement (iv) that we have

$$\begin{cases} f([a,b]) \subseteq [a,b], \\ [a,b]_f = \{u^{**}\}. \end{cases}$$

ACCEPTED MANUSCRIPT

Therefore, $\omega(\varphi) \cup \omega(A) \subseteq \omega(\{\varphi\} \cup A) \subseteq \omega(X_{[a,b]}) \subseteq X_{[a,b]_f} = \{u^{**}\}$ due to Theorem 3.2, and hence statement (v) holds.

4 Examples

In this section, we present several reaction-diffusion systems with different "nonlinear force of infection" arising from population biology to illustrate the results obtained in Section 3.

Example 4.1. *Consider systems* (1.5) *with the "nonlinear force of infection"* $g(\cdot)$ *having the Ricker form as below*

$$g(u) = \beta u e^{-\delta u}, \quad for \quad u \in \mathbb{R}^+. \tag{4.1}$$

The nonlinear function g *is a widely used birth function (for fish population dynamics and for blowfly population, see for example, the biological significance of* g *can be found in [9, 10, 18, 20, 25, 27, 28, 38].*

Theorem 4.1. *Let* $g(\cdot)$ *be defined by* (4.1)*. Then the following statements hold:*

(i) If $\beta a_{12} \le a_{11}a_{22}$*, then* 0 *is globally asymptotically stable in* X_+*; furthermore,* 0 *is a global attractor in* X_+*, i.e.,* 0 *attracts any bounded set contained in* X_+*.*

(ii) If $a_{11}a_{22} < \beta a_{12} \le e^2 a_{11}a_{22}$*, then* $\begin{pmatrix} \frac{1}{\delta}\ln\frac{\beta a_{12}}{a_{11}a_{22}} \\ \frac{a_{11}}{\delta a_{12}}\ln\frac{\beta a_{12}}{a_{11}a_{22}} \end{pmatrix}$ *is globally asymptotically stable in* $X_+ \setminus \{0\}$*, and attracts any closed bounded set contained in* $Int(X_+)$*.*

Proof. Clearly, $f(u) = \frac{\beta a_{12}}{a_{11}a_{22}} u e^{-\delta u}$ for all $u \in \mathbb{R}_+$.

(i) Using a simple computation, one can see that assumption (I) holds. It is obvious that statement (i) follows from corresponding conclusions of Theorem 3.4.

(ii) Applying the proof of Theorem 4.2 in [39], one can see that assumption (II) holds. The statement (ii) can be proved by Theorem 3.5 immediately.

Example 4.2. *Consider systems* (1.5) *with the "nonlinear force of infection"* $g(\cdot)$ *has the form as below*

$$g(u) = \frac{bu}{1+u^n}, \quad for \quad u \in \mathbb{R}^+. \tag{4.2}$$

The form of nonlinear function g *was originally used by Mackey and Glass in [21] to model the blood cell production in an ordinary differential equation model. Since then modified models have been studied by many researchers. See, for example, [39, 17, 41]*

Theorem 4.2. *Let* $g(\cdot)$ *be defined by* (4.2)*. Then the following statements hold:*

(i) If $ba_{12} \le a_{11}a_{22}$*, then* 0 *is a global attractor in* X_+*.*

(ii) If $b > \frac{a_{11}a_{22}}{a_{12}}$*, and furthermore* $0 < n \le 2$ *or* $\frac{ba_{12}}{a_{11}a_{22}} \le \frac{n}{n-2}$ *for* $n > 2$*, then* $\begin{pmatrix} [\frac{ba_{12}}{a_{11}a_{22}}-1]^{\frac{1}{n}} \\ \frac{a_{11}}{a_{12}}[\frac{ba_{12}}{a_{11}a_{22}}-1]^{\frac{1}{n}} \end{pmatrix}$ *is globally asymptotically stable in* $X_+ \setminus \{0\}$*, and attracts any closed bounded set contained in* $Int(X_+)$*.*

ACCEPTED MANUSCRIPT

Proof. Clearly, $f(u) = \frac{ba_{12}}{a_{11}a_{22}}\frac{u}{1+u^n}$ for all $u \in \mathbb{R}_+$. Using the proof of Theorem 4.5 in [39], one can prove statements (i) and (ii) immediately by Theorems 3.4 and 3.5.

Example 4.3. *Consider systems* (1.5) *with the "nonlinear force of infection"* $g(\cdot)$ *having the Allee-type form as below*

$$g(u) = \beta u^2 e^{-au}, \quad for \quad u \in \mathbb{R}^+. \tag{4.3}$$

The nonlinear function g *is used to reflect the so called Allee effect in the population model [15, 1, 19, 29].*

Theorem 4.3. *Let* $g(\cdot)$ *be defined by* (4.3)*. Then we have the following results:*

(i) If $ae > \frac{a_{12}\beta}{a_{11}a_{22}}$*, then* 0 *is a global attractor in* X_+*.*

(ii) If $ae = \frac{a_{12}\beta}{a_{11}a_{22}}$*, then* 0 *is globally asymptotically stable in* $X_{[0,\frac{1}{a}]} \setminus \{\binom{\frac{1}{a}}{\frac{a_{11}}{a_{12}a}}\}$*, and attracts any closed subset contained in* $X_{[0,\frac{1}{a})}$*;* $\binom{\frac{1}{a}}{\frac{a_{11}}{a_{12}a}}$ *is globally asymptotically stable in* $X_{[0,\theta]}$*, and attracts any bounded subset contained in* $X_{[0,\theta]}$*, where* $\theta = (f|_{[\frac{2}{a},+\infty)})^{-1}(\frac{1}{a})$*; and furthermore,* $\omega(X_{[0,\frac{1}{a}]})$ *attracts any bounded subset contained in* X_+*.*

(iii) If $ae < \frac{a_{12}\beta}{a_{11}a_{22}}$*, then there exist* $u^*, u^{**} > 0$ *such that the following statements hold:*

(1) $u^* < u^{**}$ *and* $\{u > 0 : g(u) = \frac{a_{11}a_{22}}{a_{12}}u\} = \{u^*, u^{**}\}$*;*

(2) 0 *is globally asymptotically stable in* $X_{[0,u^*]} \setminus \{\binom{u^*}{\frac{a_{11}}{a_{12}}u^*}\}$*, and attracts any closed subset contained in* $X_{[0,u^*)}$*;*

(3) $\binom{u^*}{\frac{a_{11}}{a_{12}}u^*}$ *is unstable in* X_+*;*

(4) If $\frac{3a_{12}\beta}{e^2a_{11}a_{22}} \le ae$ *and* $\frac{16\beta^3a_{12}^3}{a^4e^4a_{11}^3a_{22}^3}e^{-\frac{4a_{12}\beta}{aa_{11}a_{22}e^2}} > u^*$*, then* $\binom{u^{**}}{\frac{a_{11}}{a_{12}}u^{**}}$ *is globally asymptotically stable in* $X_{[u^*,u_1^*]} \setminus \{\binom{u^*}{\frac{a_{11}}{a_{12}}u^*}\}$*, and attracts any closed subset contained in* $X_{(u^*,u_1^*)}$*, where* $u_1^* \triangleq (f|_{[\frac{2}{a},+\infty)})^{-1}(u^*)$*.*

Proof. In view of the discussions after Proposition 4.1 in [15], we easily see that (i) follows from Theorem 3.4, while (ii) follows from Theorem 3.6.

(iii) Suppose $ae < \frac{a_{12}\beta}{a_{11}a_{22}}$. By Proposition 4.1 in [15], we easily see the assumption (IV) holds, in particular, (iii)-(1) holds. Clearly, (iii)-(2) and (iii)-(3) follow from Theorem 3.7-(i) and -(ii), respectively.

Note that $\frac{2a_{12}\beta}{ea_{11}a_{22}} \le ae$ implies $\frac{16\beta^3a_{12}^3}{a^4e^4a_{11}^3a_{22}^3}e^{-\frac{4a_{12}\beta}{aa_{11}a_{22}e^2}} > u^*$, that is, $f^2(c) > u^*$, where $c = \frac{2}{a}$. Now we shall prove (iii)-(4) by discussing the two possible cases.

Case 1: $\frac{2a_{12}\beta}{ea_{11}a_{22}} \le ae$.

In this case, using the proof of Theorem 4.2-(ii) in [15], we have $\frac{2}{a} \ge u^{**}$. Thus, (iii)-(4) follows from Theorem 3.7-(iii) immediately.

Case 2: $\frac{3a_{12}\beta}{e^2a_{11}a_{22}} \le ae \le \frac{2a_{12}\beta}{ea_{11}a_{22}}$ and $\frac{16\beta^3a_{12}^3}{a^4e^4a_{11}^3a_{22}^3}e^{-\frac{4a_{12}\beta}{aa_{11}a_{22}e^2}} > u^*$.

In this case, using the discussions of Theorem 4.1-(3) in [19], we know that all the assumptions in Theorem 3.7-(v) hold. Thus, (iii)-(4) follows from Theorem 3.7-(v) immediately.

ACCEPTED MANUSCRIPT

5 Discussion

With the help of iterative property of interval mapping and dynamical system methods, we have obtained some interesting results on the global stability, positively invariant set, and attractive basins of the Neumann problems. In this paper, we only consider the homogeneous Neumann boundary condition, how to study the positively invariant set and attractive basins for homogeneous Dirichlet boundary problems is a challenging yet interesting problem. We point out that as far as the basin of attraction is concerned, it is very crucial to analyze the influence of model parameters on zero solution, positive solution and invariant set. Additionally, how to study the existence of connection orbit between steady state and invariant set is another important and challenging problem. We shall leave these problems as future research.

Acknowledgments

We are extremely grateful to the anonymous reviewer and the editor for their valuable comments and suggestions which have contributed considerably to the improved presentation of this paper.

References

[1] F. Brauer, C. Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology, Springer-Verlag, New York, 2001.

[2] V. Capasso, Mathematical Structures of Epidemic Systems, corrected 2nd printing, Lecture Notes Biomath., Vol. 97, Springer-Verlag, Heidelberg, 2009.

[3] V. Capasso L. Maddalena, A nonlinear diffusion system modelling the spread of orofaecal diseases, Nonlinear phenomena in mathematical sciences. (V. Lakshmikantham, ed.) New York: Academic Press, 1981.

[4] K. Cooke, P. van den Driessche, X. Zou, Interaction of maturation delay and nonlinear birth in population and epidemic models. J. Math. Biol., **39** (1999), 332-352.

[5] V. Capasso, L. Maddalena, Convergence to equilibrium states for a reaction-diffusion system modeling the spatial spread of a class of bacterial and viral diseases. J. Math. Biol., **13** (1981), 173-184.

[6] V. Capasso, E. Wilson, Analysis of reaction-diffusion system modeling manenvironment man epidemics. SIAM. J. Appl. Math., **57** (1997), 327-346.

[7] T. Faria, Asymptotic stability for delayed logistic type equations. Math. Comput. Modelling, **43** (2006), 433-445.

[8] W. Fitzgibbon, Semilinear functional differential equations in Banach space. J. Differential Equations, **29** (1978), 1-14.

ACCEPTED MANUSCRIPT

[9] W. Gurney, S. Blythe, R. Nisbet, Nicholson's blowflies revisited. Nature, **287** (1980), 17-21.

[10] I. Győri, S. Trofimchuk, Global attractivity in $x'(t) = -\delta x(t)+pf(x(t-\tau))$. Dynam. Systems Appl., **8** (1999), 197-210.

[11] A. Gourley, J. Wu, Delayed nonlocal diffusive systems in biological invasion and disease spread. Nonlinear dynamics and evolution equations, 137–200 Fields Inst. Commun., 48 Amer. Math. Soc., Providence, RI 2006.

[12] J. Hale, Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs 25, American Mathematical Society, Providence, RI, 1988.

[13] J. Hale, S. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[14] C. Hsu, T. Yang, Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models. Nonlinearity, **26** (2013), 121-139.

[15] C. Huang, Z. Yang, T. Yi, X. Zou. On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities, J. Differential Equations, **256** (2014), 2101-2114.

[16] A. Ivanov, A. Sharkovsky, Oscillations in singularly perturbed delay equations, Dynam. Report., **1** (1991), 164-224.

[17] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics. Academic, London, 1993.

[18] E. Liz, Four theorems and one conjecture on the global asymptotic stability of delay differential equations. In The first 60 years of nonlinear analysis of Jean Mawhin (eds M. Delgado et al.), (2004), 117-129. Singapore: World Scientific.

[19] E. Liz, R. Alfonso, Delayed population models with Allee effects and exploitation. Math. Biosci. Eng., **12** (2015), 83-97.

[20] E. Liz, G. Röst, On the global attractivity of delay differential equations with unimodel feedback. Discrete Contin. Dyn. Syst., **24** (2009), 1215-1224.

[21] M. Mackey, L. Glass, Oscillation and chaos in physiological control systems. Science, **197** (1977), 287-289.

[22] J. Mallet-Paret, R. Nussbaum, Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation. Ann. Math. Pura Appl., **145** (1986), 33-128.

[23] R. Martin, H. Smith, Abstract functional-differential equations and reaction-diffusion systems. Trans. Amer. Math. Soc., **321**(1990), 1-44.

ACCEPTED MANUSCRIPT

[24] R. Martin, H. Smith, Reaction-diffusion systems with time delays: monotonicity, invariance, comparison and convergence. J. Reine Angew. Math., **413**(1991), 1-35.

[25] A. Nicholson, An outline of the dynamics of animal populations. Aust. J. Zool., **2** (1954), 9-65.

[26] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, Mathematics for life science and medicine, vol. 2, pp. 97–122. Berlin, Germany: Springer-Verlag, 2007.

[27] G. Röst, J. Wu, Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback. Proc. R. Soc. Lond. Ser. A, **463** (2007), 2655-2669.

[28] H. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. Mathematical Surveys and Monographs, 41, American Mathematical Society, Providence, RI, 1995.

[29] A. Terry, Impulsive adult culling of a tropical pest with a stage-structured life cycle. Nonlinear Anal. Real World Appl., **11** (2010), 645-664.

[30] C. Travis, G. Webb, Existence and stability for partial functional differential equations. Trans. Amer. Math. Soc., **200** (1974), 395-418.

[31] H. Thieme, X. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. J. Differential Equations, **195** (2003), 430-470.

[32] J. Wu, Theory and Applications of Partial Functional-Differential Equations. Applied Mathematical Sciences 119, Springer-Verlag, New York, 1996.

[33] S. Wu, S. Liu, Asymptotic speed of spread and traveling fronts for a nonlocal reaction-diffusion model with distributed delay, Appl. Math. Model., **33** (2009), 2757-2765.

[34] S. Wu, S. Liu, Existence and uniqueness of traveling waves for non-monotone integral equations with application, J. Math. Anal. Appl., **365** (2010), 729-741.

[35] S. Wu, H. Hsu, Y. Xiao, Global attractivity, spreading speeds and traveling waves of delayed nonlocal reaction-diffusion systems. J. Differential Equations **258** (2015), 1058-1105.

[36] D. Xu, X. Zhao, Erratum to "Bistable waves in an epidemic model". J. Dynam. Diff. Eqns., **27** (2005), 219-247.

[37] T. Yi, Y. Chen, J. Wu, The global asymptotic behavior of nonlocal delay reaction diffusion equation with unbounded domain. Z. Angew. Math. Phys, **63** (2012), 793-812.

[38] T. Yi, X. Zou, Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: A non-monotone case, J. Differential Equations, **245** (2008), 3376-3388.

ACCEPTED MANUSCRIPT

[39] T. Yi, X. Zou, Map dynamics versus dynamics of associated delay reaction-diffusion equations with a Neumann condition. Proc. R. Soc. Lond. Ser. A, **466** (2010), 2955-2973.

[40] X. Zhao, W. Wang, Fisher waves in an epidemic model. Discrete Contin. Dyn. Syst. B, **4** (2004), 1117-1128.

[41] X. Zhao, Global attractivity in a class of nonmonotone reaction-diffusion equations with time delay. Can. Appl. Math. Q., **17** (2009), 271-281.

[42] C. Pao, W. Ruan, Existence and dynamics of quasilinear parabolic systems with time delays. J. Differential Equations, **258** (2015), 3248-3285.

ACCEPTED MANUSCRIPT