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The exponential behavior of a stochastic globally modified Cahn–Hilliard–Navier–Stokes model with multiplicative noise

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A B S T R A C T

In this article, we study the stability of weak solutions to a stochastic version of a globally modified coupled Cahn–Hilliard–Navier–Stokes model with multiplicative noise. The model consists of the globally modified Navier–Stokes equations for the velocity, coupled with an Cahn–Hilliard model for the order (phase) parameter. We prove that under some conditions on the forcing terms, the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions. We also prove a result related to the stabilization of these equations.

1. Introduction

It is well accepted that the incompressible Navier–Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [16]. For instance, this approach is used in [2] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [3,15–17]. In the isothermal compressible case, the existence of a global weak solution is proved in [14]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen–Cahn equation, [16]. The associated initial and boundary value

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problem was studied in [16] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [16] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [26]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [16]. As noted in [15], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in which these patters organize themselves into parallel layers (see, e.g. [21] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn–Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier–Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [18]). This is a system of equations where an incompressible Navier–Stokes equation for the (mean) velocity v is coupled with a convective Cahn–Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

The long-time behavior of flows is a very interesting and important problem in the theory of fluid dynamic. As the vast literature shows [1,4,5,12,13,19,22,23,26,28], the problem has been receiving very much attention over the last three decades.

Another interesting question is to analyze the effects produced on a deterministic system by some stochastic or random disturbances appearing in the problem. This problem has been studied for the NS model, [6,7]. In [6], the authors studied the stability of the stationary solutions of the stochastic 2D NS equations. In particular, they proved that the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions under some restrictions on the viscosity and the forcing terms. In [7], the authors generalized to the results of [6] to a class of dissipative nonlinear systems that include the 3D Lagrangian average NS equations.

Our work is motivated by the above references. We study the stability of weak solutions to the stochastic 3D globally modified CH-NS (GMCHNS) model with multiplicative noise. In particular, we proved that the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions under some restrictions on the viscosity and the forcing terms. Let us note that the coupling between the Navier–Stokes and the Cahn–Hilliard systems makes the analysis of the control problem more involved.

The article is divided as follows. In the next section, we introduce the stochastic 3D GMCHNS model and its mathematical setting. The third section studies the stability of weak solutions. As in [6], applying the Itô formula, we study the stability of stationary solutions to the stochastic 3D GMCHNS model. We also prove in the fourth section a result related to the stabilization of these equations.

2. The stochastic GMCHNS model and its mathematical setting

2.1. Governing equations

In this article, we consider a modified version of the coupled CH-NS model with multiplicative noise. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^3$. Then, we consider the following coupled CH-NS system

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla p - \mathcal{K}\mu\nabla\phi = g_0^1(t) + g_1^1(v,\phi) + g_2^1(t,v,\phi)\dot{W}_t^1, \\ \text{div } v = 0, \\ \frac{\partial \phi}{\partial t} + v\cdot\nabla\phi - \Delta\mu = g_0^2(t) + g_1^2(v,\phi) + g_2^2(t,v,\phi)\dot{W}_t^2, \\ \mu = -\epsilon\Delta\phi + \alpha f(\phi), \end{cases} \tag{2.1}$$

in $\mathcal{M}\times[0,T]$.

In (2.1), the unknown functions are the velocity $v = (v_1, \cdots v_d)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The external volume force $g_1(v,\phi) \equiv (g_1^1(t,v,\phi), g_1^2(t,v,\phi))$, $g_0(t) \equiv (g_0^1, g_0^2)(t)$, are given. The term $g_2(t,v,\phi)\dot{W}_t \equiv (g_2^1(t,v,\phi)\dot{W}_t^1, g_2^2(t,v,\phi)\dot{W}_t^2)$ represents random external forces depending eventually on (v,ϕ) where $\dot{W}_t \equiv (\dot{W}_t^1, \dot{W}_t^2)$ denotes the time derivative of a cylindrical Wiener process. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}_1(\phi) = \int_{\mathcal{M}} \left(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\right) ds, \tag{2.2}$$

where, e.g., $F(x) = \int_0^x f(\zeta)d\zeta$. Here, the constants $\nu > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively. Here ϵ, $\alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ϵ is related with the thickness of the interface separating the two fluids. Hereafter, as in [16] we assume that $\epsilon \le \alpha$.

A typical example of potential F is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type $F(x) = \gamma_1 x^4 - \gamma_2 x^2$, γ_1, γ_2 being positive constants. As noted in [15], $(2.1)_1$ can be replaced by

$$\frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla\tilde{p} = -\mathcal{K}\text{div }(\nabla\phi\otimes\nabla\phi) + g_0^1(t) + g_1^1(v,\phi) + g_2^1(t,v,\phi)\dot{W}_t^1, \tag{2.3}$$

where $\tilde{p} = p - \mathcal{K}(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi))$, since $\mathcal{K}\mu\nabla\phi = \nabla(\mathcal{K}(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi))) - \mathcal{K}\text{div }(\nabla\phi\otimes\nabla\phi)$. The stress tensor $\nabla\phi\otimes\nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for the model, as in [15] we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta\phi = \partial_\eta\mu = 0, \quad \text{on } \partial\mathcal{M}\times[0,T], \tag{2.4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial\mathcal{M}$. These conditions ensure the mass conservation in he deterministic case. In fact, for $g_0^2 = 0$, $g_1^2 = 0$, $g_2^2 = 0$, from $\partial_\eta\mu = 0$ on $\partial\mathcal{M}\times[0,T]$, we have the conservation of the following quantity

$$\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \phi(x,t)dx, \tag{2.5}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle, \ \forall t\in[0,T]. \tag{2.6}$$

Hereafter, we assume that g_0^2, g_1^2 and g_2^2 are chosen such that (2.6) is satisfied, which is the case if we assume that

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$$\langle g_0^2(t)\rangle = 0,\ \langle g_1^2(u,\psi)\rangle = 0,\ \langle g_2^2(u,\psi)\dot{W}_t^2\rangle = 0,\ \forall t \geq 0,\ (u,\psi) \in \mathcal{H}, \tag{2.7}$$

where $\mathcal{H}$ is defined by (2.25) below.

Concerning the boundary condition for v, we assume the Dirichlet (no-slip) boundary condition

$$v = 0, \quad \text{on } \partial\mathcal{M} \times (0,\infty). \tag{2.8}$$

Therefore we assume that there is no relative motion at the fluid–solid interface.

The initial condition is given by

$$(v,\phi)(0) = (v_0,\phi_0) \text{ in } \mathcal{M}. \tag{2.9}$$

Now, we define the function $F_N : \Re^+ \to \Re^+$ by

$$F_N(r) = \min\{1, N/r\},\ r \in \Re^+, \tag{2.10}$$

for some (fixed) $N \in \Re^+$. We recall from [8] the following properties of F_N.

Lemma 2.1. *The function F_N satisfies:*

$$\begin{aligned}
&|F_N(p) - F_N(r)| \leq \frac{|p-r|}{r},\ \forall p,r \in \Re^+,\ r \neq 0,\\
&|F_N(\|v_1\|) - F_N(\|v_2\|)| \leq \frac{\|v_1 - v_2\|}{\|v_2\|},\ \forall v_1,v_2 \in V_1,\ v_2 \neq 0,\\
&|F_M(p) - F_N(r)| \leq \frac{|M-N|}{r} + \frac{|p-r|}{r},\ \forall p,r,M,N \in \Re^+,\ r \neq 0.
\end{aligned} \tag{2.11}$$

Now we consider the following 3D GMCHNS

$$\begin{cases}
\dfrac{\partial v}{\partial t} - \nu\Delta v + F_N(\|v\|)[(v\cdot\nabla)v] + \nabla p - \mathcal{K}\mu\nabla\phi = g_0^1(t) + g_1^1(v,\phi) + g_2^1(t,v,\phi)\dot{W}_t^1,\\
\text{div } v = 0,\\
\dfrac{\partial \phi}{\partial t} + v\cdot\nabla\phi - \Delta\mu = g_0^2(t) + g_1^2(v,\phi) + g_2^2(t,v,\phi)\dot{W}_t^2,\\
\mu = -\epsilon\Delta\phi + \alpha f(\phi),
\end{cases} \tag{2.12}$$

in $\mathcal{M} \times (0,+\infty)$, where $\|v\|$ is a norm defined below.

The deterministic version of the GMCHNSE (2.12) was studied in [25], where the author proved the existence and uniqueness of strong solutions as well as the existence of a global $\mathcal{U}$-attractor. In [24], the author proved the existence and final fractal dimension of a pullback attractor in the space $\mathcal{U}$ for a three dimensional system of a non-autonomous GMCHNSE model. In [11], the authors studied the stochastic GMACNSE model (2.12) in a 3D bounded domain and proved the existence and uniqueness of a strong solutions in the sense of stochastic analysis and PDE sense.

Let us recall that the GMCHNSE model was inspired from the globally modified Navier–Stokes equations (GMNSE) proposed in [8]. As noted in [8] in the case of the GMNSE, the GMCHNSE are indeed globally modified. The factors $F_N(\|v\|)$ and $F_N(\|(v,\phi)\|_{\mathcal{U}})$ depend respectively on the norms $\|v\|$ and $\|(v,\phi)\|_{\mathcal{U}}$. They prevent large values of $\|v\|$ and $\|(v,\phi)\|_{\mathcal{U}}$ dominating the dynamics. Just like the GMNSE, the GMACNSE violates the basic laws of mechanics, but mathematically the model is well defined. See also [10] for other modifications of the nonlinear term in the NSE.

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2.2. Mathematical setting

We first recall from [15] a weak formulation of (2.1), (2.4), (2.8)–(2.9). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^2$). We also assume that $f \in \mathcal{C}^2(\Re)$ satisfies

$$\begin{cases} \lim\limits_{|x|\to+\infty} f'(x) > 0, \\ |f^{(i)}(x)| \le c_f(1+|x|^{2-i}), \ \forall x \in \Re, \ i = 0,1,2, \end{cases} \tag{2.13}$$

where c_f is some positive constant.

We now recall from [15] the functional set up of the model (2.1), (2.4), (2.8), (2.9).

If X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in \mathcal{C}_c^\infty(\mathcal{M}) : \quad \text{div } u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^3$ and $(H_0^1(\mathcal{M}))^3$ respectively. The scalar product in H_1 is denoted by $(\cdot,\cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u,v)) = \sum_{i=1}^{3}(\partial_{x_i}u, \partial_{x_i}v)_{L^2}, \quad \|u\| = ((u,u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 v = -\mathcal{P}\Delta v, \ \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray–Helmotz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

Hereafter, we set

$$H_2 = L^2(\mathcal{M}), \ V_2 = H^1(\mathcal{M}), \ H = H_1 \times H_2, \ V = V_1 \times V_2. \tag{2.14}$$

Then we introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_1\phi = -\Delta\phi, \ \forall \phi \in D(A_1) = \{\phi \in H^2(\mathcal{M}), \ \partial_\eta\phi = 0, \quad \text{on } \partial\mathcal{M}\}, \tag{2.15}$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot|_{L^2} + |\langle \cdot \rangle|_{L^2}$, which is equivalent to the H^2-norm. Also we define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2-functions with null mean

$$B_n\phi = -\Delta\phi, \ \forall \phi \in D(B_n) = D(A_1) \cap L_0^2(\mathcal{M}). \tag{2.16}$$

Note that B_n^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_n^s, for any $s \in \Re$, noting that $|B_n^{s/2} \cdot|_{L^2}$, $s > 0$, is an equivalent norm to the canonical H^s-norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.

We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_0)$ into H, $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_1) \cap H^3(\mathcal{M}))$ into H_1, respectively. More precisely, we set

$$
\begin{aligned}
&(B_0(u,v),w) = \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w),\ \forall u,v,w\in D(A_0),\\
&(B_1(u,\phi),\rho) = \int_{\mathcal{M}} [(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho),\ \forall u\in D(A_0),\ \phi,\rho\in D(A_1),\\
&(R_0(\mu,\phi),w) = \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]dx = b_1(w,\phi,\mu),\ \forall w\in D(A_0),\ \phi\in D(A_1)\cap H^3(\mathcal{M}),\ \mu\in L^2(\mathcal{M}).
\end{aligned}
\tag{2.17}
$$

Note that

$$R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi.$$

We recall that B_0, B_1 and R_0 satisfy the following estimates

$$
\begin{aligned}
&|b_0(u,v,w)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_0v|_{L^2}|w|_{L^2},\ \forall u\in V_1, v\in D(A_0)), w\in H_1,\\
&|B_0(u,v)|_{V_1^*} \le c|u|_{L^2}^{1/4}\|u\|^{3/4}|v|_{L^2}^{1/4}\|v\|^{3/4},\ \forall u,v\in V_1,\\
&|B_0(u,v)|_{L^2} \le c\|u\|\|v\|^{1/2}|A_0v|_{L^2}^{1/2},\ \forall u\in V_1, v\in D(A_0),
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
&|b_1(u,\phi,\psi)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_1\phi|_{L^2}|\psi|_{L^2},\ \forall u\in V_1, \phi\in D(A_1)), \psi\in H_2,\\
&|B_1(u,\phi)|_{V_2^*} \le c|u|_{L^2}^{1/4}\|u\|^{3/4}|\phi|_{L^2}^{1/4}\|\phi\|^{3/4},\ \forall u\in V_1, \phi\in V_2,\\
&|B_1(v,\phi)|_{L^2} \le c\|v\|\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2},\ \forall v\in V_1, \phi\in D(A_1),
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
&|R_0(A_1\phi,\rho)|_{V_1^*} \le c\|\rho\|^{1/2}|A_1\rho|_{L^2}^{1/2}|A_1\phi|_{L^2},\ \forall \phi,\rho\in D(A_1),\\
&|R_0(A_1\phi,\rho)|_{L^2} \le c|A_1\phi|_{L^2}|A_1\phi|_{L^2}^{1/2}|A_1^{3/2}\phi|_{L^2}^{1/2},\ \forall \phi\in D(A_1), \rho\in D(A_1^{3/2}).
\end{aligned}
\tag{2.20}
$$

Hereafter we set

$$b_0^N(u,v,w) = F_N(\|v\|)b_0(u,v,w),\quad \langle B_0^N(u,v),w\rangle = b_0^N(u,v,w),\ \forall u,v,w\in V_1. \tag{2.21}$$

It follows from (2.18)–(2.20) and (2.27) that (see [8] for more details)

$$
\begin{aligned}
&b_0^N(u,v,v) = 0,\ \forall u,v\in V_1,\\
&|b_0^N(u,v,w)| \le cN\|u\|\|w\|,\ \forall u,v,\in V_1,\\
&\|B_0^N(u,v)\|_{V_1^*} \le cN\|u\|,\ \forall u,v\in V_1.
\end{aligned}
\tag{2.22}
$$

We recall that (due to the mass conservation) we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = M_0,\ \forall t>0. \tag{2.23}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0,\ \forall t>0. \tag{2.24}$$

We set

$$\mathcal{H} = H_1\times D(A_1^{1/2}). \tag{2.25}$$

The space $\mathcal{H}$ is a complete metric space with respect to the norm

$$|(v,\phi)|^2_{\mathcal{H}} = |v|^2_{L^2} + \epsilon|\nabla\phi|^2_{L^2}. \tag{2.26}$$

We define the Hilbert space $\mathcal{U}$ by

$$\mathcal{U} = V_1 \times D(A_1^{3/2}), \tag{2.27}$$

endowed with the scalar product whose associated norm is

$$\|(v,\phi)\|^2_{\mathcal{U}} = \|v\|^2 + \epsilon|A_1^{3/2}\phi|^2_{L^2}. \tag{2.28}$$

We will denote by $\lambda_1 > 0$ a positive constant such that

$$\lambda_1|(w,\psi)|^2_{\mathcal{H}} \le \|(w,\psi)\|^2_{\mathcal{U}},\ \forall(w,\psi) \in \mathcal{U}. \tag{2.29}$$

We will also denote by c a generic positive constant that depends on the domain $\mathcal{M}$.

Let $(\Omega, \mathcal{P}, \mathcal{J})$ be a probability space on which an increasing and right continuous family $\{\mathcal{J}_t\}_{t\in[0,\infty)}$ of complete sub σ-algebra of $\mathcal{J}$ is defined. Let $\beta_n(t)$ $(n = 1, 2, 3, \cdots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathcal{P}, \mathcal{J})$. We set

$$W_t(t) = \sum_{n=1}^{\infty} \sqrt{\lambda'_n}\beta_n(t)e_n,\ t \ge 0, \tag{2.30}$$

where λ'_n $(n = 1, 2, 3, \cdots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty}\lambda'_n < \infty$, and $\{e_n\}$ $(n = 1, 2, 3, \cdots)$ is a complete orthogonal basis in the real and separable Hilbert space K. Let $Q \in L(K, K)$ be the operator defined by $Qe_n = \lambda'_n e_n$. The above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

Thus, we consider the stochastic GMCHNS model written in the following abstract mathematical setting:

$$\begin{cases} \dfrac{dv}{dt} + \nu A_0 v + B_0^N(v,v) - R_0(\epsilon A_1\phi, \phi) = g_0^1(t) + g_1^1(v,\phi) + g_2^1(t,v,\phi)\dot{W}_t^1 \text{ in } V_1^*, \\ \dfrac{d\phi}{dt} + A_1\mu + B_1(v,\phi) = g_0^2(t) + g_1^2(v,\phi) + g_2^2(t,v,\phi)\dot{W}_t^2,\ \mu = \epsilon A_1\phi + \alpha f(\phi) \ \text{ in } V_2^*, \\ (v,\phi)(0) = (v_0,\phi_0) \in \mathcal{H}, \end{cases} \tag{2.31}$$

or equivalently

$$\begin{cases} v(t) + \displaystyle\int_0^t (\nu A_0 v(s) + B_0^N(v(s),v(s)))ds = v_0 + \int_0^t R_0(\epsilon A_1\phi(s),\phi(s)))ds \\ + \displaystyle\int_0^t (g_0^1(s) + g_1^1((v,\phi)(s)))ds + \int_0^t g_2^1(s,v(s),\phi(s))dW_s^1, \\ \phi(t) + \displaystyle\int_0^t (A_1\mu(s) + B_1(v(s),\phi(s)) = \phi_0 + \int_0^t (g_0^2(s) + g_1^2((v,\phi)(s)))ds + \int_0^t g_2^2(s,v(s),\phi(s))dW_s^2, \\ \mu = \epsilon A_1\phi + \alpha f(\phi), \end{cases} \tag{2.32}$$

$\mathbb{P}$-a.s, and for all $t \in [0,T]$, where

$$g_0 \equiv (g_0^1, g_0^2) \in L^2(0,\infty,\mathcal{H}),\ g_1 \equiv (g_1^1, g_1^2) : \mathcal{U} \to \mathcal{H},\ g_2 \equiv (g_2^1, g_2^2) : [0,\infty) \times \mathcal{U} \to L(K,\mathcal{H}). \tag{2.33}$$

Remark 2.1. In the formulation (2.31) or (2.32), the term $\mu\nabla\phi$ is replaced by $A_1\nabla\phi$. This is justified since $f'(\phi)\nabla\phi$ is the gradient $F(\phi)$ and can be incorporated into the pressure gradient, see [15] for details.

Definition 2.1. A stochastic process $(v,\phi)(t)$, $t \geq 0$ is said to be a weak solution to (2.31) or (2.32) if

i) $(v,\phi)(t)$ is $\mathcal{J}_t$-adapted,
ii) $(v,\phi)(t) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{U})$ almost surely for all $T > 0$,
iii) (v,ϕ) satisfies (2.32) as an identity in $\mathcal{U}^*$, almost surely, for $t \in [0,\infty)$.

Note that (2.32) implies that almost surely, $(v,\phi) \in \mathcal{C}(0,T;\mathcal{U}^*)$ and since $(v,\phi)(\cdot)$ is also bounded in $\mathcal{H}$, as in [26,27] we can check that (v,ϕ) is almost surely in $\mathcal{C}(0,T;\mathcal{H}_{weak})$, the space of $\mathcal{H}$-valued weakly continuous functions on $[0,T]$.

Hereafter, we assume that f satisfies the additional condition

$$\begin{aligned}&\langle \alpha A_1 f(\psi), \epsilon A_1\psi\rangle = \langle \alpha A_1^{1/2} f(\psi), \epsilon A_1^{3/2}\psi\rangle \geq -\kappa_0\epsilon |A_1^{3/2}\psi|^2_{L^2}, \quad \forall \psi \in D(A_1^{3/2}),\\ &\langle \alpha A_1 f(\phi_1) - A_1 f(\phi_2), \epsilon A_1(\phi_1-\phi_2)\rangle \geq -\kappa_0\epsilon |A_1^{3/2}(\phi_1-\phi_2)|^2_{L^2}, \quad \forall \phi_1,\phi_2 \in D(A_1^{3/2}),\end{aligned} \tag{2.34}$$

where $\kappa_0 > 0$ is a fixed constant.

We also set

$$\alpha_1 = \min(\nu, \epsilon - \kappa_0) > 0. \tag{2.35}$$

3. The exponential stability of solutions

In this section we discuss the moment exponential stability and almost sure exponential stability of weak solutions to (2.31) assuming that they exist. We discuss the long-time behavior of the weak solutions $(v,\phi)(t)$ under some conditions. As in [6], applying the Itô formula, we study the stability of stationary solutions to the stochastic 3D GMCHNS model.

We will use the notation

$$\begin{aligned}&\|g_2(t,v,\phi)\|^2_{L^2(\mathcal{H})} \equiv tr(g_2(t,v,\phi)Qg_2(t,v,\phi)^*),\\ &\langle (x_1,x_2),(y_1,y_2)\rangle = (x_1,y_1)_{L^2} + \langle x_2,y_2\rangle, \ \forall (x_1,x_2),(y_1,y_2) \in \mathcal{H}.\end{aligned} \tag{3.1}$$

In this section, we assume that

$$g_0 = (g_0^1, g_0^2) \in \mathcal{H} \ \text{ and } g_1 = (g_1^1, g_1^2) : \mathcal{U} \to \mathcal{H} \tag{3.2}$$

satisfies

$$g_1(0,0) = 0, \ \|g_1(v_1,\phi_1) - g_1(v_2,\phi_2)\|_{\mathcal{U}^*} \leq L_1\|(v_1,\phi_1)-(v_2,\phi_2)\|_{\mathcal{U}}, \ \forall (v_1,\phi_1),(v_2,\phi_2) \in \mathcal{U}, \tag{3.3}$$

for some fixed constant $L_1 > 0$.

A stationary solution to (2.31) is a (v^*,ϕ^*) such that

$$\begin{cases} \nu A_0 v^* + B_0^N(v^*,v^*) - R_0(\epsilon A_1\phi^*,\phi^*) = g_0^1 + g_1^1(v^*,\phi^*),\\ \epsilon A_1^2\phi^* + \alpha A_1 f(\phi^*) + B_1(v^*,\phi^*) = g_0^2 + g_1^2(v^*,\phi^*).\end{cases} \tag{3.4}$$

To be more precise, by stationary solution, we mean an element $u^* = (v^*,\phi^*) \in \mathcal{U}$ that satisfies $(3.4)_1$ and $(3.4)_2$ in V_1^* and V_2^* respectively.

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3.1. Existence and uniqueness of stationary solution

Theorem 3.1. *Under the above assumptions and notations, if*

$$\alpha_1 - L_1 > 0, \tag{3.5}$$

then (3.4) has at least one solution u^*, which is in fact in $D(A_0) \times D(A_1^2)$. Moreover, any such stationary solution $u^* = (v^*, \phi^*)$ satisfies

$$\|(v^*, \phi^*)\|_{\mathcal{U}} \leq (\alpha_1 - L_1)^{-1} \|g_0\|_{\mathcal{U}^*} \equiv K_1. \tag{3.6}$$

Furthermore, if

$$\alpha_1 - (L_1 + 3cK_1) > 0, \tag{3.7}$$

then stationary solution is unique.

Proof. To prove (3.6), by multiplying $(3.4)_1$ by v^* and $(3.4)_2$ by $\epsilon A_1 \phi^*$ to derive that

$$\begin{aligned} &\nu\|v^*\|^2 + \epsilon^2 |A_1^{3/2}\phi^*|_{L^2}^2 + \langle \alpha A_1^{1/2} f(\phi^*), \epsilon A_1^{3/2}\phi^*\rangle = \langle g_0^1 + g_1^1(v^*, \phi^*), v^*\rangle + \langle g_0^2 + g_1^2(v^*, \phi^*), \epsilon A_1 \phi^*\rangle \\ &\leq \|g_0\|_{\mathcal{U}^*} \|(v^*, \phi^*)\|_{\mathcal{U}} + L_1 \|(v^*, \phi^*)\|_{\mathcal{U}}^2, \end{aligned} \tag{3.8}$$

which gives (assuming $(2.34)_1$)

$$(\alpha_1 - L_1)\|(v^*, \phi^*)\|_{\mathcal{U}}^2 \leq \|g_0\|_{\mathcal{U}^*} \|(v^*, \phi^*)\|_{\mathcal{U}}, \tag{3.9}$$

where

$$\alpha_1 = \min(\nu, \epsilon - \kappa_0) > 0.$$

We derive that

$$\|(v^*, \phi^*)\|_{\mathcal{U}} \leq (\alpha_1 - L_1)^{-1} \|g_0\|_{\mathcal{U}^*} \equiv K_1, \tag{3.10}$$

and (3.6) is proved

For the existence, let $\{(w_i, \psi_i), i = 1, 2, 3, \cdots\} \subset \mathcal{U}$ be an orthonormal basis of $\mathcal{H}$, where $\{w_i, i = 1, 2 \cdots\}$, $\{\psi_i, i = 1, 2 \cdots\}$ are eigenvectors of A_0 and A_1 respectively. We set $\mathcal{U}_m = \operatorname{span}\{(w_1, \psi_1), \cdots (w_m, \psi_m)\}$. We define the operator $\mathcal{Z}_m : \mathcal{U}_m \to \mathcal{U}_m$ by:

$$\begin{aligned} &\langle \mathcal{Z}_m u_1, u_2\rangle = \langle \nu A_0 v_1, v_2\rangle + \epsilon\langle A_1^2 \phi_1, \epsilon A_1 \phi_2\rangle + \langle B_0^N(v_1, v_1), v_2\rangle + \langle B_1(v_1, \phi_1), \epsilon A_1 \phi_2\rangle \\ &- \langle R_0(\epsilon A_1 \phi_1, \phi_1), v_2\rangle + \langle \alpha A_1 f(\phi_1), \epsilon A_1 \phi_2\rangle - \langle g_0^1 + g_1^1(v_1, \phi_1), v_2\rangle - \langle g_0^2 + g_1^2(v_1, \phi_1), \epsilon A_1 \phi_2\rangle, \end{aligned} \tag{3.11}$$

for $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in \mathcal{U}_m$.

Since the right hand side is a continuous linear map from $\mathcal{U}_m$ to $\Re$, by the Riesz theorem, each $\mathcal{Z}_m u_1 \in \mathcal{U}_m$ is well defined. We will check that $\mathcal{Z}_m$ is continuous.

Let $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in \mathcal{U}_m$. We set $u = (w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)$. For $u_3 = (v_3, \phi_3) \in \mathcal{U}_m$, we have

$$\begin{aligned}
&\langle \mathcal{Z}_m u_1 - \mathcal{Z}_m u_2, u_3\rangle = \langle \nu A_0 w, v_3\rangle + \epsilon\langle A_1^2\psi, \epsilon A_1\phi_3\rangle + \langle B_0^N(v_1,v_1) - B_0^N(v_2,v_2), v_3\rangle \\
&+\langle B_1(v_1,\phi_1) - B_1(v_2,\phi_2), \epsilon A_1\phi_3\rangle - \langle R_0(\epsilon A_1\phi_1,\phi_1) - R_0(\epsilon A_1\phi_2,\phi_2), v_3\rangle \\
&+\langle \alpha A_1 f(\phi_1) - \alpha A_1 f(\phi_2), \epsilon A_1\phi_3\rangle - \langle g_1^1(v_1,\phi_1) - g_1^1(v_2,\phi_2), v_3\rangle - \langle g_1^2(v_1,\phi_1) - g_1^2(v_2,\phi_2), \epsilon A_1\phi_3\rangle.
\end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}
B_0^N(v_1,v_1) - B_0^N(v_2,v_2) &= F_N(\|v_1\|)B_0(w,v_1) + F_N(\|v_2\|)B_0(v_2,w) \\
&+(F_N(\|v_1\|) - F_N(\|v_2\|))B_0(v_2,v_1),
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\langle B_0^N(v_1,v_1) - B_0^N(v_2,v_2), v_3\rangle = F_N(\|v_1\|)b_0(w,v_1,v_3) \\
&+F_N(\|v_2\|)b_0(v_2,w,v_3) + (F_N(\|v_1\|) - F_N(\|v_2\|))b_0(v_2,v_1,v_3) \equiv I_1 + I_2 + I_3.
\end{aligned} \tag{3.14}$$

We have

$$\begin{aligned}
|I_1| &= F_N(\|v_1\|)|b_0(w,v_1,v_3)| \le cN\|w\|\|v_3\|, \\
|I_2| &= F_N(\|v_2\|)|b_0(v_2,w,v_3)| \le cN\|w\|\|v_3\|, \\
|I_3| &= |(F_N(\|v_1\|) - F_N(\|v_2\|))b_0(v_2,v_1,v_3)| \le cN\|w\|\|v_3\|\|v_1\|,
\end{aligned} \tag{3.15}$$

$$\langle \nu A_0 w, v_3\rangle + \epsilon\langle A_1^2\psi, \epsilon A_1\phi_3\rangle \le c\|u_1 - u_2\|_{\mathcal{U}}\|u_3\|_{\mathcal{U}}, \tag{3.16}$$

$$\begin{aligned}
&|\langle B_1(v_1,\phi_1) - B_1(v_2,\phi_2), \epsilon A_1\phi_3\rangle| = |b_1(w,\phi_1,\epsilon A_1\phi_3) + b_1(v_2,\psi,\epsilon A_1\phi_3)| \\
&\le c\epsilon\|w\||A_1\phi_1|_{L^2}|A_1\phi_3|_{L^2} + c\epsilon\|v_2\||A_1\psi|_{L^2}|A_1\phi_3|_{L^2},
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&|\langle R_0(\epsilon A_1\phi_1,\phi_1) - R_0(\epsilon A_1\phi_2,\phi_2), v_3\rangle| = |b_1(v_3,\phi_1,\epsilon A_1\psi) + b_1(v_3,\psi,\epsilon A_1\phi_2)| \\
&\le c\epsilon\|v_3\||A_1\phi_1|_{L^2}|A_1\psi|_{L^2} + c\epsilon\|v_3\||A_1\psi|_{L^2}|A_1\phi_2|_{L^2},
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&|\langle \alpha A_1 f(\phi_1) - \alpha A_1 f(\phi_2), \epsilon A_1\phi_3\rangle| = \alpha|\langle A_1^{1/2}(f(\phi_1) - f(\phi_2)), \epsilon A_1^{3/2}\phi_3\rangle| \\
&\le \epsilon M_2(|A_1\phi_1|_{L^2}, |A_1\phi_2|_{L^2})|A_1^{3/2}\phi_3|_{L^2}|A_1^{3/2}\psi|_{L^2},
\end{aligned} \tag{3.19}$$

where hereafter M_2 denotes some monotone non-decreasing function depending only on the function f.

It follows from (3.12)–(3.19) that

$$|\langle \mathcal{Z}_m u_1 - \mathcal{Z}_m u_2, u_3\rangle| \le c\left[1 + \|v_2\| + |A_1\phi_1|_{L^2} + |A_1\phi_2|_{L^2} + M_2(|A_1\phi_1|_{L^2}, |A_1\phi_2|_{L^2})\right]\|u_1 - u_2\|_{\mathcal{U}}\|u_3\|_{\mathcal{U}}, \tag{3.20}$$

which gives

$$\|\mathcal{Z}_m u_1 - \mathcal{Z}_m u_2\|_{\mathcal{U}^*} \le c\left[1 + \|v_2\| + |A_1\phi_1|_{L^2} + |A_1\phi_2|_{L^2} + M_2(|A_1\phi_1|_{L^2}, |A_1\phi_1|_{L^2})\right]\|u_1 - u_2\|_{\mathcal{U}}, \tag{3.21}$$

which proves that $\mathcal{Z}_m : \mathcal{U}_m \to \mathcal{U}^*$ is continuous.

For $u = (v,\phi) \in \mathcal{U}_m$, we have

$$\begin{aligned}
\langle \mathcal{Z}_m u, u\rangle &= \nu\|v\|^2 + \epsilon^2|A_1^{3/2}\phi|_{L^2}^2 + \langle \alpha A_1 f(\phi), \epsilon A_1\phi\rangle - \langle g_0 + g_1(v,\phi), (v, \epsilon A_1\phi)\rangle \\
&\ge (\alpha_1 - L_1)\|(v,\phi)\|_{\mathcal{U}}^2 - \|g_0\|_{\mathcal{U}^*}\|(v,\phi)\|_{\mathcal{U}}.
\end{aligned} \tag{3.22}$$

Thus, if we take $K_1 = (\alpha_1 - L_1)^{-1}\|g_0\|_{\mathcal{U}^*}$, we obtain $\langle \mathcal{Z}_m u, u\rangle \ge 0,\ \forall u = (v,\phi) \in \mathcal{U}_m$ with $\|u\|_{\mathcal{U}} = K_1$.

Consequently by a Corollary of the Brouwer's fixed point theorem (see page 53 of [20]), for each $m \geq 1$, there exists $u_m = (v_m, \phi_m) \in \mathcal{U}_m$ such that $\mathcal{Z}_m u_m = 0$ with $\|u_m\|_{\mathcal{U}} \leq K_1$.

From (3.4), we have

$$\begin{aligned} &\nu|A_0 v_m|^2_{L^2} + \epsilon^2 |A_1^2 \phi_m|^2_{L^2} = -\langle B_0^N(v_m, v_m), A_0 v_m\rangle + \langle R_0(\epsilon A_1\phi_m, \phi_m), A_0 v_m\rangle \\ &-\langle B_1(v_m, \phi_m), \epsilon A_1^2 \phi_m\rangle - \alpha\langle A_1 f(\phi_m), \epsilon A_1^2\phi_m\rangle + \langle g_0 + g_1(v_m, \phi_m), (A_0 v_m, \epsilon A_1^2 \phi_m)\rangle. \end{aligned} \tag{3.23}$$

Note that (see [8,15,16,25])

$$|-\langle B_0^N(v_m, v_m), A_0 v_m\rangle| \leq \frac{\nu}{8}|A_0 v_m|^2_{L^2} + c\|v_m\|^2. \tag{3.24}$$

Using (2.18)–(2.20) and interpolation, we can check that (see [24,25])

$$\begin{aligned} |\langle R_0(\epsilon A_1\phi_m, \phi_m), A_0 v_m\rangle| &= |b_1(A_0 v_m, \phi_m, \epsilon A_1 \phi_m)| \leq c\epsilon |A_0 v_m|_{L^2} \|\phi_m\|^{1/2} |A_1\phi_m|^{1/2}_{L^2} |A_1^{3/2}\phi_m|_{L^2} \\ &\leq c\epsilon |A_0 v_m|_{L^2} \|\phi_m\|^{3/4} |A_1^{3/2}\phi_m|^{5/4}_{L^2} \leq c\epsilon |A_0 v_m|_{L^2} \|\phi_m\|^{3/4} |A_1\phi_m|^{5/8}_{L^2} |A_1^2\phi_m|^{5/8}_{L^2} \\ &\leq c\epsilon |A_0 v_m|_{L^2} \|\phi_m\|^{3/4} \|\phi_m\|^{5/16} |A_1^{3/2}\phi_m|^{5/16}_{L^2} |A_1^2\phi_m|^{5/8}_{L^2} \\ &\leq \frac{\nu}{8}|A_0 v|^2_{L^2} + \frac{\epsilon^2}{8}|A_1^2\phi|^2_{L^2} + c\|\phi_m\|^{34}. \end{aligned} \tag{3.25}$$

Similarly, we have (see [24,25])

$$\begin{aligned} |b_1(v_m, \phi_m, \epsilon A_1^2\phi)| &\leq c\epsilon |v_m|^{1/2}_{L^2} \|v_m\|^{1/2} |A_1\phi_m|_{L^2} |A_1^2\phi_m|_{L^2} \\ &\leq c\epsilon |v|^{1/2}_{L^2} \|v_m\|^{1/2} \|\phi_m\|^{21/32} |A_1^{3/2}\phi_m|^{1/32}_{L^2} |A_1^2\phi_m|^{21/16}_{L^2} \\ &\leq \frac{\nu}{8}|A_0 v_m|^2_{L^2} + \frac{\epsilon^2}{8}|A_1^2\phi_m|^2_{L^2} + c|v_m|^{32/5}_{L^2} \|\phi_m\|^{42/5}. \end{aligned} \tag{3.26}$$

As in [24,25], we also have

$$\alpha|\langle A_1 f(\phi_m), \epsilon A_1^2\phi_m\rangle| = \alpha|\langle f''\phi_m)(A_1^{1/2}\phi_m)^2 + f'(\phi_m)A_1\phi_m, \epsilon A_1^2\phi_m\rangle| \leq J_1 + J_2. \tag{3.27}$$

We note that from (2.13) with $i = 2$, we have

$$\begin{aligned} J_1 \equiv \alpha|\langle f''(\phi_m)(A_1^{1/2}\phi_m)^2, \epsilon A_1^2\phi\rangle| &\leq c\epsilon \int_{\mathcal{M}} |A_1^{1/2}\phi_m|^2 |A_1^2\phi_m| dx \\ &\leq c\epsilon |A_1^{1/2}\phi_m|^2_{L^4} |A_1^2\phi_m|_{L^2} \\ &\leq c\epsilon \|\phi_m\|^{1/2} |A_1\phi_m|^{3/2}_{L^2} |A_1^2\phi_m|_{L^2} \\ &\leq c\epsilon \|\phi_m\|^{5/4} |A_1^{3/2}\phi_m|^{3/4}_{L^2} |A_1^2\phi_m|_{L^2} \\ &\leq \frac{\epsilon^2}{16}|A_1^2\phi_m|^2_{L^2} + c\|\phi_m\|^{10}. \end{aligned} \tag{3.28}$$

Similarly, we have

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$$
\begin{aligned}
J_2 \equiv \alpha|\langle f'(\phi_m)A_1\phi_m, \epsilon A_1^2\phi_m\rangle| &\le c\epsilon \int_{\mathcal{M}} (1+|\phi_m|)|A_1\phi||A_1^2\phi_m|dx \\
&\le c\epsilon|A_1\phi_m|_{L^2}^2|A_1^2\phi_m|_{L^2} + c\epsilon\|\phi_m\||A_1\phi_m|_{L^3}|A_1^2\phi|_{L^2} \\
&\le c\epsilon\|\phi_m\|^{1/2}|A_1^2\phi_m|_{L^2}^{3/2} + c\epsilon\|\phi_m\|\|\phi_m\|^{1/2}|A_1^2\phi_m|_{L^2}^{7/4} \\
&\le \frac{\epsilon^2}{16}|A_1^2\phi_m|_{L^2}^2 + c\|\phi_m\|^2 + c\|\phi_m\|^{12}.
\end{aligned} \tag{3.29}
$$

It follows that

$$
\alpha|\langle A_1 f(\phi_m), \epsilon A_1^2\phi_m\rangle| \le \frac{\epsilon^2}{8}|A_1^2\phi_m|_{L^2}^2 + c\|\phi_m\|^{10} + c\|\phi_m\|^2 + c\|\phi_m\|^{12}. \tag{3.30}
$$

We also have

$$
\begin{aligned}
|\langle g_1(v_m,\phi_m), (A_0v_m, \epsilon A_1^2\phi_m)\rangle| &\equiv |\langle g_1^1(v_m,\phi_m), A_0v_m\rangle + \langle g_1^2(v_m,\phi_m), \epsilon A_1^2\phi_m\rangle| \\
&\le \tfrac{\nu}{8}|A_0v_m|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_1^2\phi_m|_{L^2}^2 + c|g_1|_{L^2}^2,
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
|\langle g_0, (A_0v_m, \epsilon A_1^2\phi_m)\rangle| &\equiv |\langle g_0^1, A_0v_m\rangle + \langle g_0^2, \epsilon A_1^2\phi_m\rangle| \\
&\le \tfrac{\nu}{8}|A_0v_m|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_1^2\phi_m|_{L^2}^2 + c|g_0|_{L^2}^2.
\end{aligned} \tag{3.32}
$$

It follows from (3.23)–(3.32) that

$$
|A_0v_m|_{L^2}^2 + \epsilon^2|A_1^2\phi_m|_{L^2}^2 \le C, \tag{3.33}
$$

where $C > 0$ is independent of $m \ge 1$.

From (3.33), we deduce that the sequence $u_m = (v_m, \phi_m)$ is bounded in $D(A_0)\times D(A_1^2)$ and consequently, we can extract a subsequence (still) denoted $u_m = (v_m, \phi_m)$ that converges weakly in $D(A_0) \times D(A_1^2)$ and strongly in $\mathcal{U}$ to an element $u^* = (v^*, \phi^*) \in D(A_0) \times D(A_1^2)$. As in [8], by passing to the limit in (3.11) we can check that $u^* = (v^*, \phi^*)$ is a stationary solution to (3.4).

For the uniqueness, let (v_1^*, ϕ_1^*), (v_2^*, ϕ_2^*) be two solutions and $(w, \psi) = (v_1^*, \phi_1^*) - (v_2^*, \phi_2^*)$. Then (w, ψ) satisfies

$$
\begin{cases}
\nu A_0 w + B_0^N(v_1^*, v_1^*) - B_0^N(v_2^*, v_2^*) - R_0(\epsilon A_1\phi_2^*, \psi) - R_0(\epsilon A_1\psi, \phi_1^*) = g_1^1(v_1^*, \phi_1^*) - g_1^1(v_2^*, \phi_2^*), \\
\epsilon A_1^2\psi + \alpha A_1 f(\phi_1^*) - \alpha A_1 f(\phi_2^*) + B_1(v_2^*, \psi) + B_1(w, \phi_1^*) = g_1^2(v_1^*, \phi_1^*) - g_1^2(v_2^*, \phi_2^*).
\end{cases} \tag{3.34}
$$

Note that

$$
\begin{aligned}
&\langle B_0^N(v_1^*, v_1^*) - B_0^N(v_2^*, v_2^*), w\rangle = F_N(\|v_1^*\|)b_0(w, v_1^*, w) \\
&+F_N(\|v_2^*\|)b_0(v_2^*, w, w) + (F_N(\|v_1^*\|) - F_N(\|v_2^*\|))b_0(v_2^*, v_1^*, w).
\end{aligned} \tag{3.35}
$$

As in (3.13)–(3.19), we can check that

$$
\begin{aligned}
&|F_N(\|v_1^*\|)b_0(w, v_1^*, w)| \le c\|v_1^*\|\|w\|^2, \\
&|(F_N(\|v_1^*\|) - F_N(\|v_2^*\|))||b_0(v_2^*, v_1^*, w)| \le c\|v_1^*\|\|w\|^2,
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
&\langle R_0(\epsilon A_1\psi, \phi_1^*), w\rangle = \langle B_1(w, \phi_1^*), \epsilon A_1\psi\rangle, \\
&|\langle B_0^N(v_1^*, v_1^*) - B_0^N(v_2^*, v_2^*), w\rangle| \le c\|v_1^*\|\|w\|^2, \\
&|\langle R_0(\epsilon A_1\phi_2^*, \psi), w\rangle| \le c\epsilon|A_1\psi|_{L^2}|A_1\phi_2^*|_{L^2}\|w\|, \\
&|\langle B_1(v_2^*, \psi), \epsilon A_1\psi\rangle| \le c\epsilon|A_1\psi|_{L^2}^2\|v_2^*\|,
\end{aligned} \tag{3.37}
$$

$$|\langle \alpha A_1 f(\phi_1^*) - \alpha A_1 f(\phi_2^*), \epsilon A_1\psi\rangle| \geq -\kappa_0\epsilon |A_1^{3/2}\psi|_{L^2}^2 \tag{3.38}$$

$$\begin{aligned}|\langle g_1^1(v_1^*,\phi_1^*) - g_1^1(v_2^*,\phi_2^*), w\rangle + \langle g_1^2(v_1^*,\phi_1^*) - g_1^2(v_2^*,\phi_2^*), \epsilon A_1\psi\rangle| &\equiv |\langle g_1(v_1^*,\phi_1^*) - g_1(v_2^*,\phi_2^*), (w, \epsilon A_1\psi)\rangle| \\ &\leq L_1\|(w,\psi)\|_{\mathcal{U}}^2.\end{aligned} \tag{3.39}$$

Multiplying $(3.34)_1$ and $(3.34)_2$ by w and $\epsilon A_1\psi$ respectively and using (3.38)–(3.39) yields

$$\nu\|w\|^2 + \epsilon^2|A_1^{3/2}\psi|_{L^2}^2 - \kappa_0\epsilon|A_1^{3/2}\psi|_{L^2}^2 \leq c(\|v_1^*\| + \epsilon^{1/2}|A_1\phi_2^*|_{L^2} + \|v_2^*\| + L_1)\|(w,\psi)\|_{\mathcal{U}}^2, \tag{3.40}$$

which gives

$$(\alpha_1 - (L_1 + 3cK_1))\|(w,\psi)\|_{\mathcal{U}}^2 \leq 0, \tag{3.41}$$

and $\|(w,\psi)\|_{\mathcal{U}} = 0$ assuming (3.7). and the theorem is proved. □

Remark 3.1. We note that condition (3.5) is satisfied if $L_1 > 0, \kappa_0$ are small enough and $\alpha_1 > 0$ is large enough. Condition (3.7) is satisfied if $\alpha_1 > 0$ is large enough, L_1 and $|g_0|_{\mathcal{H}}$ are small enough.

3.2. Stability of the steady state solutions

We study in this section the stability of the steady state solutions. We assume that conditions (3.5) and (3.7) are satisfied so that (3.4) has a unique solution (v^*,ϕ^*). We first recall from [9] some preliminary definitions.

Definition 3.1. We say that a weak solution $(v,\phi)(t)$ to (2.31) converges to $(v^*,\phi^*) \in \mathcal{H}$ exponentially in the mean square if there exists $\eta > 0$ and $M_0 = M_0((v,\phi)(0)) > 0$ such that

$$\mathbb{E}|(v,\phi)(t) - (v^*,\phi^*)|_{\mathcal{H}}^2 \leq M_0 e^{-\eta t},\ t \geq 0. \tag{3.42}$$

If (v^*,ϕ^*) is a solution to (3.4), we say that (v^*,ϕ^*) is exponentially stable in the mean square provided that every weak solution to (2.31) converges to (v^*,ϕ^*) exponentially in the mean square with the same exponential order $\eta > 0$.

Definition 3.2. We say that a weak solution $(v,\phi)(t)$ to (2.31) converges to $(v^*,\phi^*) \in \mathcal{H}$ almost surely exponentially if there exists $\eta > 0$ such that

$$\lim_{t\to\infty} \frac{1}{t}\log|(v,\phi)(t) - (v^*,\phi^*)|_{\mathcal{H}} \leq -\eta. \tag{3.43}$$

If (v^*,ϕ^*) is a solution to (3.4), we say that (v^*,ϕ^*) is almost surely exponentially stable provided that every weak solution to (2.31) converges to (v^*,ϕ^*) almost surely exponentially with the same constant $\eta > 0$.

Theorem 3.2. *We assume that g_1 satisfies (3.2)–(3.3) and g_2 satisfies*

$$\|g_2(t,v,\phi)\|_{L^2(\mathcal{H})}^2 \leq \varphi(t) + (\zeta + \delta(t))|(v,\phi)(t) - (v^*,\phi^*)|_{\mathcal{H}}^2, \tag{3.44}$$

where $\zeta > 0$ is a constant and $\varphi(t),\delta(t)$ are nonnegative integrable functions such that there exist real numbers $\rho > 0$, $M_\delta \geq 1$, $M_\varphi \geq 1$ with

$$\varphi(t) \leq M_\varphi e^{-\rho t},\ \delta(t) \leq M_\delta e^{-\rho t},\ t \geq 0. \tag{3.45}$$

Let $(v^*, \phi^*) \in \mathcal{U}$ be the unique solution to (3.4) and let

$$\lambda_1^{-1}\zeta + c_1\|(v^*, \phi^*)\|_{\mathcal{U}} + 2L_1 - 2\alpha_1 < 0, \tag{3.46}$$

where c_1 is defined by (3.53) below.

Then any weak solution $(v, \phi)(t)$ to (2.31) converges to (v^*, ϕ^*) exponentially in the mean square. More precisely, there exist real numbers $\eta \in (0, \rho)$, $M_0 \equiv M_0((v, \phi)(0)) > 0$ such that

$$\mathbb{E}|(v, \phi)(t) - (v^*, \phi^*)|^2_{\mathcal{H}} \leq M_0 e^{-\eta t}, \ \forall t > 0. \tag{3.47}$$

Proof. First we choose $\eta \in (0, \rho)$ such that

$$\lambda_1^{-1}(\zeta + \eta) + c_1\|(v^*, \phi^*)\|_{\mathcal{U}} + 2L_1 - 2\alpha_1 < 0. \tag{3.48}$$

Let us set

$$(w, \psi) = (v^*, \phi^*) - (v, \phi).$$

Applying Itô's formula to $e^{\eta t}|(w, \psi)(t)|^2_{\mathcal{H}}$ gives

$$\begin{aligned}
&e^{\eta t}|(w, \psi)(t)|^2_{\mathcal{H}} = |(w, \psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}|(w, \psi)(s)|^2_{\mathcal{H}} ds - 2\nu \int_0^t e^{\eta s}\langle A_0 v(s), w(s)\rangle ds \\
&-2\epsilon^2 \int_0^t e^{\eta s}\langle A_1^2\phi(s), A_1\psi(s)\rangle ds - 2\int_0^t e^{\eta s}\langle B_0^N(v(s), v(s)) - R_0(\epsilon A_1\phi(s), \phi(s)), w(s)\rangle ds \\
&-2\int_0^t e^{\eta s}\langle B_1(v(s), \phi(s)) + \alpha A_1 f(\phi(s)), \epsilon A_1\psi(s)\rangle ds + 2\int_0^t e^{\eta s}\langle g_0 + g_1(v(s), \phi(s)), (w, \epsilon A_1\psi)(s)\rangle ds \\
&+2\int_0^t e^{\eta s}\langle g_2(v(s), \phi(s)), (w, \epsilon A_1\psi)(s)\rangle dW_s + \int_0^t e^{\eta s}\|g_2(s, v(s), \phi(s))\|^2_{L^2(\mathcal{H})} ds.
\end{aligned} \tag{3.49}$$

We also know that (v^*, ϕ^*) satisfies

$$\begin{aligned}
&\int_0^t e^{\eta s}\langle \nu A_0 v^* + B_0^N(v^*, v^*) - R_0(\epsilon A_1\psi^*, \phi^*), w(s)\rangle ds \\
&+ \int_0^t e^{\eta s}\langle \epsilon A_1^2\phi^* + B_1(v^*, \phi^*) + \alpha A_1 f(\phi^*), \epsilon A_1\psi(s)\rangle ds \\
&= \int_0^t e^{\eta s}\langle g_0^1 + g_1^1(v^*, \phi^*), w(s)\rangle ds + \int_0^t e^{\eta s}\langle g_0^2 + g_1^2(v^*, \phi^*), \epsilon A_1\psi(s)\rangle ds \\
&\equiv \int_0^t e^{\eta s}\langle g_0 + g_1(v^*, \phi^*), (w, \epsilon A_1\psi)(s)\rangle ds.
\end{aligned} \tag{3.50}$$

Using (3.49)–(3.50), we derive that

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$$
\begin{aligned}
e^{\eta t}\mathbb{E}|(w,\psi)(t)|^2_{\mathcal{H}} &= \mathbb{E}|(w,\psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\psi)(s)|^2_{\mathcal{H}}ds - 2\nu\int_0^t e^{\eta s}\mathbb{E}\|w(s)\|^2 ds \\
&-2\epsilon^2\int_0^t e^{\eta s}\mathbb{E}|A_1^{3/2}\psi(s)|^2_{L^2}ds - 2\int_0^t e^{\eta s}\mathbb{E}F_N(\|v^*\|)b_0(w,v^*,w)ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(\|v^*\|)-F_N(\|v\|))b_0(v,v^*,w)ds + 2\int_0^t e^{\eta s}\mathbb{E}b_1(w,\psi,\epsilon A_1\phi^*)ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}b_1(v^*,\psi,\epsilon A_1\psi)ds - 2\alpha\int_0^t e^{\eta s}\mathbb{E}\langle A_1 f(\phi^*)(s) - A_1 f(\phi), \epsilon A_1\psi(s)\rangle ds \\
&+\int_0^t e^{\eta s}\mathbb{E}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds + 2\int_0^t e^{\eta s}\mathbb{E}\langle g_1(v^*(s),\phi^*(s)) - g_1(v,\phi),(w,\epsilon A_1\psi)(s)\rangle ds.
\end{aligned}
\tag{3.51}
$$

Note that

$$
\begin{aligned}
&2F_N(\|v^*\|)|b_0(w,v^*,w)| \le c_1 N\|(w,\phi)\|^2_{\mathcal{U}}, \\
&2|F_N(\|v^*\|) - F_N(\|v\|)||b_0(v,v^*,w)| \le c_1\|v^*\|\|(w,\psi)\|^2_{\mathcal{U}} \\
&2|b_1(v^*,\psi,\epsilon A_1\psi)| \le c_1\epsilon\|v^*\||A_1\psi|^2_{L^2} \le c_1\|(v^*,\phi^*)\|_{\mathcal{U}}\|(w,\psi)\|^2_{\mathcal{U}}, \\
&2|b_1(w,\psi,\epsilon A_1\phi^*)| \le c\epsilon\|w\||A_1\psi|_{L^2}|A_1\phi^*|^2_{L^2} \le c\|(v^*,\phi^*)\|_{\mathcal{U}}\|(w,\psi)\|^2_{\mathcal{U}}, \\
&-\alpha\langle A_1 f(\phi^*) - A_1 f(\phi), \epsilon A_1\psi(s)\rangle \le \epsilon\kappa_0|A_1^{3/2}\psi|^2_{L^2}, \\
&|\langle g_1(v^*(s),\phi^*(s)) - g_1(v,\phi),(w,\epsilon A_1\psi)\rangle| \le L_1\|(w,\psi)\|^2_{\mathcal{U}}.
\end{aligned}
\tag{3.52}
$$

It follows that

$$
\begin{aligned}
&2F_N(\|v^*\|)|b_0(w,v^*,w)| + |b_1(v_*,\psi,\epsilon A_1\psi)| + 2|F_N(\|v^*\|) - F_N(\|v\|)||b_0(v,v^*,w)| + 2|b_1(w,\psi,\epsilon A_1\phi^*)| \\
&-2\alpha\langle A_1 f(\phi^*) - A_1 f(\phi), \epsilon A_1\psi(s)\rangle + 2|\langle g_1(v^*(s),\phi^*(s)) - g_1(v,\phi),(w,\epsilon A_1\psi)\rangle| \\
&\le [c_1\|(v^*,\phi^*)\|_{\mathcal{U}} + 2L_1]\,\|(w,\psi)\|^2_{\mathcal{U}} + 2\epsilon\kappa_0|A_1^{3/2}\psi|^2_{L^2},
\end{aligned}
\tag{3.53}
$$

for some $c_1 > 0$.

We derive from (3.49)–(3.52) that

$$
\begin{aligned}
e^{\eta t}\mathbb{E}|(w,\psi)(t)|^2_{\mathcal{H}} &\le \mathbb{E}|(w,\psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\psi)(s)|^2_{\mathcal{H}}ds \\
&-2\alpha_1\int_0^t e^{\eta s}\mathbb{E}\|(w,\psi)(s)\|^2_{\mathcal{U}}ds + [c_1\|(v^*,\phi^*)\|_{\mathcal{U}} + 2L_1]\int_0^t e^{\eta s}\mathbb{E}\|(w,\psi)(s)\|^2_{\mathcal{U}}ds \\
&+\int_0^t e^{\eta s}(\varphi(s) + (\zeta + \delta(s))\mathbb{E}|(w,\psi)(s)|^2_{\mathcal{H}})ds.
\end{aligned}
\tag{3.54}
$$

Note that in (3.54), we use the fact that

$$
\nu\|w\|^2 + \epsilon^2|A_1^{3/2}\psi|^2_{L^2} - \epsilon\kappa_0|A_1^{3/2}\psi|^2_{L^2} \ge \alpha_1\|(w,\psi)\|^2_{\mathcal{U}}. \tag{3.55}
$$

We recall that

$$-2\alpha_1 + [c_1\|(v^*,\phi^*)\|_{\mathcal{U}} + 2L_1] + \lambda_1^{-1}(\eta+\zeta) < 0. \tag{3.56}$$

It follows from (3.54)–(3.56) that

$$e^{\eta t}\mathbb{E}(|(w,\psi)(t)|^2_{\mathcal{H}}) \leq \mathbb{E}(|(w,\psi)(0)|^2_{\mathcal{H}}) + \int_0^t e^{\eta s}(\varphi(s) + \delta(s)|(w,\psi)(s)|^2_{\mathcal{H}})ds. \tag{3.57}$$

Using the Gronwall lemma, we derive that there exists $M_0 > 0$ such that

$$\mathbb{E}(|(w,\psi)(t)|^2_{\mathcal{H}}) \leq M_0 e^{-\eta t}, \ \forall t > 0, \tag{3.58}$$

which proves (3.47). □

Theorem 3.3. *The hypothesis are the same as in Theorem 3.2. Then any weak solution $(v,\phi)(t)$ to (2.31) converges to the stationary solution (v^*,ϕ^*) of (3.4) almost surely exponentially.*

Proof. Let N_1 be a positive integer and $(w,\psi) = (v_1^*,\phi_1^*) - (v_2^*,\phi_2^*)$. By the Itô formula, for any $t \geq N_1$ we have

$$\begin{aligned}
&|(w,\psi)(t)|^2_{\mathcal{H}} = |(w,\psi)(N_1)|^2_{\mathcal{H}} - 2\nu\int_{N_1}^t \|w(s)\|^2 ds - 2\epsilon^2\int_{N_1}^t |A_1^{3/2}\psi(s)|^2_{L^2}ds \\
&-2\int_{N_1}^t (F_N(\|v^*\|)b_0(w,v^*,w) - b_1(v^*,\psi,\epsilon A_1\psi))ds \\
&-2\int_{N_1}^t (F_N(\|v^*\|) - F_N(\|v\|))b_0(v,v^*,w)ds + 2\int_{N_1}^t b_1(w,\psi,\epsilon A_1\phi^*))ds \\
&-2\alpha\int_{N_1}^t \langle A_1 f(\phi^*) - A_1 f(\phi), \epsilon A_1\psi\rangle ds + 2\int_{N_1}^t \langle g_1(v^*(s),\phi^*(s)) - g_1(v,\phi),(w,\epsilon A_1\psi)(s)\rangle ds \\
&+\int_{N_1}^t \|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds + 2\int_{N_1}^t \langle (w,\epsilon A_1\psi)(s), g_2(s,v(s),\phi(s))dW_s\rangle
\end{aligned} \tag{3.59}$$

By the Burkholder–Davis–Gundy lemma, we have

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{N_1\leq t\leq N_1+1}\int_{N_1}^t \langle (w,\epsilon A_1\psi)(s), g_2(s,v(s),\phi(s))dW_s\rangle\right] \\
&\leq \eta_1\left[\mathbb{E}\int_{N_1}^{N_1+1} |(w,\psi)(s)|^2_{\mathcal{H}}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds\right]^{1/2} \\
&\leq \eta_1\left[\mathbb{E}\left(\sup_{N_1\leq t\leq N_1+1}|(w,\psi)(t)|^2_{\mathcal{H}}\int_{N_1}^{N_1+1}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds\right)\right]^{1/2} \\
&\leq \eta_2\int_{N_1}^{N_1+1}\mathbb{E}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds + \frac{1}{2}\mathbb{E}\sup_{N_1\leq t\leq N_1+1}|(w,\psi)(s)|^2_{\mathcal{H}},
\end{aligned} \tag{3.60}$$

where $\eta_1 > 0, \eta_2 > 0$ are some constants.

Therefore as in (3.49)–(3.54), we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{N_1\le t\le N_1+1}|(w,\psi)(t)|^2_{\mathcal{H}}\right]\le \mathbb{E}|(w,\psi)(N_1)|^2_{\mathcal{H}}-2\alpha_1\int_{N_1}^{N_1+1}\mathbb{E}\|(w,\psi)(s)\|^2_{\mathcal{U}}ds\\
&+\left[\zeta\lambda_1^{-1}+c_1\|(v^*,\phi^*)\|_{\mathcal{U}}+2L_1\right]\int_{N_1}^{N_1+1}\mathbb{E}\|(w,\psi)(s)\|^2_{\mathcal{U}}ds \\
&+\eta_0\int_{N_1}^{N_1+1}\mathbb{E}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})}ds+\frac{1}{2}\mathbb{E}\sup_{N_1\le t\le N_1+1}|(w,\psi)(t)|^2_{\mathcal{H}},
\end{aligned}
\tag{3.61}
$$

for some $\eta_0>0$.

It follows (3.44), (3.48) and (3.61) that

$$
\frac{1}{2}\mathbb{E}\sup_{N_1\le t\le N_1+1}|(w,\psi)(t)|^2_{\mathcal{H}}\le \mathbb{E}|(w,\psi)(N_1)|^2_{\mathcal{H}}+\eta_0\int_{N_1}^{N_1+1}(\varphi(s)+(\zeta+\delta(s))\mathbb{E}|(w,\psi)(s)|^2_{\mathcal{H}})ds. \tag{3.62}
$$

Since

$$
\varphi(t)\le M_\varphi e^{-\rho t},\ \delta(t)\le M_\delta e^{-\rho t},\ \eta\in(0,\rho),\ M_\varphi\ge 1,\ M_\delta\ge 1, \tag{3.63}
$$

it follows from Theorem 3.2 that there exist $M_1=M_1((v,\phi)(0))\ge 1$ such that

$$
\mathbb{E}\left(\sup_{N_1\le t\le N_1+1}|(w,\psi)(t)|^2_{\mathcal{H}}\right)\le M_1e^{-\eta N_1}, \tag{3.64}
$$

and the proof of the theorem follows from the Borel–Cantelli lemma as in [6] (see also [7]). □

Theorem 3.4. *Let $(v^*,\phi^*)\in\mathcal{U}$ be the unique solution to (3.4). Furthermore, we assume that*

$$
\begin{aligned}
&g_2(v^*,\phi^*)=0,\ \forall t\ge 0,\\
&\|g_2(t,v_1,\phi_1)-g_2(v_2,\phi_2)\|_{L^2(\mathcal{H})}\le c_g|(v_1,\phi_1)-(v_2,\phi_2)|_{\mathcal{H}},\ \forall(v_1,\phi_1),(v_2,\phi_2)\in\mathcal{H}.
\end{aligned}
\tag{3.65}
$$

If

$$
-2\alpha_1+c_g\lambda_1^{-1}+c_1\|(v^*,\phi^*)\|_{\mathcal{U}}+2L_1<0, \tag{3.66}
$$

then any weak solution to (2.31) converges to (v^*,ϕ^*) exponentially in the mean square. That is, there exists $\eta>0$ such that

$$
\mathbb{E}|(v,\phi)(t)-(v^*,\phi^*)|^2_{\mathcal{H}}\le \mathbb{E}|(v_0,\phi_0)-(v^*,\phi^*)|^2_{\mathcal{H}}e^{-\eta t},\ \forall t\ge 0. \tag{3.67}
$$

Moreover, the path-wise exponential stability with probability one of (v^*,ϕ^*) also holds true.

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Proof. Let $(w,\psi)=(v_1^*,\phi_1^*)-(v_2^*,\phi_2^*)$. We start with the equality

$$\begin{aligned}
&v^*-v(t)=v^*-v(0)-\int_0^t\left[\nu A_0(v^*-v)-B_0^N(v,v)+B_0^N(v^*,v^*)\right]ds\\
&+\int_0^t\left[R_0(\epsilon A_1\phi^*,\phi^*)-R_0(\epsilon A_1\phi,\phi))+g_1(v^*,\phi^*)-g_1(v,\phi))\right]ds\\
&+\int_0^t(g_2^1(s,v^*,\phi^*)-g_2^1(s,v,\phi))dW_s^1, \qquad (3.68)\\
&\phi^*-\phi(t)=\phi^*-\phi(0)-\epsilon\int_0^t A_1^2(\phi^*-\phi)ds-\int_0^t\left[(B_1(v^*,\phi^*)-B_1(v,\phi)\right]ds\\
&-\alpha\int_0^t\left[A_1f(\phi^*)-A_1f(\phi)\right]ds+\int_0^t(g_2^2(v^*,\phi^*)-g_2^2(v,\phi))dW_s^2.
\end{aligned}$$

Let $\eta>0$ small enough and fixed later. By the Itô formula, we have

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(w,\psi)(t)|_{\mathcal{H}}^2=\mathbb{E}|(w,\psi)(0)|_{\mathcal{H}}^2+\int_0^t\eta e^{\eta s}\mathbb{E}|(w,\psi)(s)_{L^2}^2ds-2\nu\int_0^t e^{\eta s}\mathbb{E}\|w(s)\|^2ds\\
&-2\epsilon^2\int_0^t e^{\eta s}\mathbb{E}|A_1^{3/2}\psi(s)|_{L^2}^2ds-2\int_0^t e^{\eta s}\mathbb{E}F_N(\|v^*\|)b_0(w,v^*,w)ds\\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(\|v^*\|)-F_N(\|v\|))b_0(v,v^*,w)ds+2\epsilon\int_0^t e^{\eta s}\mathbb{E}b_1(w,\psi,\epsilon A_1\phi^*)ds-2\epsilon\int_0^t e^{\eta s}\mathbb{E}b_1(v^*,\psi,\epsilon A_1\psi)ds\\
&+2\int_0^t e^{\eta s}\mathbb{E}\langle g_1(v^*(s),\phi^*(s))-g_1(v,\phi),(w,\epsilon A_1\phi)(s)\rangle ds+\int_0^t e^{\eta s}\mathbb{E}\|g_2(v^*(s),\phi^*(s))-g_2(v,\phi)\|_{L^2(\mathcal{H})}^2ds\\
&-2\int_0^t e^{\eta s}\mathbb{E}\langle\alpha A_1f(\phi^*)-\alpha A_1f(\phi),\epsilon A_1\psi(s)\rangle ds.
\end{aligned}$$
(3.69)

It follows from (3.69) and (2.18)–(2.20) that

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(w,\psi)(t)|_{\mathcal{H}}^2\le\mathbb{E}|(w,\psi)(0)|_{\mathcal{H}}^2+\int_0^t\eta e^{\eta s}\mathbb{E}|(w,\psi)(s)|_{\mathcal{H}}^2ds\\
&+\left[-2\alpha_1+2c_g\lambda_1^{-1}+c_1\|(v^*,\phi^*)\|_{\mathcal{U}}+2L_1\right]\int_0^t e^{\eta s}\mathbb{E}|(w,\psi)(s)_{\mathcal{H}}^2ds \qquad (3.70)\\
&\le\mathbb{E}|(w,\psi)(0)|_{\mathcal{H}}^2+(\eta+\kappa_2\lambda_1)\int_0^t e^{\eta s}|(w,\psi)(s)|_{\mathcal{H}}^2ds\le\mathbb{E}|(w,\psi)(0)|_{\mathcal{H}}^2,
\end{aligned}$$

where

$$\kappa_2\equiv-2\alpha_1+c_g\lambda_1^{-1}+c_1\|(v^*,\phi^*)\|_{\mathcal{U}}+2L_1<0, \qquad (3.71)$$

and η is chosen such that

$$\eta + \kappa_2 \lambda_1 < 0.$$

It follows from (3.70) that

$$\mathbb{E}e^{\eta t}|(w,\psi)(t)|^2_{\mathcal{H}} \le \mathbb{E}|(w,\psi)(0)|^2_{\mathcal{H}}, \tag{3.72}$$

and the proof of the first part of the theorem follows as that of Theorem 3.2. The rest of the theorem is proved using a similar method to the one in the proof of Theorem 3.3. □

Theorem 3.5. *We assume that $g_0 \equiv 0$ and there exists a constant $\zeta > 0$ such that*

$$\|g_2(t,v,\phi)\|^2_{L^2(\mathcal{H})} \le \varphi(t) + (\zeta + \delta(t))|(v,\phi)|^2_{\mathcal{H}}, \tag{3.73}$$

where $\varphi(t)$, $\delta(t)$ satisfy (3.45). We also suppose that $g_1 : [0,\infty) \times \mathcal{U} \to \mathcal{U}^*$ satisfies

$$\langle g_1(t,v,\phi),(v,\epsilon A_1\phi)\rangle \le \alpha(t) + (c_3 + \beta(t))|(v,\phi)|^2_{\mathcal{H}}, \tag{3.74}$$

where $c_3 > 0$, $\alpha(t)$, $\beta(t)$ are integrable functions such that there exist real numbers $\rho > 0$, $M_\alpha \ge 1$, $M_\beta \ge 1$, with

$$\alpha(t) \le M_\alpha e^{-\rho t},\ \ \beta(t) \le M_\beta e^{-\rho t},\ \ t \ge 0. \tag{3.75}$$

Furthermore, let

$$2\alpha_1 > \zeta\lambda_1^{-1} + 2c_3\lambda_1^{-1}. \tag{3.76}$$

Then any weak solution $(v,\phi)(t)$ to (2.31) converges to zero almost surely exponentially.

Proof. Let $\eta \in (0,\rho)$ be such that

$$2\alpha_1 > \lambda_1^{-1}(\zeta + \eta) + 2c_3\lambda_1^{-1}. \tag{3.77}$$

Then we have

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(v,\phi)(t)|^2_{\mathcal{H}} = \mathbb{E}|(v,\phi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(v,\phi)(s)|^2_{\mathcal{H}} ds \\
&-2\nu\int_0^t e^{\eta s}\mathbb{E}\|v(s)\|^2 ds - 2\epsilon^2\int_0^t e^{\eta s}\mathbb{E}|A_1^{3/2}\phi(s)|^2_{L^2} ds - 2\epsilon\int_0^t e^{\eta s}\mathbb{E}\langle A_1 f(\phi),\epsilon A_1\phi\rangle ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}\langle g_1(v(s),\phi(s)),(v,\epsilon A_1\phi)(s)\rangle ds + \int_0^t e^{\eta s}\mathbb{E}\|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})} ds \\
&\le \mathbb{E}|(v,\phi)(0)|^2_{\mathcal{H}} + (-2\alpha_1 + \lambda_1^{-1}(\zeta+\eta) + 2c_3\lambda_1^{-1})\int_0^t e^{\eta s}\mathbb{E}\|(v,\phi)(s)\|^2_{\mathcal{U}} ds \\
&+\int_0^t e^{\eta s}\mathbb{E}(2\alpha(s) + \varphi(s) + (\beta(s)+\delta(s))|(v,\phi)(s)|^2_{\mathcal{H}}) ds \\
&\le \mathbb{E}|(v,\phi)(0)|^2_{\mathcal{H}} + \int_0^t e^{\eta s}\mathbb{E}(2\alpha(s) + \varphi(s) + (\beta(s)+\delta(s))|(v,\phi)(s)|^2_{\mathcal{H}}) ds,
\end{aligned} \tag{3.78}$$

which gives

$$\mathbb{E}e^{\eta t}|(v,\phi)(t)|_{\mathcal{H}}^2 \leq \mathbb{E}|(v,\phi)(0)|_{\mathcal{H}}^2 + \int_0^t e^{\eta s}(\varphi(s)+2\alpha(s)+(2\beta(s)+\delta(s))\mathbb{E}|(v,\phi)(s)|_{\mathcal{H}}^2 ds. \tag{3.79}$$

By the Gronwall lemma, we obtain that any weak solution to (2.31) converges to zero exponentially in the mean square. We can then finish the proof using the same method as in the proof of Theorem 3.3. □

4. Stabilization of the 3D GMCHNS model (2.31)

Hereafter, we briefly discuss the stabilization of the 3D GMCHNS model (2.31). As noted in [6,7], in order to produce a stabilization effect, it is enough to consider a one dimensional Wiener process for that purpose.

Hereafter, we suppose that $g_0 \in \mathcal{H}$ and g_2 is given by

$$g_2(t,v,\phi) = \sigma(v^* - v, \phi^* - \phi), \ \forall (v,\phi) \in \mathcal{H},$$

for some $\sigma \in \Re$. We also assume that

$$|g_1(v_1,\phi_1) - g_1(v_2,\phi_2)|_{\mathcal{H}} \leq L_1|(v_1,\phi_1)-(v_2,\phi_2)|_{\mathcal{H}}, \ \forall (v_1,\phi_1),(v_2,\phi_2) \in \mathcal{H}, \quad g_1(0,0) \neq 0. \tag{4.1}$$

Lemma 4.1. *Let $(v^*,\phi^*) \in \mathcal{U}$ be the unique solution to (3.4). If g_1 satisfies (4.1) and*

$$2\alpha_1 - [c_1\|(v^*,\phi^*\|_{\mathcal{U}} + 2L_1] > 0, \tag{4.2}$$

where L_1 is the Lipschitz constant of g_1 given in (4.1), then the stationary solution (v^*,ϕ^*) to (3.4) is exponentially stable.

Proof. We will only sketch the proof as it is similar to the proof of Theorem 10.2 of [26]. Let (v,ϕ) be a solution to the deterministic system:

$$\begin{cases} \dfrac{dv}{dt} + \nu A_0 v + B_0^N(v,v) - R_0(\epsilon A_1\phi,\phi) = g_0^1 + g_1^1(v,\phi), \\ \dfrac{d\phi}{dt} + A_1\mu + B_1(v,\phi) = g_0^2 + g_1^2(v,\phi), \quad \mu = \epsilon A_1\phi + \alpha f(\phi), \\ (v,\phi)(0) = (v_0,\phi_0). \end{cases} \tag{4.3}$$

Let

$$(w,\psi) = (v^*,\phi^*) - (v,\phi).$$

Then (w,ψ) satisfies

$$\begin{cases} \dfrac{dw}{dt} + \nu A_0 w + F_N(\|v\|)B_0(v,w) + F_N(\|v^*\|)B_0(w,v^*)(F_N(\|v^*\|) - F_N(\|v\|))B_0(v,v^*) \\ -R_0(\epsilon A_1\phi^*,\psi) - R_0(\epsilon A_1\psi,\phi) = g_1^1(v^*,\phi^*) - g_1^1(v,\phi), \\ \dfrac{d\psi}{dt} + \epsilon A_1^2\psi + B_1(w,\phi) + B_1(v^*,\psi) + \alpha A_1 f(\phi^*) - \alpha A_1 f(\phi) = g_1^2(v^*,\phi^*) - g_1^2(v,\phi), \\ (w,\psi)(0) = (v^*,\phi^*) - (v_0,\phi_0). \end{cases} \tag{4.4}$$

Let

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$$y = |(w,\psi)|_{\mathcal{H}}^2.$$

Then, multiplying $(4.4)_1$ by w, $(4.4)_2$ by $\epsilon A_1 \psi$ and adding the resulting equalities, we derive as in (3.49)–(3.51) that

$$\frac{dy}{dt} + 2\alpha_1 \|(w,\psi)\|_{\mathcal{U}}^2 \leq [c_1 \|(v^*, \phi^*\|_{\mathcal{U}} + 2L_1] \, \|(w,\psi)\|_{\mathcal{U}}^2. \tag{4.5}$$

Assuming that

$$\sigma_0 \equiv 2\alpha_1 - [c_1 \|(v^*, \phi^*\|_{\mathcal{U}} + 2L_1] > 0, \tag{4.6}$$

we derive that

$$\frac{dy}{dt} + \kappa_2 y \leq 0, \tag{4.7}$$

where

$$\kappa_2 \equiv \lambda_1 \sigma_0 > 0. \tag{4.8}$$

It follows that

$$y(t) \leq y(0) e^{-\kappa_2 t}, \ \forall t \geq 0, \tag{4.9}$$

and the lemma is proved. □

If the Lipschitz constant L_1 of g_1 is sufficiently large such that $\kappa_2 < 0$, then we do not know if (v^*, ϕ^*) is exponentially stable or not. However, the following result related to the stabilization of the 3D GMCHNS systems holds true.

Theorem 4.2. *We assume that g_1 satisfies (4.1). Let $(v^*, \phi^*) \in \mathcal{U}$ be the unique solution to (3.4). Let $\kappa_2 < 0$, where κ_2 is given by (4.8). Assume that σ is any real number such that*

$$\lambda_1 \kappa_2 + \sigma^2 > 0. \tag{4.10}$$

Then there exists $\Omega_0 \subset \Omega$, $\mathcal{P}(\Omega_0) = 0$, such that for $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that

$$|(v,\phi)(t) - (v^*, \phi^*)|_{\mathcal{H}}^2 \leq |(v,\phi)(0) - (v^*, \phi^*)|_{\mathcal{H}}^2 e^{-\eta t}, \ \forall t \geq T(\omega), \tag{4.11}$$

where $\eta > 0$ is given below and $(v,\phi)(t)$ is any weak solution to (2.31) with the function g_2 given by

$$g_2(t,x,y) = \sigma(v^* - x, \phi^* - y), \ \forall (x,y) \in \mathcal{H}. \tag{4.12}$$

Proof. Let

$$(w,\psi)(t) = (v^*, \phi^*) - (v,\phi)(t).$$

Applying the Itô formula to $|(w,\psi)(t)|^2_{\mathcal{H}}$, we derive as in (3.49)–(3.51) that

$$\begin{aligned}
&|(w,\psi)(t)|^2_{\mathcal{H}} = |(w,\psi)(0)|^2_{\mathcal{H}} - 2\nu\int_0^t \|w(s)\|^2 ds - 2\epsilon^2\int_0^t |A_1^{3/2}\psi(s)|^2_{L^2} ds \\
&-2\int_0^t (F_N(\|v^*\|) - F_N(\|v\|))b_0(v,v^*,w)ds \\
&-2\int_0^t F_N(\|v^*\|)b_0(w,v^*,w)ds - 2\int_0^t b_1(v^*,\psi,\epsilon A_1\psi)ds + 2\int_0^t b_1(w,\psi,\epsilon A_1\phi^*)ds \\
&-2\alpha\int_0^t \langle A_1 f(\phi^*) - A_1 f(\phi), \epsilon A_1\psi\rangle ds + \int_0^t \|g_2(s,v(s),\phi(s))\|^2_{L^2(\mathcal{H})} ds \\
&+2\int_0^t \langle (w,\epsilon A_1\psi), g_2(s,v,\phi)dW_s(s)\rangle + 2\int_0^t \langle g_1(v^*,\phi^*) - g_1(v,\phi),(w,\epsilon A_1\psi)\rangle ds.
\end{aligned} \tag{4.13}$$

Using (3.52), we also have

$$\begin{aligned}
&-2\nu\|w(s)\|^2 - 2\epsilon^2|A_1\psi(s)|^2_{L^2} + 2|F_N(\|v^*\|) - F_N(\|v\|)||b_0(v,v^*,w)| \\
&+2F_N(\|v^*\|)|b_0(w,v^*,w)| + 2|b_1(v^*,\psi,\epsilon A_1\psi)| + 2|b_1(w,\psi,\epsilon A_1\phi^*)| \\
&-2\alpha\int_0^t \langle A_1 f(\phi^*) - A_1 f(\phi), \epsilon A_1\psi\rangle \\
&\le [-2\alpha_1 + c_1\|(v^*,\phi^*\|_{\mathcal{U}} + 2L_1]\,\|(w,\psi)(s)\|^2_{\mathcal{U}} \\
&\le [-2\alpha_1 + c_1\|(v^*,\phi^*\|_{\mathcal{U}} + 2L_1]\,\lambda_1|(w,\psi)(s)|^2_{\mathcal{H}}.
\end{aligned} \tag{4.14}$$

Let

$$2\eta = \lambda_1\kappa_2 + \sigma^2 > 0, \tag{4.15}$$

where κ_2 is given by (4.8).

It follows from (4.13)–(4.15) that

$$\begin{aligned}
&\log|(w,\psi)(t)|^2_{\mathcal{H}} = \int_0^t \frac{1}{|(w,\psi)(t)|^2_{\mathcal{H}}}\left(-2\nu\|w(s)\|^2 - 2\epsilon^2|A_1^{3/2}\psi(s)|^2_{L^2} + \sigma^2|(w,\psi)(s)|^2_{\mathcal{H}}\right) ds \\
&+\log|(w,\psi)(0)|^2_{\mathcal{H}} - \int_0^t \frac{2}{|(w,\psi)(s)|^2_{\mathcal{H}}}\left(b_1(v^*,\psi,\epsilon A_1\psi) - b_1(w,\psi,\epsilon A_1\phi^*)\right) ds \\
&-\int_0^t \frac{2}{|(w,\psi)(s)|^2_{\mathcal{H}}}\left(F_N(\|v^*\|)b_0(w,v^*,w) + (F_N(\|v^*\|) - F_N(\|v\|))b_0(v,v^*,w)\right) ds \\
&+\int_0^t \frac{2}{|(w,\psi)(s)|^2_{\mathcal{H}}}\left(\langle g_1(v^*,\phi^*) - g_1(v,\phi),(w,\epsilon A_1\psi)\rangle - \alpha\langle A_1 f(\phi^*) - A_1 f(\phi),\epsilon A_1\psi\rangle\right) ds \\
&+2\int_0^t \frac{\sigma|(w,\psi)(s)|^2_{\mathcal{H}}}{|(w,\psi)(s)|^2_{\mathcal{H}}} dW_s(s) - \frac{1}{2}\int_0^t \frac{4\sigma^2|(w,\psi)(s)|^4_{\mathcal{H}}}{|(w,\psi)(s)|^4_{\mathcal{H}}} ds \\
&\le \log|(w,\psi)(0)|^2_{\mathcal{H}} - 2\eta t + 2\sigma W_t(t).
\end{aligned} \tag{4.16}$$

Since almost surely we have

$$\lim_{t\to\infty} \frac{W_t(t)}{t} = 0,$$

we can find $\Omega_0 \subset \Omega$ with $\mathcal{P}(\Omega_0) = 0$ such that for each $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that for all $t \geq T(\omega)$, we have

$$\frac{2\sigma W_t(t)}{t} \leq \eta. \tag{4.17}$$

Therefore, we obtain that for $t \geq T(\omega)$ we derive from (4.16) that

$$\log |(w,\psi)(t)|^2_{\mathcal{H}} \leq \log |(w,\psi)(0)|^2_{\mathcal{H}} - \eta t, \tag{4.18}$$

which proves (4.11). □

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