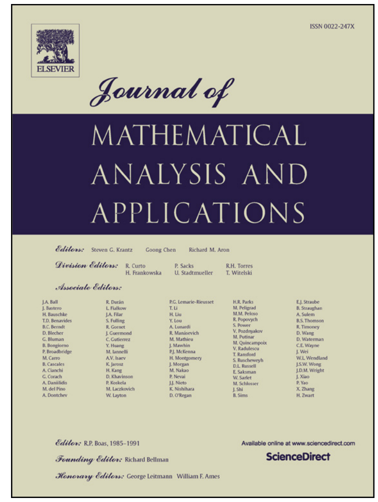

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Limiting Distributions of Spectral Radii for Product of Matrices from the Spherical Ensemble

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Abstract. Consider the product of m independent $n \times n$ random matrices from the spherical ensemble for $m \geq 1$. The spectral radius is defined as the maximum absolute value of the n eigenvalues of the product matrix. When $m = 1$, the limiting distribution for the spectral radii has been obtained by Jiang and Qi (2017). In this paper, we investigate the limiting distributions for the spectral radii in general. When m is a fixed integer, we show that the spectral radii converge weakly to distributions of functions of independent Gamma random variables. When $m = m_n$ tends to infinity as n goes to infinity, we show that the logarithmic spectral radii have a normal limit.

Keywords: limiting distribution, spectral radius, spherical ensemble, product ensemble, random matrix

1 Introduction

In the last few decades, random matrix theory has expanded very quickly and found applications in many areas such as heavy-nuclei (Wigner, 1955), condensed matter physics (Beenakker, 1997), number theory (Mezzadri and Snaith, 2005), wireless communications (Couillet and Debbah, 2011), and high dimensional statistics (Johnstone (2001, 2008) and Jiang (2009)), just to mention a few. Interested readers are referred to the Oxford Handbook of Random Matrix Theory edited by Akemann, Baik and Francesco (2011) for more references and a wide range of applications in both mathematics and physics.

The study of the largest eigenvalues of Hermitian random matrices has been very active after the discovery of the so-called Tracy-Widom distributions. For the three Hermitian matrices including Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble, Tracy and Widom (1994, 1996) have proved that the largest eigenvalues converge in distribution to some distributions, now known as the Tracy-Widom laws. Later developments in this direction can be found in Baik et al. (1999), Tracy and Widom (2002), Johansson (2007), Johnstone (2001, 2008) and Jiang (2009), and Ramírez et al. (2011).

The study of non-Hermitian matrices, initiated by Ginibre (1965) for Gaussian random matrices, has attracted much attention as well, and applications are found in areas such as quantum chromodynamics, chaotic quantum systems and growth processes; see, e.g., Akemann, Baik and Francesco (2011) for more descriptions. For non-Hermitian matrices, the largest absolute values of their eigenvalues are refereed to as the spectral radii. Rider (2003, 2004) and Rider and Sinclair (2014) consider the real, complex and symplectic Ginibre ensembles. In particular, for the complex Ginibre ensemble, Rider (2003) shows that the spectral radius converges in distribution to the Gumbel distribution. Jiang and Qi (2017) investigate the limiting distributions for the spectral radii for the spherical ensemble, truncation of circular unitary ensemble and product of independent matrices with entries being independent complex standard normal random variables. These limiting distributions are no longer the Tracy-Widom laws. Gui and Qi (2018) further extend Jiang and Qi's (2017) result for the truncations of circular unitary ensemble. A common feature for all these random matrices is the intrinsic independence structure for the absolute values of their eigenvalues, which is shared by certain determinantal point processes; see e.g., Hough et al. (2009).

Let $m \geq 1$ be an integer and assume $\mathbf{X}_1, \cdots, \mathbf{X}_m$ are m independent and identically distributed (i.i.d.) $n \times n$ random matrices. The product of the m matrices is an $n \times n$ random matrix, denoted by

$$\mathbf{X}^{(m)} = \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_m. \tag{1.1}$$

The product of random matrices have been applied in wireless telecommunication, disordered spin chain, the stability of large complex system, quantum transport in disordered wires, among others. See Ipsen (2015) for a survey of applications.

Some recent interests focus on the study of the limiting properties of the product ensemble $\mathbf{X}^{(m)}$, including the limit of the empirical spectral distributions and the spectral radii. For example, Götze and Tikhomirov (2010), Bordenave (2011), O'Rourke and Soshnikov (2011) and O'Rourke *et al.* (2015) have investigated the limiting empirical spectral distribution for the product from the complex Ginibre ensemble when m is fixed, Götze, Kösters and Tikhomirov (2015) and Zeng (2016) have obtained the limits of the empirical spectral distribution for the product from the spherical ensemble when m is fixed, and Chang and Qi (2017) obtain the the limit of the empirical distributions based on scaled eigenvalues when $m = m_n$ changes with n. The universality of convergence for the empirical spectral distribution is also obtained by Bordenave (2011) and Götze, Kösters and Tikhomirov (2015) when m is a fixed integer.

When the n^2 entries of $\mathbf{X}_1$ are i.i.d. complex standard normal random variables, the limiting distribution for the spectral radii of $\mathbf{X}^{(m_n)}$ depends on the limits of m_n/n. Three different types of limiting distributions are obtained in Jiang and Qi (2017) when $\lim_{n\to\infty} m_n/n = 0$, $\lim_{n\to\infty} m_n/n = \alpha \in (0, \infty)$, and $\lim_{n\to\infty} m_n/n = \infty$.

Assume that $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ random matrices and all of the $2n^2$ entries of the matrices are i.i.d. standard complex normal random variables. A spherical ensemble is defined as $\mathbf{X} := \mathbf{A}^{-1}\mathbf{B}$; see e.g., Hough *et al.* (2009). Denote $\mathbf{z}_1, \cdots, \mathbf{z}_n$ as the eigenvalues of $\mathbf{X}$. Then it follows from Krishnapur (2009) that the joint probability density function of the n eigenvalues is given by

$$C_1 \cdot \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{k=1}^{n} \frac{1}{(1+|z_k|^2)^{n+1}}, \tag{1.2}$$

where C_1 is a normalizing constant.

In this paper, we consider the product of m independent matrices from the spherical ensemble. We are interested in the limiting distributions of the spectral radii for the product ensemble $\mathbf{X}^{(m)}$ when n goes to infinity. We also allow that $m = m_n$ changes with n.

Let $\mathbf{X}_1, \cdots, \mathbf{X}_m$ be m independent and identically distributed $n \times n$ random matrices that have the same distribution as $\mathbf{X}$ defined above. The product ensemble $\mathbf{X}^{(m)}$ is defined as in (1.1). Then we have from Adhikari *et al.* (2016) that the n eigenvalues $\mathbf{z}_1, \cdots, \mathbf{z}_n$ of $\mathbf{X}^{(m)}$ have a joint probability density function

$$C_m \cdot \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{k=1}^{n} w_m(z_k), \tag{1.3}$$

where C_m is a normalizing constant and $w_m(z)$ can be expressed in terms of a Meijer G-function. A recursive formula for w_m is obtained by Zeng (2016) as follows

$$w_{k+1}(z) = 2\pi \int_0^\infty w_k(\frac{z}{r}) \frac{1}{(1+r^2)^{n+1}} \frac{d\,r}{r}$$

for $k \ge 1$ with initial $w_1(z) = \dfrac{1}{(1+|z|^2)^{n+1}}$. Clearly, (1.2) is a special case of (1.3) when $m = 1$.

When $m = 1$, the limiting distribution has been obtained in Jiang and Qi (2017). In this paper, our objective is to obtain the limiting distributions for the spectral radii for the product ensemble $\mathbf{X}^{(m)}$ in the following two cases: (a) $m \ge 1$ is a fixed integer, and (b) $m = m_n$ tends to infinity as n goes to infinity. We will show that the limiting distributions of the spectral radii can be expressed as the distributions of functions of independent Gamma random variables when m is fixed, and the limiting distributions for the logarithmic spectral radii are normal when $m = m_n$ diverges as $n \to \infty$.

The rest of the paper is organized as follows. The main results of the paper are introduced in Section 2, and their proofs are given in Section 3.

2 Main Results

We assume that the product $\mathbf{X}^{(m)}$ defined in (1.1) is the product of m i.i.d. random matrices from the spherical ensemble. Note that the eigenvalues $\mathbf{z}_1, \cdots, \mathbf{z}_n$ of $\mathbf{X}^{(m)}$ are complex random variables with the joint density distribution function given in (1.3). The spectral radius of $\mathbf{X}^{(m)}$ is defined as

$$M_n = \max_{1\le j\le n} |\mathbf{z}_j|. \tag{2.1}$$

Let $\{E_{ijk},\ i \ge 1,\ j \ge 1,\ k \ge 1\}$ be i.i.d random variables with standard exponential distribution (ie., Gamma(1) distribution). Set $\Gamma_{ij}[k_1 : k_2] = \sum_{k=k_1}^{k_2} E_{ijk}$ for any $k_2 \ge k_1 \ge 1$, $i \ge 1$, $j \ge 1$, and denote $\Gamma_{ij} = \Gamma_{ij}[1 : i] = \sum_{k=1}^{i} E_{ijk}$ for $i \ge 1,\ j \ge 1$. Then $\Gamma_{ij}, j \ge 1, i \ge 1$ are independent random variables and Γ_{ij} has a Gamma(i) distribution with density function $x^{i-1}e^{-x}I(x > 0)/\Gamma(i)$, where $I(A)$ denotes the indicator function of set A, and $\Gamma(x)$ denotes the Gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

We have the following two theorems on the limiting distributions of the spectral radius M_n for the product ensemble $\mathbf{X}^{(m)}$. The two theorems reveal two different types of limiting distributions according to whether m is fixed or divergent.

Theorem 2.1. *Assume $m \ge 1$ is a fixed integer. Then*

$$\frac{M_n}{n^{m/2}} \xrightarrow{d} \max_{1\le i<\infty} \frac{1}{\prod_{j=1}^m \Gamma_{ij}^{1/2}} \quad \text{as } n \to \infty, \tag{2.2}$$

where $\xrightarrow{d}$ *denotes convergence in distribution.*

Theorem 2.2. *Assume that* $m = m_n \to \infty$ *as* $n \to \infty$. *Then we have*

$$\frac{\log M_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty, \tag{2.3}$$

where $\mu_n = \frac{m_n}{2} \sum_{k=1}^{n-1} \frac{1}{k}$ *and* $\sigma_n^2 = m_n \pi^2/24$.

Remark 1. The limiting distributions are expressed in terms of functions of independent Gamma random variables in Theorem 2.1. The random variable on the right-hand side of (2.2) is well defined. See Lemma 3.3 for a proof.

Remark 2. There is no explicit form for the distribution of the random variable defined on the right-hand side of (2.2) except the case $m = 1$. In fact, if we define $H_i(x) = e^{-x} \sum_{j=0}^{i-1} \frac{x^j}{j!}$ for $i \geq 1$, then for any $x > 0$

$$P(\frac{1}{\Gamma_{i1}^{1/2}} \leq x) = P(\Gamma_{i1} \geq x^{-2}) = H_i(x^{-2}) \quad \text{for } i \geq 1,$$

and consequently, the distribution of the random variable on the right-hand side of (2.2) when $m = 1$ is

$$H(x) = P(\max_{1 \leq i < \infty} \frac{1}{\Gamma_{i1}^{1/2}} \leq x) = P(\max_{1 \leq i < \infty} \frac{1}{\Gamma_{i1}^{1/2}} \leq x) = \prod_{i=1}^{\infty} H_i(x^{-2}), \quad x > 0.$$

This is exactly what Jiang and Qi (2017) have obtained in their Theorem 1. Meanwhile, they have verified that $1 - H(x) \sim \frac{1}{x^2}$ as $x \to \infty$, and therefore, $H(x)$ is a heavy-tailed distribution.

Remark 3. In Theorem 2.2, the limiting distributions are obtained for logarithmic spectral radius $\log M_n$. It is possible to show that there do not exist real constants a_n and $b_n > 0$ such that $(M_n - a_n)/b_n$ converges in distribution to a non-degenerate distribution function.

3 Proofs

First, we will introduce some notation, and then present some important lemmas. The proofs of the two main results are given afterwards.

Let $\stackrel{d}{=}$ and $\stackrel{p}{\to}$ denote equality in distribution and convergence in probability. For a sequence of random variables X_n, $n \geq 1$ and any sequence of positive constants a_n, $n \geq 1$, notation $X_n = o_p(a_n)$ means $X_n/a_n \stackrel{d}{\to} 0$ as $n \to \infty$. Notation $X_n = O_p(a_n)$ implies that $\lim_{c\to\infty} \limsup_{n\to\infty} P(|X_n/a_n| > c) = 0$. In particular, if X_n/a_n converges in distribution, then we have $X_n = O_p(a_n)$.

Let $U_1, \cdots, U_n$ be independent random variables uniformly distributed over $(0, 1)$ and define $U_{(1)} \leq \cdots \leq U_{(n)}$ as the order statistics of $U_1, \cdots, U_n$.

Assume that $\{s_{j,r},\ 1 \leq r \leq m,\ 1 \leq j \leq n\}$ are independent random variables, and the density of $s_{j,r}$ is proportional to $\frac{y^{j-1}}{(1+y)^{n+1}} I(y > 0)$ for $1 \leq r \leq m$, $1 \leq j \leq n$.

Recall that $\Gamma(x)$ denotes the Gamma function. Write $\psi(x) = \Gamma'(x)/\Gamma(x)$, $x > 0$, which is called the digamma function. Since $\psi(x) = \frac{d}{dx} \log \Gamma(x)$, we have

$$\frac{\Gamma(b)}{\Gamma(a)} = \exp\Big(\log \Gamma(b) - \log \Gamma(a)\Big) = \exp\left(\int_a^b \psi(x)dx\right) \quad \text{for } a > 0,\ b > 0. \tag{3.1}$$

Lemma 3.1. *Let random variable Y have a Gamma(α) distribution and $X = \log(Y)$. Then the moment generating function of X is given by*

$$\beta(t) := \mathbb{E}(e^{tX}) = \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)} \quad \textit{for } t > -\alpha.$$

Moreover, $\mathbb{E}(X) = \psi(\alpha)$ and $Var(X) = \psi'(\alpha)$.

Proof. Note $\beta(t) = \mathbb{E}(Y^t)$. We have for $t > -\alpha$

$$\beta(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^t y^{\alpha-1} e^{-y} dy = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+t-1} e^{-y} dy = \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)}.$$

Then $\mathbb{E}(X) = \beta'(0) = \Gamma'(\alpha)/\Gamma(\alpha) = \psi(\alpha)$. Further, we have $\mathbb{E}(X^2) = \beta''(0) = \Gamma''(\alpha)/\Gamma(\alpha)$. Hence we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \psi^2(\alpha) = \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)^2 = \frac{d}{dt}\frac{\Gamma'(t)}{\Gamma(t)}\Big|_{t=\alpha} = \psi'(\alpha).$$

This completes the proof of the lemma. ■

Next, we collect some properties of the bigamma function $\psi(x)$.

Lemma 3.2. *For the bigamma function $\psi(x)$ we have*

a. (Formulas 6.3.18 in Abramowitz and Stegun (1972))

$$\psi(x) = \log x - \frac{1}{2x} + O\Big(\frac{1}{x^2}\Big) \quad \text{as} \quad x \to \infty. \tag{3.2}$$

b. (Formula 6.3.2 in Abramowitz and Stegun (1972))

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad \text{for } n \geq 2,$$

where $\gamma = 0.57721\cdots$ is the Euler constant.

c. (Formula 6.4.10 in Abramowitz and Stegun (1972))

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2}, \qquad x > 0.$$

From Lemmas 3.2 and 3.1 we have

$$\psi'(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \psi(n) - \psi(1) = \sum_{k=1}^{n-1} \frac{1}{k}$$

and

$$\mathbb{E}(\log \Gamma_{ij}) = \psi(i), \qquad \mathrm{Var}(\log \Gamma_{ij}) = \psi'(i).$$

Therefore, the constants μ_n and σ_n^2 in Theorem 2.2 can be rewritten as

$$\mu_n = m_n(\psi(n) - \psi(1)) \quad \text{and} \quad \sigma_n^2 = m_n \pi^2/24 = \frac{m_n \psi'(1)}{4}.$$

Lemma 3.3. *For each fixed integer $m \geq 1$, the random variable*

$$M := \lim_{n\to\infty} \max_{1 \leq i \leq n} \frac{1}{\prod_{j=1}^m \Gamma_{ij}^{1/2}}$$

is well defined, and $P(M < \infty) = 1$.

Proof. Since $\max_{1\le i\le n}\frac{1}{\prod_{j=1}^m \Gamma_{ij}^{1/2}}$ is non-decreasing in n with probability one, the limit M exists and $M>0$. Note that

$$M^4=\max_{1\le i<\infty}\frac{1}{\prod_{j=1}^m \Gamma_{ij}^2}\le\sum_{i=1}^{\infty}\frac{1}{\prod_{j=1}^m \Gamma_{ij}^2}=\sum_{i=1}^{\infty}\prod_{j=1}^m\frac{1}{\Gamma_{ij}^2}. \tag{3.3}$$

It follows from Lemma 3.2 that $\int_{x-2}^x \psi(t)dt\ge 1.5\log x$ for all large $x\ge i_0$ for some integer $i_0\ge 3$. Therefore, it follows from Lemma 3.1 and equation (3.1) that for $i\ge i_0$

$$\mathbb{E}(\frac{1}{\Gamma_{ij}^2})=\frac{\Gamma(i-2)}{\Gamma(i)}=\exp\Big(-\int_{i-2}^i\psi(t)dt\Big)\le\exp(-1.5\log(i))=i^{-1.5}.$$

By using the independence of Γ_{ij} we have

$$\mathbb{E}(\sum_{i=i_0}^{\infty}\prod_{j=1}^m\frac{1}{\Gamma_{ij}^2})=\sum_{i=i_0}^{\infty}\mathbb{E}(\prod_{j=1}^m\frac{1}{\Gamma_{ij}^2})\le\sum_{i=i_0}^{\infty}\prod_{j=1}^m\mathbb{E}(\frac{1}{\Gamma_{ij}^2})\le\sum_{i=i_0}^{\infty}\prod_{j=1}^m\mathbb{E}(\frac{1}{\Gamma_{ij}^2})\le\sum_{i=i_0}^{\infty}i^{-1.5m}<\infty,$$

and hence, $P(\sum_{i=i_0}^{\infty}\prod_{j=1}^m\frac{1}{\Gamma_{ij}^2})<\infty)=1$, which together with (3.3) implies $P(M<\infty)=1$. This completes the proof of the lemma. ■

Lemma 3.4. *$g(|\mathbf{z}_1|^2,\cdots,|\mathbf{z}_n|^2)$ and $g(\prod_{j=1}^m s_{1,j},\cdots,\prod_{j=1}^m s_{n,j})$ have the same distribution function for any symmetric function $g(x_1,\cdots,x_n)$.*

See Lemma 2.1 in Zeng (2006).

Lemma 3.5. *For $1\le j\le m, 1\le i\le n$, $s_{i,j}$ and $\frac{U_{(i)}}{1-U_{(i)}}$ are identically distributed.*

See the proof of Lemma 2.3 in Zeng (2016).

Lemma 3.6. *$(U_{(1)},U_{(2)},\cdots,U_{(n)})$ and $(\frac{S_1}{S_{n+1}},\frac{S_2}{S_{n+1}},\cdots,\frac{S_n}{S_{n+1}})$ have the same joint distribution, where $S_k=\sum_{j=1}^k E_j$ for $k\ge 1$, and $E_j,\ j\ge 1$ are independent random variables with the standard exponential distribution*

See, e.g., equation (2.2.1) on page 12 in Ahsanullah and Nevzorov (2015).

Lemma 3.7. *For any fixed $m\ge 1$*

$$\max_{1\le i\le n,1\le j\le m}|\frac{\Gamma_{ij}[2:(n+1)]}{n}-1|\to 0 \qquad \textit{with probability one}$$

as $n\to\infty$.

Proof. For any $i, j \geq 1$, $\Gamma_{ij}[2:(n+1)]$ is the sum of n i.i.d. random variables with a Gamma(1) distribution, we have

$$P(\max_{1\leq i\leq n, 1\leq j\leq m} |\frac{\Gamma_{ij}[2:(n+1)]}{n} - 1| > \varepsilon) \leq mnP(|\frac{\Gamma_{11}[2:(n+1)]}{n} - 1| > \varepsilon)$$

for any $\varepsilon > 0$. Since $\Gamma_{ij}[2:(n+1)]$ is a partial sum from a sequence of i.i.d. standard exponential random variables with $\mathbb{E}(\Gamma_{111}) = 1$ and $\mathbb{E}(\Gamma_{111}^3) = 6 < \infty$, we have from Theorem 3 in Baum and Katz (1963) that

$$\sum_{n=1}^{\infty} nP(|\frac{\Gamma_{11}[2:(n+1)]}{n} - 1| > \varepsilon) < \infty \quad \text{for any } \varepsilon > 0,$$

which implies that

$$\sum_{n=1}^{\infty} P(\max_{1\leq i\leq n, 1\leq j\leq m} |\frac{\Gamma_{ij}[2:(n+1)]}{n} - 1| > \varepsilon) < \infty \quad \text{for any } \varepsilon > 0.$$

Then the lemma follows from Borel-Cantelli lemma. ■

By setting $g(x_1, \cdots, x_n) = \max_{1\leq i\leq n} x_i$ in Lemma 3.4 we have that $M_n^2 = \max_{1\leq i\leq n} |\mathbf{z}_i|^2$ and $\max_{1\leq i\leq n} \prod_{j=1}^m s_{i,j} = \max_{1\leq i\leq n} \prod_{j=1}^m s_{n+1-i,j}$ have the same distribution.

From Lemma 3.6, $\Gamma_{ij}/\Gamma_{ij}[1:(n+1)] = \Gamma_{ij}[1:i]/\Gamma_{ij}[1:(n+1)]$ is identically distributed as $U_{(i)}$. Since $1 - U_{(i)}$ has the same distribution as $U_{(n+1-i)}$, we have

$$1 - \frac{\Gamma_{ij}}{\Gamma_{ij}[1:(n+1)]} = \frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}[1:(n+1)]}$$

has the same distribution as $U_{(n+1-i)}$. Then it follows from Lemma 3.5 that

$$\frac{\frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}[1:(n+1)]}}{1 - (1 - \frac{\Gamma_{ij}}{\Gamma_{ij}[1:(n+1)]})} = \frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}$$

has the same distribution as $s_{n+1-i,j}$ for any $j \geq 1$. Note that $\frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}$, $i \geq 1$, $j \geq 1$ are independent random variables. Therefore, $\prod_{j=1}^m \frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}$ has the same distribution as $\prod_{j=1}^m s_{n-i+1,j}$, and $\max_{1\leq i\leq n} |\mathbf{z}_i|^2$ and $\max_{1\leq i\leq n} \prod_{j=1}^m \frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}$ have the same distribution. This implies

$$M_n \overset{d}{=} \max_{1\leq i\leq n} \prod_{j=1}^m \sqrt{\frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}}. \tag{3.4}$$

Now we define

$$V_i = \prod_{j=1}^{m} \sqrt{\frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}}}$$

for $1 \le i \le n$. Then we have for any $i \in \{1, \cdots, n-1\}$

$$V_i = \prod_{j=1}^{m} \sqrt{\frac{\Gamma_{ij}[(i+1):(n+1)]}{\Gamma_{ij}[1:i]}} \ge \prod_{j=1}^{m} \sqrt{\frac{\Gamma_{ij}[(i+2):(n+1)]}{\Gamma_{ij}[1:(i+1)]}} \stackrel{d}{=} V_{i+1},$$

which implies

$$P(V_i \le x) \text{ is non-decreasing in } i \in \{1, \cdots, n\} \tag{3.5}$$

for any $x \in \mathbb{R}$.

Proof of Theorem 2.1. It follows from Lemma 3.7 that as $n \to \infty$

$$R_n := \max_{1\le j\le m} \max_{1\le i\le n} \frac{\Gamma_{ij}[2:(n+1)]}{n} \to 1, \quad r_n := \min_{1\le j\le m} \min_{1\le i\le n} \frac{\Gamma_{ij}[2:(n+1)]}{n} \to 1 \tag{3.6}$$

with probability one.

Define for $r \ge 1$

$$W_r = \max_{1\le i\le r} \left(\prod_{j=1}^{m} \sqrt{\frac{\Gamma_{ij}[(i+1):(n+1)]}{n}} \prod_{j=1}^{m} \frac{1}{\Gamma_{ij}^{1/2}} \right).$$

and set $Z_r = \max_{1\le i\le r} \prod_{j=1}^{m} \frac{1}{\Gamma_{ij}^{1/2}}$. Then we have from (3.4) that $M_n/n^{m/2} \stackrel{d}{=} W_n$. To show the theorem, it suffices to prove that $W_n \to M$ with probability one. Let $k \ge 1$ be any fixed integer. Then we have

$$W_n \le \max_{1\le i\le n} \prod_{j=1}^{m} \sqrt{\frac{\Gamma_{ij}[2:(n+1)]}{n}} Z_n \le R_n^{m/2} M,$$

which together with (3.6) yields

$$\limsup_{n\to\infty} W_n \le M.$$

For any fixed $k \ge 2$, we have for all large n

$$W_n \ge W_k \ge \left(r_n - \frac{\max_{1\le i\le k} \max_{1\le j\le m} \Gamma_{ij}[2:k]}{n} \right)^{m/2} Z_k.$$

Again, in view of (3.6) we have that $\liminf_{n\to\infty} W_n \geq Z_k$ with probability one. Hence, by letting $k \to \infty$, and using Lemma 3.3 we get that $\liminf_{n\to\infty} W_n \geq M$. Therefore, we conclude that $\liminf_{n\to\infty} W_n = \limsup_{n\to\infty} W_n = M$ with probability one. ■

Proof of Theorem 2.2. In view of (3.4) we have

$$P(\log M_n \leq \mu_n + \sigma_n x) = \prod_{i=1}^{n} a_{ni}(x) \tag{3.7}$$

for every $x \in \mathbb{R}$, where $a_{ni}(x) = P(\log V_i \leq \mu_n + \sigma_n x)$ for $1 \leq i \leq n$. Moreover, it follows from (3.5) that for each $x \in \mathbb{R}$,

$$a_{ni}(x) \text{ is non-decreasing in } i \in \{1, \cdots, n\}. \tag{3.8}$$

Our goal is to show that the limit on the right-hand side of (3.7) is $\Phi(x)$, which is defined as the cumulative distribution of a standard normal random variable. It suffices to show that

$$\lim_{n\to\infty} a_{n1}(x) = \Phi(x) \tag{3.9}$$

and

$$\lim_{n\to\infty} \prod_{i=2}^{n} a_{ni}(x) = 1 \tag{3.10}$$

for every $x \in \mathbb{R}$.

Note that (3.9) is equivalent to

$$\frac{\log V_1 - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1). \tag{3.11}$$

For each $i \geq 1$, $\log(\Gamma_{ij})$, $j = 1, \cdots, m_n$ are i.i.d. random variables with mean $\psi(i)$ and variance $\psi'(i)$. Then we have

$$\frac{\sum_{j=1}^{m_n} \log(\Gamma_{ij}) - m_n\psi(i)}{\sqrt{m_n \psi'(i)}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty \tag{3.12}$$

by the classic central limit theorem, and as $n \to \infty$

$$\frac{1}{\sqrt{m_n}} \left(\sum_{j=1}^{m_n} \log(\Gamma_{ij}[(i+1):(n+1)]) - m_n\psi(n+1-i) \right) \xrightarrow{p} 0 \tag{3.13}$$

since

$$
\begin{aligned}
& \mathbb{E}\left(\frac{1}{\sqrt{m_n}}(\sum_{j=1}^{m_n}\log(\Gamma_{ij}[(i+1):(n+1)]) - m_n\psi(n+1-i))\right)^2 \\
= & \frac{1}{m_n}\mathrm{Var}(\sum_{j=1}^{m_n}\log(\Gamma_{ij}[(i+1):(n+1)])) \\
= & \frac{1}{m_n}m_n\psi'(n+1-i) \\
= & O(\frac{1}{n}) \\
\to & 0 \quad \text{as } n\to\infty
\end{aligned}
$$

from Lemma 3.2 (c).

Note that for $1 \le i \le n$

$$
\log V_i = \frac{1}{2}\left(\sum_{j=1}^{m_n}\log(\Gamma_{ij}[(i+1):(n+1)]) - \sum_{j=1}^{m_n}\log(\Gamma_{ij})\right). \tag{3.14}
$$

For $i=1$, we have from (3.12) and (3.13)

$$
\begin{aligned}
& \frac{\log V_1 - \mu_n}{\sigma_n} \\
= & \frac{\sum_{j=1}^{m_n}\log(\Gamma_{1j}[2:(n+1)]) - m_n\psi(n)}{\sqrt{m_n\psi'(1)}} - \frac{\sum_{j=1}^{m_n}\log(\Gamma_{1j}) - m_n\psi(1)}{\sqrt{m_n\psi'(1)}} \\
= & -\frac{\sum_{j=1}^{m_n}\log(\Gamma_{1j}) - m_n\psi(1)}{\sqrt{m_n\psi'(1)}} + o_p(1) \\
\xrightarrow{d} & N(0,1),
\end{aligned}
$$

proving (3.11).

For $i = 2$, by using (3.12) and (3.13) and Lemma 3.2 (b) we get

$$\begin{aligned}
&\frac{\log V_2 - \mu_n}{\sigma_n} \\
= \quad & \frac{\sum_{j=1}^{m_n} \log(\Gamma_{2j}[3:(n+1)]) - m_n\psi(n-1)}{\sqrt{m_n\psi'(1)}} - \frac{\sum_{j=1}^{m_n} \log(\Gamma_{2j}) - m_n\psi(2)}{\sqrt{m_n\psi'(2)}}\sqrt{\frac{\psi'(2)}{\psi'(1)}} \\
& - \frac{m_n(\psi(2) - \psi(1) + \psi(n-1) - \psi(n))}{\sqrt{m_n\psi'(1)}} \\
= \quad & -\frac{\sqrt{m_n}(1 - \frac{1}{n-1})}{\sqrt{\psi'(1)}} + O_p(1) \\
\xrightarrow{p} \quad & -\infty,
\end{aligned}$$

which implies $1 - a_{n2}(x) = P(\log V_2 > \mu_n + \sigma_n x) \to 0$ as $n \to \infty$ for any $x \in \mathbb{R}$. Hence, we conclude from (3.8) that $\max_{2 \le i \le n}(1 - a_{ni}(x)) = 1 - a_{n2}(x) \to 0$ as $n \to \infty$.

To show (3.10), it suffices to show

$$\lim_{n\to\infty} \sum_{i=2}^{n} (1 - a_{ni}(x)) = 0 \tag{3.15}$$

since $1 - \sum_{i=2}^{n}(1 - a_{ni}(x)) \le \prod_{i=1}^{n}(1 - (1 - a_{ni}(x))) = \prod_{i=2}^{n} a_{ni}(x) \le 1$.

By applying inequality $P(X > 0) \le \mathbb{E}(e^X)$ and noting that all summands on the right-hand side of (3.14) are independent, we have from Lemma 3.1 and (3.1) that

$$\begin{aligned}
&1 - a_{ni}(x) \\
= \quad & P(\sum_{j=1}^{m_n} \log(\Gamma_{ij}[(i+1):(n+1)]) - \sum_{j=1}^{m_n} \log(\Gamma_{ij}) - m_n(\psi(n) - \psi(1)) - \sqrt{m_n\psi'(1)}x > 0) \\
\le \quad & \mathbb{E}\exp(\sum_{j=1}^{m_n} \log(\Gamma_{ij}[(i+1):(n+1)]) - \sum_{j=1}^{m_n} \log(\Gamma_{ij}) - m_n(\psi(n) - \psi(1)) - \sqrt{m_n}x) \\
= \quad & \Big(\frac{\Gamma(n+2-i)}{\Gamma(n+1-i)}\Big)^{m_n} \Big(\frac{\Gamma(i-1)}{\Gamma(i)}\Big)^{m_n} \exp\Big(-m_n(\psi(n) - \psi(1)) - \sqrt{m_n}x\Big) \\
= \quad & \exp(m_n \int_0^1 (\psi(n+1-i+t) - \psi(i-1+t))dt - m_n(\psi(n) - \psi(1)) - \sqrt{m_n}x) \\
= \quad & \exp(m_n \int_0^1 (\psi(n+1-i+t) - \psi(n) - (\psi(i-1+t) - \psi(1)))dt - \sqrt{m_n}x).
\end{aligned}$$

We have $\psi(x)$ is increasing in $x > 0$ since $\psi'(x) > 0$ from Lemma 3.2. Therefore, $\psi(n+1-i+t) - \psi(n) \leq 0$ and $\psi(i-1+t) - \psi(1) \geq \psi(i-1) - \psi(1)$ for $t \in [0,1]$ and $i \geq 3$. Thus, we have

$$1 - a_{ni}(x) \leq \exp(-m_n(\psi(i-1) - \psi(1))) = \exp(-m_n \sum_{k=1}^{i-2} \frac{1}{k} + \sqrt{m_n \psi'(1)}x)$$

for $3 \leq i \leq n$. Now we choose a positive integer $i_0 \geq 3$ such that $\frac{1}{2}\log(i_0 - 1) \geq \sqrt{\psi'(1)}|x|$. Since $\sum_{k=1}^{i-2} \frac{1}{k} \geq \log(i-1)$, we get

$$1 - a_{ni}(x) \leq \exp(-\frac{m_n}{2}\log(i-1)) = (i-1)^{-m_n/2}, \quad i_0 \leq i \leq n$$

and

$$\sum_{i=i_0}^{n}(1 - a_{ni}(x)) \leq \sum_{i=i_0}^{n}(i-1)^{-\frac{m_n}{2}} \leq \int_{i_0-2}^{n-1} \frac{1}{x^{m_n/2}} dx < \frac{2}{m_n - 2}(i_0 - 2)^{1-\frac{m_n}{2}}$$

which converges to zero as $n \to \infty$. Consequently, we have as $n \to \infty$

$$\sum_{i=2}^{n}(1 - a_{ni}(x)) \leq (i_0 - 2)(1 - a_{n2}(x)) + \sum_{i=i_0}^{n}(1 - a_{ni}(x)) \to 0,$$

which proves (3.15). This completes the proof of the theorem. ■

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