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The unit ball of an injective operator space has an extreme point

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ABSTRACT

We define an AW^*-TRO as an off-diagonal corner of an AW^*-algebra, and show that the unit ball of an AW^*-TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] affirmatively. We also show that an AW^*-TRO (respectively, an injective operator space) has an ideal decomposition, that is, it can be decomposed into the direct sum of a left ideal, a right ideal, and a two-sided ideal in an AW^*-algebra (respectively, an injective C^*-algebra). In particular, we observe that an AW^*-TRO, hence an injective operator space, has an algebrization which admits a quasi-identity.
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Recall that an operator space X is called a *triple system* or a *ternary ring of operators* (*TRO* for short) if there exists a complete isometry ι from X into a C^*-algebra such that $\iota(x)\iota(y)^*\iota(z) \in \iota(X)$ for all $x, y, z \in X$. A theorem of Ruan and Hamana (independently) states that an operator space X is injective if and only if it is an off-diagonal corner of an injective C^*-algebra, i.e., there exist an injective C^*-algebra $\mathcal{A}$ and projections $p, q \in \mathcal{A}$ (meaning $p = p^2 = p^*$ and $q = q^2 = q^*$) such that X is completely isometric to $p\mathcal{A}q$ (Theorem 4.5 in [14] and Theorem 3.2 (i) in [2]). In particular, an injective operator space is a TRO. Noting that an injective C^*-algebra is monotone complete and hence an AW^*-algebra, the Ruan–Hamana theorem motivates the following definition. (The reader is referred to [15] for a modern account of and recent progress in monotone complete C^*-algebras and AW^*-algebras.)

Definition 1. We say that an operator space X is an AW^***-TRO** if there exist an AW^*-algebra $\mathcal{A}$ and projections $p, q \in \mathcal{A}$ such that X is completely isometric to $p\mathcal{A}q$.

Remark 2.

(1) While a different definition of an AW^*-TRO was given in [12] (Definition 6.2.1), where the definition is that the linking C^*-algebra is AW^*, we think our definition is the right one. For instance, a countably-

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infinite-dimensional column Hilbert space is an injective operator space ([13]) and hence a TRO, but its linking C^*-algebra is not unital (and hence not AW^*), while the space is an AW^*-TRO using our definition. Note also that a W^*-TRO can have a linking C^*-algebra which is not a W^*-algebra.

(2) With our definition of an AW^*-TRO, all Theorems, Corollaries, and Lemmas (excluding the second assertion of Lemma 6.2.9) in Section 6.2 of [12] remain to be valid replacing TT^* by $p\mathcal{A}p$, T^*T by $q\mathcal{A}q$, and $\mathscr{L}_T = \begin{pmatrix} TT^* & T \\ T^* & T^*T \end{pmatrix}$ by $\mathscr{L}_T = \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}$ $(\subseteq M_2(\mathcal{A}))$, where $\mathcal{A}$ is an AW^*-algebra, and $p, q \in \mathcal{A}$ are projections, and T is an AW^*-TRO identified with $p\mathcal{A}q$. In addition, instead of Definition 6.3.1 in [12], if we define T to be of *type I*, *type II*, or *type III* if $\mathcal{A}$ can be chosen to be of type I, type II, or type III, respectively, then all Theorems and Lemmas (excluding item 3 of Lemma 6.3.5) in Section 6.3 of [12] also remain to be valid.

Theorem 3. *The unit ball (always assumed to be norm-closed) of an AW^*-TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] (Question 2) affirmatively.*

Proof. Let X be an AW^*-TRO. We may assume that $X = p\mathcal{A}q$, where $\mathcal{A}$ is an AW^*-algebra and $p, q \in \mathcal{A}$ are projections. By the comparison theorem in [3], there exist unique central projections $r, t, l \in \mathcal{A}$ satisfying $r + t + l = 1$ such that $rp \prec rq$, $tp \sim tq$, and $lp \succ lq$. (Here $rp \prec rq$ means $rp \preceq rq$ but $rp \nsim rq$, however, $0 \prec 0$ is allowed.) That is, there exist partial isometries $u, v, w \in \mathcal{A}$ such that $uu^* = rp$, $u^*u \leq rq$, $vv^* = tp$, $v^*v = tq$, $ww^* \leq lp$, and $w^*w = lq$. Let $e := u + v + w\,(\in p\mathcal{A}q)$, then it is easy to check that $(p - ee^*)\mathcal{A}(q - e^*e) = \{0\}$. Thus by a variation of Kadison's theorem (Theorem 1 in [4]; see Proposition 1.4.8 in [11] or Proposition 1.6.5 in [16] for the variation we need here), e is an extreme point of the unit ball of $p\mathcal{A}q$. □

From the proof above we obtain "ideal decompositions" for AW^*-TROs and injective operator spaces similar to the ones done for TROs with predual in [7]. The technique we use here is to embed an off-diagonal corner into the diagonal corners which is a modification of the technique developed in [1] and is employed in [7].

Corollary 4. *An AW^*-TRO (respectively, an injective operator space) can be decomposed into the direct sum of TROs X_T, X_L, and X_R:*

$$X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$$

so that there is a complete isometry ι from X into an AW^*-algebra (respectively, an injective C^*-algebra) in which $\iota(X_T)$, $\iota(X_L)$, and $\iota(X_R)$ are a two-sided, left, and right ideal, respectively, and

$$\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R).$$

Proof. Let X be an AW^*-TRO, and assume that $X = p\mathcal{A}q$, where $\mathcal{A}$ is an AW^*-algebra and $p, q \in \mathcal{A}$ are projections. Let $r, t, l \in p\mathcal{A}q$ as in the proof of Theorem 3, and put $X_T := tX$, $X_L := lX$, and $X_R := rX$, then $X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$. Let $\mathcal{B} := p\mathcal{A}p \overset{\infty}{\oplus} q\mathcal{A}q$ which is an AW^*-algebra since $p\mathcal{A}p$ and $q\mathcal{A}q$ are so by Theorem 2.4 in [10]. For each $x \in X$, let $x_T := tx$, $x_L := lx$, and $x_R := rx$, and define a mapping $\iota : X \to \mathcal{B}$ by $\iota(x) := (x_T + x_L)e^* \oplus e^*x_R$, where e is as in the proof of Theorem 3. Then clearly $\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R)$. We claim that ι is a complete isometry. $\|\iota(x)\| = \max\{\|(x_T + x_L)e^*\|, \|e^*x_R\|\} = \max\{\|(x_T + x_L)e^*e(x_T + x_L)^*\|^{1/2}, \|x_R^*ee^*x_R\|^{1/2}\} = \max\{\|x_Tv^*vx_T^* + x_Lw^*wx_L^*\|^{1/2}, \|x_R^*uu^*x_R\|^{1/2}\} = \max\{\|xtx^* + xlx^*\|^{1/2}, \|x^*rx\|^{1/2}\} = \max\{\|(t+l)x\|, \|rx\|\} = \|(t+l+r)x\| = \|x\|$, which shows that ι is an isometry. A similar calculation works at each matrix level, which concludes that ι is a complete isometry.

Clearly, $\iota(X_T)$, $\iota(X_L)$, and $\iota(X_R)$ are respectively a two-sided, left, and right ideals in $\mathcal{B}$, and thus we are done. The proof in the case that X is an injective operator space is identical noting that $\mathcal{B}$ is an injective C^*-algebra in this case. □

Remark 5.

(1) In the proof above it is also possible to define $\iota : X \to \mathcal{B}$ by $\iota(x) := x_L e^* \oplus e^*(x_R + x_T)$ for $x \in X$.
(2) The proof of the Theorem in [7] establishes the ideal decomposition with respect to "any" chosen extreme point of the unit ball of a TRO with predual. However, if one is not concerned with a particular extreme point, then the above argument provides a quick establishment of an ideal decomposition and a short proof of that theorem noting that a TRO X with predual can be considered as an off-diagonal corner of a von Neumann algebra $\mathcal{A}$ (see the beginning of the proof of the Theorem in [7]) and that $\mathcal{B}$ in the above proof is a von Neumann algebra in this case. The simplicity of this alternate proof is attributed to the use of the comparison theorem for projections.

The following corollary is straightforward from the corollary above. The reader is referred to [5], [6], or [9] for quasi-multipliers and algebrizations of operator spaces, and Definition 4.2 (i) in [8] for quasi-identities.

Corollary 6. *An AW*-TRO, hence an injective operator space, has an algebrization which admits a quasi-identity of norm* 1.

Proof. It is straightforward to check that $(v + w)e^* \oplus e^*u$ serves as a quasi-identity of $\iota(X)$ in the proof of Corollary 4. □

Remark 7. An element $e^* \in X^*$ in the proof of Corollary 4 can be identified as a quasi-multiplier of X noting that $X^* \subseteq \mathcal{QM}(X)$ if X is a TRO, where $\mathcal{QM}(X)$ is the quasi-multiplier space of X, and ι is the "algebrization" by e^*.

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