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The stochastic order of probability measures on ordered metric spaces

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ABSTRACT

The general notion of a stochastic ordering is that one probability distribution is smaller than a second one if the second attaches more probability to higher values than the first. Motivated by recent work on barycentric maps on spaces of probability measures on ordered Banach spaces, we introduce and study a stochastic order on the space of probability measures $\mathcal{P}(X)$, where X is a metric space equipped with a closed partial order, and derive several useful equivalent versions of the definition. We establish the antisymmetry and closedness of the stochastic order (and hence that it is a closed partial order) for the case of a partial order on a Banach space induced by a closed normal cone with interior. We also consider order-completeness of the stochastic order for a cone of a finite-dimensional Banach space and derive a version of the arithmetic-geometric-harmonic mean inequalities in the setting of the associated probability space on positive invertible operators on a Hilbert space.

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1. Introduction

The stochastic order for random variables X, Y from a probability measure space (M, P) to \mathbb{R} is defined by $X \leq Y$ if $P(X > t) \leq P(Y > t)$ for all $t \in \mathbb{R}$. This notion extends directly to random variables into \mathbb{R}^n equipped with the coordinatewise order. Alternatively one can define a stochastic order on the Borel probability measures on \mathbb{R}^n by $\mu \leq \nu$ if for each $s \in \mathbb{R}^n$, $\mu(s < t) \leq \nu(s < t)$, where $(s < t) := \{t \in \mathbb{R}^n : s < t\}$. One then has for random variables X, Y , $X \leq Y$ in the stochastic order if and only if $P_X \leq P_Y$, where P_X, P_Y are the push-forward probability measures with respect to X, Y respectively.

There are important metric spaces which are equipped with a naturally defined partial order, for example the open cone \mathbb{P}_n of positive definite matrices of some fixed dimension, where the order is the Loewner order.

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One can use the Loewner order to define an order on $\mathcal{P}(\mathbb{P}_n)$, the space of Borel probability measures, an order that we call the stochastic order, as it generalizes the case of \mathbb{R} or \mathbb{R}^n .

In this paper we broadly generalize the stochastic order to an order on the set $\mathcal{P}(X)$ of Borel probability measures on a partially ordered metric space (X, d, \leq) . We develop basic properties of this order and specialize to the setting of normal cones in Banach spaces to show that the stochastic order in that setting is indeed a partial order. A special example is the cone \mathbb{P} of positive invertible operators on a Hilbert space. The order on $\mathcal{P}(\mathbb{P})$ plays a crucial role in the study of means of operators that has recently been under active investigation.

The paper is organized as follows. Section 2 is a short preliminary on Borel measures. In Section 3 we give the general definition of the stochastic order on $\mathcal{P}(X)$ for a partially ordered metric space X and derive several useful alternative formulations. In Section 4 we show for normal cones with interior that the stochastic order on $\mathcal{P}(X)$ is indeed a partial order (the antisymmetry being the nontrivial property to establish). In Section 5 we show in the normal cone setting that the stochastic partial order is a closed order with respect to the weak topology, and hence with respect to the Wasserstein topology. In Section 6 we consider the order-completeness of $\mathcal{P}(X)$, and in Section 7 derive a version of the arithmetic-geometric-harmonic mean inequalities in the setting of the probability space $\mathcal{P}(\mathbb{P})$ on the positive invertible operators.

In what follows $\mathbb{R}^+ = [0, \infty)$.

2. Borel measures

In this section we recall some basic results about Borel measures on metric spaces that will be needed in what follows. As usual the Borel algebra on a metric space (X, d) is the smallest σ -algebra containing the open sets and a finite positive Borel measure is a countably additive measure μ defined on the Borel sets such that $\mu(X) < \infty$. We work exclusively with finite positive Borel measures, primarily those that are probability measures.

Recall that a Borel measure μ is τ -additive if $\mu(U) = \sup_{\alpha} \mu(U_{\alpha})$ for any directed union $U = \bigcup_{\alpha} U_{\alpha}$ of open sets. The measure μ is said to be *inner regular* or *tight* if for any Borel set A and $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $\mu(A) - \varepsilon < \mu(K)$. A tight finite Borel measure is also called a Radon measure.

Probability on metric spaces has been carried out primarily for separable metric spaces [3, Chapter 11], although results exist for the non-separable setting. We recall the following result, which can be more-or-less cobbled together from results in the literature; see [7] for more details.

Proposition 2.1. *A finite Borel measure μ on a metric space (X, d) has separable support. The following three conditions are equivalent:*

- (1) *The support of μ has measure $\mu(X)$.*
- (2) *The measure μ is τ -additive.*
- (3) *The measure μ is the weak limit of a sequence of finitely supported measures.*

If in addition X is complete, these are also equivalent to:

- (4) *The measure μ is inner regular.*

Proof. For a proof of separability and the equivalence of the first three conditions, see [7]. Assume that X is complete. Let μ be a finite Borel measure, and suppose (1)–(3) hold. Then the support S of μ is closed, separable and has measure 1. Let A be any Borel measurable set. Then $\mu(A \cap (X \setminus S)) = 0$ since $\mu(X \setminus S) = 0$, so $\mu(A) = \mu(A \cap S)$. Since the metric space S is a separable complete metric space, it is a standard result

that $\mu|_S$ is an inner regular measure. Thus for $\varepsilon > 0$ there exists a compact set $K \subseteq S \cap A \subseteq A$ such that $\mu(A) = \mu(A \cap S) < \mu(K) + \varepsilon$.

Conversely suppose μ is inner regular. If $\mu(S) < \mu(X)$ for the support S of μ , then for $U = X \setminus S$, $\mu(U) > 0$. By inner regularity there exists a compact set $K \subseteq U$ such that $\mu(K) > 0$. Since K misses the support of μ , for each $x \in K$, there exists an open set U_x containing x such that $\mu(U_x) = 0$. Finitely many of the $\{U_x\}$ cover K , the finite union has measure 0, so the subset K has measure 0, a contradiction. So the support of μ has measure $\mu(X)$. \square

Remark 2.2. Finite Borel measures on separable metric spaces are easily shown to be τ -additive and hence satisfy the other equivalent conditions of Proposition 2.1. Finite Borel measures that fail to satisfy the previous conclusions are rare. Indeed it is a theorem that in a complete metric space X there exists a finite Borel measure that fails to be inner regular if and only if the minimal cardinality $w(X)$ for a basis of open sets of X is a measurable cardinal; see volume 4, page 244 of [4]. The existence of measurable cardinals is an axiom independent of the basic Zermelo–Fraenkel axioms of set theory and thus if its negation is assumed, all finite Borel measures on complete metric spaces satisfy the four conditions of Proposition 2.1.

3. The stochastic order

We henceforth restrict our attention to the set of Borel probability measures on a metric space X satisfying the four conditions of Proposition 2.1 and denote this set $\mathcal{P}(X)$. For separable metric spaces the set $\mathcal{P}(X)$ consists of all Borel probability measures, which are automatically τ -additive in this case.

Definition 3.1. A *partially ordered topological space* is a topological space equipped with a closed partial order \leq , one for which $\{(x, y) : x \leq y\}$ is closed in $X \times X$.

For a nonempty subset A of a partially ordered set P , let $\uparrow A := \{y \in P : \exists x \in A, x \leq y\}$. The set $\downarrow A$ is defined in an order-dual fashion. A set A is an *upper set* if $\uparrow A = A$ and a *lower set* if $\downarrow A = A$. We abbreviate $\uparrow\{x\}$ by $\uparrow x$ and $\downarrow\{x\}$ by $\downarrow x$.

Lemma 3.2. A partially ordered topological space is Hausdorff. If K is a nonempty compact subset, then $\uparrow K$ and $\downarrow K$ are closed.

Proof. See Section VI-1 of [5]. \square

The following definition captures in the setting of ordered topological spaces the notion that higher values should have higher probability.

Definition 3.3. For a topological space X equipped with a closed partial order, the *stochastic order* on $\mathcal{P}(X)$ is defined by $\mu \leq \nu$ if $\mu(U) \leq \nu(U)$ for each open upper set U .

Proposition 3.4. Let X be a metric space equipped with a closed partial order. Then the following are equivalent for $\mu, \nu \in \mathcal{P}(X)$:

- (1) $\mu \leq \nu$;
- (2) $\mu(A) \leq \nu(A)$ for each closed upper set A ;
- (3) $\mu(B) \leq \nu(B)$ for each upper Borel set B .

Proof. Clearly (3) implies both (1) and (2).

(1) \Rightarrow (3): Let $B = \uparrow B$ be a Borel set. The $A = X \setminus B$ is also a Borel set. Let $\varepsilon > 0$. By inner regularity there exists a compact set $K \subseteq A$ such that $\nu(K) > \nu(A) - \varepsilon$. By Lemma 3.2 $\downarrow K$ is closed, and $K \subseteq \downarrow K \subseteq \downarrow A = A$. Thus $\nu(\downarrow K) > \nu(A) - \varepsilon$. The complement U of $\downarrow K$ is an open upper set. Taking complements we obtain

$$\begin{aligned}\mu(B) &= 1 - \mu(A) \leq 1 - \mu(\downarrow K) = \mu(U) \leq \nu(U) = 1 - \nu(\downarrow K) \\ &< 1 - \nu(A) + \varepsilon = \nu(B) + \varepsilon.\end{aligned}$$

Since $\mu(B) < \nu(B) + \varepsilon$ for all $\varepsilon > 0$, we conclude $\mu(B) \leq \nu(B)$.

(2) \Rightarrow (3): We can approximate any Borel upper set B arbitrarily closely from the inside with compact subsets K and their upper sets $\uparrow K$ will be closed sets that are at least as good approximations. The Borel measure ν dominates μ on these closed upper sets and hence also in the limiting case of B . \square

Remark 3.5. By taking complements one determines that each of the preceding equivalences has an equivalent version for lower sets with the inequalities in (2) and (3) reversed.

We turn now to functional characterizations of the stochastic order on $\mathcal{P}(X)$ for X a metric space equipped with a closed partial order. In the next proposition, we write $\int_X f(x) d\mu(x)$ or simply $\int_X f d\mu$ for any Borel function $f : X \rightarrow \mathbb{R}^+$ and $\mu \in \mathcal{P}(X)$, where the integral is possibly infinite. We say that f is *monotone* if $x \leq y$ in X implies $f(x) \leq f(y)$.

Proposition 3.6. *Let X be a metric space equipped with a closed partial order. Then the following are equivalent for $\mu, \nu \in \mathcal{P}(X)$:*

- (1) $\mu \leq \nu$;
- (2) for every monotone (bounded) Borel function $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$;
- (3) for every monotone (bounded) lower semicontinuous $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$.

Proof. The implications that the general case implies the bounded case in items (2) and (3) are trivial.

(1) \Rightarrow (2): Assume $\mu \leq \nu$ and let f be a non-negative monotone Borel measurable function on X . For each n , define $\delta_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\delta_n(0) = 0$, $\delta_n(t) = (i-1)/2^n$ if $(i-1)/2^n < t \leq i/2^n$ for some integer i , $1 \leq i \leq n2^n$, and $\delta_n(t) = n$ for $n < t$. Note that the ascending step function δ_n has finite image contained in \mathbb{R}^+ and that the sequence δ_n monotonically increases to the identity map on \mathbb{R}^+ . Hence $f_n := \delta_n \circ f$, the composition of δ_n and f , monotonically increases to f . One verifies directly that the step function f_n has an alternative description given by

$$f_n = \sum_{i=1}^{n2^n} \frac{1}{2^n} \chi_{f^{-1}([i/2^n, \infty))},$$

where χ_A is the characteristic function of A . Since the sequence $\{f_n\}$ converges pointwise and monotonically to f , we conclude that $\int_X f d\mu = \lim_n \int_X f_n d\mu$, and similarly for ν . Since $f^{-1}([i/2^n, \infty))$ is an upper Borel set, by Proposition 3.4 $\mu(f^{-1}([i/2^n, \infty))) \leq \nu(f^{-1}([i/2^n, \infty)))$ for each i , so $\int_X f_n d\mu \leq \int_X f_n d\nu$ for each n , and thus in the limit $\int_X f d\mu \leq \int_X f d\nu$.

(2) \Rightarrow (3): Since a lower semicontinuous function is a Borel measurable function, (3) follows immediately from (2).

(3) \Rightarrow (1): The characteristic function χ_U is bounded, lower semicontinuous, and monotone for U an open upper set and hence $\mu(U) = \int_X \chi_U d\mu \leq \int_X \chi_U d\nu = \nu(U)$. \square

Call a real function f on a partially ordered set X *antitone* if it is order reversing, i.e., $x \leq y$ implies $f(x) \geq f(y)$.

Corollary 3.7. *Let X be a metric space equipped with a closed partial order. Then the following are equivalent for $\mu, \nu \in \mathcal{P}(X)$:*

- (1) $\nu \leq \mu$;
- (2) for every antitone (bounded) Borel function $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$;
- (3) for every antitone (bounded) lower semicontinuous $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$.

Proof. Every partially ordered set has a dual order, namely the converse \geq of \leq is taken for the partial order. Let X^{od} denote the order dual of X . Note that a subset A of X is an upper set in (X, \leq) if and only if it is a lower set in X^{od} . Using Remark 3.5, one sees that $\mu \leq \nu$ with respect to (X, \leq) if and only if $\nu \leq \mu$ with respect to X^{od} . Since antitone functions convert to monotone functions in the order dual of X , the corollary follows from applying the previous proposition to the order dual. \square

Next we consider sufficient conditions for one to define the stochastic order in terms of continuous monotone functions.

Proposition 3.8. *Suppose that (X, d) is a metric space equipped with a closed partial order satisfying the property that given $x \leq y$ and $x_1 \in X$, there exists a $y_1 \geq x_1$ such that $d(y, y_1) \leq d(x, x_1)$. Then for $\mu, \nu \in \mathcal{P}(X)$ the following are equivalent:*

- (1) $\mu \leq \nu$;
- (2) for every continuous (bounded) monotone $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$;
- (3) for every continuous (bounded) antitone $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\nu \leq \int_X f d\mu$.

Proof. That (1) implies (2) follows from Proposition 3.6 and (1) implies (3) by Corollary 3.7.

(3) \Rightarrow (1): Let V be an open lower set with complement A , a closed upper set. For each $n \in \mathbb{N}$, define $f_n : X \rightarrow [0, 1]$ by $f_n(x) = \min\{nd(x, A), 1\}$ and note that f_n is a continuous function into $[0, 1]$. To show f_n is antitone, we note for any $x \leq y$ and $x_1 \in A$, there exists a $y_1 \geq x_1$ such that $d(y, y_1) \leq d(x, x_1)$. It follows from $x_1 \leq y_1$ that $y_1 \in A$, hence $d(y, A) \leq d(x, x_1)$, and thus $d(y, A) \leq d(x, A)$ since x_1 was an arbitrary point of A . Hence

$$f_n(y) = \min\{nd(y, A), 1\} \leq \min\{nd(x, A), 1\} = f_n(x).$$

It follows directly from the definition of f_n that the sequence $\{f_n\}$ is an monotonically increasing sequence with supremum χ_V . Thus

$$\nu(V) = \int_X \chi_V d\nu = \lim_n \int_X f_n d\nu \leq \lim_n \int_X f_n d\mu = \int_X \chi_V d\mu = \mu(V).$$

Since V was an arbitrary open lower set, $\mu \leq \nu$ by Remark 3.5.

(2) \Rightarrow (1): Property (2) implies that $\int_X f d\mu \leq \int_X f d\nu$ for every continuous antitone function $f : X^{od} \rightarrow \mathbb{R}^+$. By the preceding paragraph $\nu \leq \mu$ with respect to X^{od} , i.e., $\mu \leq \nu$ with respect to (X, \leq) . \square

Definition 3.9. A topological space equipped with a closed order is called *monotone normal* if given a closed upper set A and a closed lower set B such that $A \cap B = \emptyset$, there exist an open upper set $U \supseteq A$ and an open lower set $V \supseteq B$ such that $U \cap V = \emptyset$.

Remark 3.10. Assume that (X, d) satisfies the property stated in Proposition 3.8 and also its dual version that given $x \leq y$ and $y_1 \in X$, there is an $x_1 \in X$ such that $x_1 \leq y_1$ and $d(x, x_1) \leq d(y, y_1)$. Then (X, \leq) is monotone normal as in the above definition. Indeed, for any closed upper set A and any closed lower set B with $A \cap B = \emptyset$, one can easily verify that the open sets

$$U := \{x \in X : d(x, A) < d(x, B)\} \quad \text{and} \quad V := \{x \in X : d(x, A) > d(x, B)\}$$

satisfy $U \supseteq A$, $V \supseteq B$ and $U \cap V = \emptyset$. One deduces that U is an upper set and V a lower from the hypothesized property and its dual. Also, we remark that an open cone in a Banach space as considered in Section 5 satisfies the above two properties (see Remark 5.3).

Proposition 3.11. Suppose that (X, d) is a metric space equipped with a closed partial order for which the space is monotone normal. Then for $\mu, \nu \in \mathcal{P}(X)$ the following are equivalent:

- (1) $\mu \leq \nu$;
- (2) for every continuous (bounded) monotone $f : X \rightarrow \mathbb{R}^+$, $\int_X f d\mu \leq \int_X f d\nu$.

Proof. In light of Proposition 3.6 we need only show condition (2) implies condition (1). Suppose there exists some open upper set U such that $\nu(U) < \mu(U)$. By inner regularity there exists a compact set $K \subseteq U$ such that $\nu(U) < \mu(K) \leq \mu(U)$. The closed upper set $A = \uparrow K \subseteq U$ also satisfies $\nu(U) < \mu(A) \leq \mu(U)$. Since X is monotone normal, a modification of the usual proof of Urysohn's Lemma yields a continuous monotone function $f : X \rightarrow [0, 1]$ such that $f(A) = 1$ and $f(X \setminus U) = 0$; see for example [5, Exercise VI-1.16]. We then have

$$\mu(A) = \int_X \chi_A d\mu \leq \int_X f d\mu \leq \int_X f d\nu \leq \int_X \chi_U d\nu = \nu(U),$$

a contradiction to our choice of A . \square

Let (X, \mathcal{M}) be a measurable space, a set X equipped with a σ -algebra \mathcal{M} , and (Y, d) a metric space. A function $f : X \rightarrow Y$ is measurable if $f^{-1}(A) \in \mathcal{M}$ whenever $A \in \mathcal{B}(Y)$, the algebra of Borel subsets of Y . For f to be measurable, it suffices that $f^{-1}(U) \in \mathcal{M}$ for each open subset U of Y . Hence continuous functions are measurable in the case X is a metrizable space and $\mathcal{M} = \mathcal{B}(X)$. A measurable map $f : X \rightarrow Y$ between metric spaces induces the push-forward map $f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ defined by $(f_*\mu)(B) = \mu(f^{-1}(B))$ for $\mu \in \mathcal{P}(X)$ and $B \in \mathcal{B}(Y)$. Note for f continuous that $\text{supp}(f_*\mu) = \overline{f(\text{supp}(\mu))}$, the closure of the image of the support of μ .

The next two propositions will be useful in the later sections.

Proposition 3.12. Let X be a metric space and Y a metric space equipped with a closed partial order. If $f, g : X \rightarrow Y$ are measurable maps such that $f(x) \leq g(x)$ for all $x \in X$, then $f_*\mu \leq g_*\mu$ for every $\mu \in \mathcal{P}(X)$.

Proof. If U is an open upper set in Y , then it is obvious from the assumption on f, g that $f^{-1}(U) \subseteq g^{-1}(U)$, so $(f_*\mu)(U) \leq (g_*\mu)(U)$. Hence $f_*\mu \leq g_*\mu$. \square

Proposition 3.13. Suppose that X, Y are metric spaces equipped with a closed partial order. Let $f : X \rightarrow Y$ be a measurable map which is monotone in the sense that $x \leq y$ implies $f(x) \leq f(y)$ for $x, y \in X$. Then if $\mu, \nu \in \mathcal{P}(X)$ and $\mu \leq \nu$, then $f_*\mu \leq f_*\nu$.

Proof. Let U be an upper Borel set in Y . From the monotonicity property of f it is easy to see that $f^{-1}(U)$ is an upper Borel set in X . Hence $\mu(f^{-1}(U)) \leq \nu(f^{-1}(U))$ follows, which means that $f_*\mu \leq f_*\nu$. \square

4. Normal cones

Let E be a Banach space containing an open cone C such that its closure \overline{C} is a proper cone, i.e., $\overline{C} \cap (-\overline{C}) = \{0\}$. The cone \overline{C} defines a closed partial order on E by $x \leq y$ if $y - x \in \overline{C}$. The cone \overline{C} is called *normal* if there is a constant K such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

For $x \leq y$ in E , the *order interval* $[x, y]$ is given by

$$[x, y] := \{w \in E : x \leq w \leq y\} = (x + \overline{C}) \cap (y - \overline{C}).$$

Note that $(x + C) \cap (y - C)$ is an open subset contained in $[x, y]$. A subset B is *order convex* if $[x, y] \subseteq B$, wherever $x, y \in B$ and $x \leq y$. An alternative formulation of normality postulates the existence of a basis of order convex neighborhoods at 0 and hence by translation at all points (see Section 19.1 of [2]); here neighborhood of x means a subset containing x in its interior.

Proposition 4.1. *Let E be a separable Banach space with an open cone C such that \overline{C} is normal. Then restricted to C , the σ -algebra generated by all its open upper sets is the Borel algebra of C .*

Proof. Let \mathcal{A} denote the σ -algebra of subsets of C generated by the collection of all open upper sets contained in C . Fix some point $u \in C$. Let $x \in C$. Set $r_n = 1/n$ for $n \in \mathbb{N}$. Then for each n , $x \in (x - r_n u) + C$, an open upper set, and $\uparrow x = \bigcap_n [(x - r_n u) + C]$. In fact, for any y in the intersection $y - (x - r_n u) = (y - x) + r_n u \in C$ and hence the limit $y - x$ is in \overline{C} , i.e., $y \in x + \overline{C} = \uparrow x$. The converse inclusion is obvious since $r_n u + \overline{C} \subseteq C$ so that $x + \overline{C} \subseteq (x - r_n u) + C$. Thus $\uparrow x$ is a countable intersection of open upper sets, hence in \mathcal{A} .

Since $\downarrow x$ is closed in E , $C \cap (E \setminus \downarrow x)$ is an open upper set. Thus its complement in C , which is $C \cap \downarrow x$ is in \mathcal{A} . Hence for $x \leq y$ in C , we note that $[x, y] = \uparrow x \cap \downarrow y = \uparrow x \cap C \cap \downarrow y \in \mathcal{A}$.

Now let U be a nonempty open subset of C . Using the alternative characterization of normality, we may pick for each $x \in U$ an order convex neighborhood N_x of x that is contained in U . For some ε small enough $x - \varepsilon u, x + \varepsilon u \in N_x$, and hence the order interval $[x - \varepsilon u, x + \varepsilon u] \subseteq N_x$. Let $B_x := (x - \varepsilon u + C) \cap (x + \varepsilon u - C)$, an open subset contained in $[x - \varepsilon u, x + \varepsilon u]$. The collection $\{B_x : x \in C\}$ is an open cover of U , which by the separability of E (and hence U) has a countable subcover $\{B_{x_n}\}$. The corresponding $[x_n - \varepsilon_n u, x_n + \varepsilon_n u]$ then also form a countable cover of U , and since from the preceding paragraph each order interval is in \mathcal{A} , it follows that $U \in \mathcal{A}$. Thus \mathcal{A} contains all open sets of C , and hence must be the Borel algebra. \square

We next recall E. Dynkin's $\pi - \lambda$ theorem. Let X be a set. A π -system is a collection of subsets of X closed under finite intersection. A λ -system is a collection with X as a member that is closed under complementation and under countable unions of pairwise disjoint members of the system. An important observation is that a λ -system that is also a π -system is a σ -algebra.

Theorem 4.2. (*Dynkin's $\pi - \lambda$ Theorem*) *If a π -system is contained in a λ -system, then the σ -algebra generated by the π -system is contained in the λ -system.*

The stochastic order on $\mathcal{P}(X)$ for a metric space X equipped with a closed order is easily seen to be reflexive and transitive, but anti-symmetry is much more difficult to derive. We now have available the tools we need to show for the open cone C that the stochastic order on $\mathcal{P}(C)$ is a partial order.

Theorem 4.3. *Let E be a Banach space containing an open cone C such that \overline{C} is a normal cone. Then the stochastic order on $\mathcal{P}(C)$ is a partial order.*

Proof. We first consider the case that E is separable. Let $\mu, \nu \in \mathcal{P}(C)$ be such that $\mu \leq \nu$ and $\nu \leq \mu$. We consider the set \mathcal{A} of all Borel sets B such that $\mu(B) = \nu(B)$. By definition of the stochastic order, $U \in \mathcal{A}$ for each open upper set U , and the collection of open upper sets is closed under finite intersection, i.e., is a π -system. Since μ and ν are σ -additive measures, it follows that the collection \mathcal{A} is closed under complementation and union of pairwise disjoint countable families, so \mathcal{A} is a λ -system. By Dynkin's $\pi - \lambda$ theorem the σ -algebra generated by the open upper sets is contained in \mathcal{A} , but by Proposition 4.1 this is the Borel algebra. Hence $\mu = \nu$ on the Borel algebra, that is to say $\mu = \nu$.

We turn now to the general case in which E may not be separable. In this case, however, both S_μ , the support of μ , and S_ν , the support of ν , are separable (Proposition 2.1). Then also the smallest closed Banach subspace F containing $S_\mu \cup S_\nu$ will be separable, and the restrictions $\mu|_F, \nu|_F \in \mathcal{P}(C \cap F)$. Since $C \cap F$ is an open cone in F with closure a normal cone, by the first part of the proof $\mu(B) = \nu(B)$ for all Borel subsets contained in $C \cap F$. Since $S_\mu \cup S_\nu \subseteq C \cap F$, for any Borel set $B \subseteq C$,

$$\mu(B) = \mu(B \cap S_\mu) = \mu(B \cap (S_\mu \cup S_\nu)) = \nu(B \cap (S_\mu \cup S_\nu)) = \nu(B \cap S_\nu) = \nu(B).$$

Thus $\mu = \nu$. \square

Remark 4.4. The techniques of the proof readily extend to any open upper set of E , in particular to E itself. Indeed, Proposition 4.1 and Theorem 4.3 hold when restricted to any open upper set in place of C . So the stochastic order on $\mathcal{P}(E)$ arising from the conic order of E is also a partial order.

5. The Thompson metric

We continue in the setting that E is a Banach space and C is an open cone with its closure \overline{C} a normal cone. A. C. Thompson [14] has proved that C is a complete metric space with respect to the *Thompson part metric* defined by

$$d_T(x, y) = \max\{\log M(x/y), \log M(y/x)\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\} = |x|_y$. Furthermore, the metric topology on C arising from the Thompson metric agrees with relative topology inherited from E . Hence we may consider $\mathcal{P}(C)$ on the metric space (C, d_T) .

The contractivity of addition in C with respect to the Thompson metric has been observed in various settings and studied in some detail in [8]. We need only the basic formulation.

Lemma 5.1. *Addition is contractive on C with respect to the Thompson metric in the sense that for all $x, y, z \in C$, $d_T(x + z, y + z) \leq d_T(x, y)$.*

Proposition 5.2. *The cone C equipped with the Thompson metric satisfies the property that given $x \leq y$ and $x_1 \in C$, there exists a $y_1 \geq x_1$ such that $d_T(y, y_1) \leq d_T(x, x_1)$. Hence for $\mu, \nu \in \mathcal{P}(C)$, $\mu \leq \nu$ in the stochastic order if and only if for every continuous (bounded) monotone $f : C \rightarrow \mathbb{R}^+$, $\int_C f d\mu \leq \int_C f d\nu$.*

Proof. Suppose $x \leq y$ and $x_1 \in C$. The contractivity of the Thompson metric (Lemma 5.1) implies for $y_1 = x_1 + (y - x)$ that

$$d_T(y, y_1) = d_T(x + (y - x), x_1 + (y - x)) \leq d_T(x, x_1).$$

The last assertion of the proposition now follows from Proposition 3.8. \square

Remark 5.3. Here is a second proof of Proposition 5.2. Assume that $x \leq y$ in C . For every $x_1 \in C$ let $\alpha = d_T(x, x_1)$ so that $e^{-\alpha}x \leq x_1 \leq e^{\alpha}x$. Set $y_1 = e^{\alpha}y$; then $y_1 \geq e^{\alpha}x \geq x_1$ and $y \leq y_1 = e^{\alpha}y$, so $d_T(y, y_1) \leq \alpha = d_T(x, x_1)$. Similarly one can show the dual version mentioned in Remark 3.10. In fact, for every $y_1 \in C$ let $\beta = d_T(y, y_1)$ and $x_1 := e^{-\beta}x$; then $x_1 \leq e^{-\beta}y \leq y_1$ and $e^{-\beta}x = x_1 \leq x$ so that $d_T(x, x_1) \leq \beta = d_T(y, y_1)$.

Recall that one of the characterizations of the weak topology on any metric space, in particular on $\mathcal{P}(C)$, is that a net $\mu_\alpha \rightarrow \mu$ weakly if and only if $\lim_\alpha \int_C f d\mu_\alpha \rightarrow \int_C f d\mu$ for all continuous bounded functions into \mathbb{R} (or \mathbb{R}^+); see [1, 3].

Proposition 5.4. *The stochastic partial order is a closed subset of $\mathcal{P}(C) \times \mathcal{P}(C)$ endowed with the product weak topology.*

Proof. Let $\mu_\alpha \rightarrow \mu$ and $\nu_\alpha \rightarrow \nu$ weakly in $\mathcal{P}(C)$, where $\mu_\alpha \leq \nu_\alpha$ for each α . From Proposition 5.2 for $f : C \rightarrow \mathbb{R}^+$ continuous bounded and monotone

$$\int_C f d\mu = \lim_\alpha \int_C f d\mu_\alpha \leq \lim_\alpha \int_C f d\nu_\alpha = \int_C f d\nu.$$

Thus again from Proposition 5.2, $\mu \leq \nu$. \square

Let (X, d) be a complete metric space, and for $p \in [1, \infty)$ let $\mathcal{P}^p(X) := \{\mu \in \mathcal{P}(X) : \int_X d(x, y)^p d\mu(y) < \infty\}$, the set of τ -additive Borel probability measures on X with finite p th moment (defined independently of the choice of $x \in X$). The p -Wasserstein metric d_p^W on $\mathcal{P}^p(X)$ is defined by

$$d_p^W(\mu, \nu) := \left[\inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y)^p d\pi(x, y) \right]^{1/p}, \quad \mu, \nu \in \mathcal{P}^p(X), \quad (5.1)$$

where $\Pi(\mu, \nu)$ is the set of all couplings for μ, ν , i.e., $\pi \in \mathcal{P}(X \times X)$ whose marginals are μ and ν . Recall (see, e.g., [13]) that $\mathcal{P}^p(X)$ is a complete metric space with the metric d_p^W , and that the Wasserstein convergence implies weak convergence.

In the following, the space $\mathcal{P}^p(C)$ with the p -Wasserstein metric d_p^W for $1 \leq p < \infty$ will be defined with respect to the Thompson metric $d = d_T$. Then we have the following corollary of the preceding proposition.

Corollary 5.5. *The stochastic partial order is a closed subset of $\mathcal{P}^1(C) \times \mathcal{P}^1(C)$ endowed with the product Wasserstein topology (induced by d_1^W).*

We recall the notion of a contractive barycentric map.

Definition 5.6. Let (X, d) be a complete metric space. A map $\beta : \mathcal{P}^1(X) \rightarrow X$ is called a *contractive barycentric map* if

- (i) $\beta(\delta_x) = x$ for all $x \in X$;
- (ii) $d(\beta(\mu), \beta(\nu)) \leq d_1^W(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}^1(X)$.

For a closed partial order \leq on X , $\beta : \mathcal{P}^1(X) \rightarrow X$ is said to be *monotonic* if $\beta(\mu) \leq \beta(\nu)$, whenever $\mu \leq \nu$.

A complete partially ordered metric space equipped with a monotonic contractive barycenter has become an important object of study in recent years.

Corollary 5.7. *Let $\beta : \mathcal{P}^1(C) \rightarrow C$ be a monotonic barycentric map, and let $f, g : C \rightarrow C$ be Lipschitzian maps with respect to d_T such that $f(x) \leq g(x)$ for all $x \in C$. Then for every $\mu \in \mathcal{P}^1(C)$, we have $f_*\mu, g_*\mu \in \mathcal{P}^1(C)$ and $\beta(f_*\mu) \leq \beta(g_*\mu)$.*

Proof. Since f, g are Lipschitz with respect to d_T , it is obvious that $\psi_*\mu \in \mathcal{P}^1(C)$ if $\mu \in \mathcal{P}^1(C)$. Then the assertion follows from Proposition 3.12 and the monotonicity property of β . \square

For instance, every translation $\tau_a(x) = a + x$, $a \in \overline{C}$, satisfies $x \leq \tau_a(x)$ for all $x \in C$ and also is non-expansive for the Thompson metric. Hence $\beta(\mu) \leq \beta(a + \mu)$ for all $\mu \in \mathcal{P}^1(C)$, where $a + \mu := (\tau_a)_*\mu$.

6. Order-completeness

In this section we always assume that the Banach space E is *finite-dimensional* (hence separable) and, as in Section 4, C is an open cone in E whose closure \overline{C} is a proper cone. Note (see Section 19.1 of [2]) that the finite dimensionality assumption automatically implies that \overline{C} is a normal cone. We consider C as a complete metric space equipped with the Thompson metric d_T . The p -Wasserstein metric d_p^W on $\mathcal{P}^p(C)$, $1 \leq p < \infty$, is given in (5.1) with $d = d_T$.

The next elementary lemma is given just for completeness.

Lemma 6.1.

- (1) *For each $x, y \in C$, the order interval $[x, y] = (x + \overline{C}) \cap (y - \overline{C})$ is a compact subset of C .*
- (2) *For any $u \in C$, $\bigcup_{k=1}^{\infty} [k^{-1}u, ku] = C$.*

Proof. (1): Since $x + \overline{C} \subset C + \overline{C} \subset C$, $[x, y] \subset C$. It is also clear that $[x, y]$ is a closed subset of E . Since \overline{C} is a normal cone, we see that if $z \in [x, y]$ then $\|z\| \leq K\|y\|$. Hence, $[x, y]$ is a bounded closed subset of E . Since E is finite-dimensional, $[x, y]$ is compact in E and so is in (C, d) .

(2): Let $u, x \in C$. For $k \in \mathbb{N}$ sufficiently large, $x - k^{-1}u \in C$ and $u - k^{-1}x \in C$ so that $x \in (k^{-1}u + C) \cap (ku - C)$. Therefore, $x \in [k^{-1}u, ku]$, which implies the assertion. \square

Before showing order-completeness, it is convenient to derive the compactness of order intervals in $\mathcal{P}(C)$ as well as in $\mathcal{P}^p(C)$.

Proposition 6.2. *Let $\nu_1, \nu_2 \in \mathcal{P}(C)$ with $\nu_1 \leq \nu_2$.*

- (1) *The order interval $[\nu_1, \nu_2] := \{\mu \in \mathcal{P}(C) : \nu_1 \leq \mu \leq \nu_2\}$ is compact in the weak topology.*
- (2) *Let $1 \leq p < \infty$. If $\nu_1, \nu_2 \in \mathcal{P}^p(C)$, then $[\nu_1, \nu_2] \subset \mathcal{P}^p(C)$ and it is compact in the d_p^W -topology.*

Proof. (1): Choose any $u \in C$. For every $\epsilon > 0$ Lemma 6.1 (2) implies that there exists a $k \in \mathbb{N}$ such that $(\nu_1 + \nu_2)(C \setminus [k^{-1}u, ku]) < \epsilon$. We write $C \setminus [k^{-1}u, ku] = U_k \cup V_k$, where $U_k := \{x \in C : x \not\leq ku\}$ and $V_k := \{x \in C : x \not\geq k^{-1}u\}$. It is clear that U_k is an upper open set while V_k is a lower open set. Hence, if $\mu \in [\nu_1, \nu_2]$, then we have

$$\begin{aligned} \mu(C \setminus [k^{-1}u, ku]) &\leq \mu(U_k) + \mu(V_k) \leq \nu_2(U_k) + \nu_1(V_k) \\ &\leq (\nu_1 + \nu_2)(C \setminus [k^{-1}u, ku]) < \epsilon \end{aligned}$$

(for $\mu(V_k) \leq \nu_1(V_k)$, see Remark 3.5). By Lemma 6.1 (1), this says that $[\nu_1, \nu_2]$ is tight, and so it is relatively compact in $\mathcal{P}(C)$ in the weak topology due to Prokhorov's theorem (see [1]). Since $[\nu_1, \nu_2]$ is closed in the weak topology by Proposition 5.4, $[\nu_1, \nu_2]$ is compact in the weak topology.

(2): Next, assume that $\nu_1, \nu_2 \in \mathcal{P}^p(C)$ for $p \in [1, \infty)$. First we prove the following “tightness” condition:

$$\lim_{R \rightarrow \infty} \sup_{\mu \in [\nu_1, \nu_2]} \int_{d(x,u) > R} d(x,u)^p d\mu(x) = 0 \quad (6.1)$$

for some $u \in C$. Choose any $u \in C$. For every $R \geq 0$ set

$$\begin{aligned} U_R &:= \{x \in C : M(x/u) > e^R, M(x/u) \geq M(u/x)\}, \\ V_R &:= \{x \in C : M(u/x) > e^R, M(x/u) < M(u/x)\}. \end{aligned}$$

Then it is immediate to see that

$$\{x \in C : d(x,u) > R\} = U_R \cup V_R \quad (\text{disjoint sum}).$$

Hence, for any $\mu \in \mathcal{P}(C)$ we have

$$\int_{d(x,u) > R} d(x,u)^p d\mu(x) = \int_C 1_{U_R}(x) d(x,u)^p d\mu(x) + \int_C 1_{V_R}(x) d(x,u)^p d\mu(x). \quad (6.2)$$

When $x \in U_R$ and $x \leq y \in C$, since $M(x/u) \leq M(y/u)$ and $M(u/x) \geq M(u/y)$, we find that $M(y/u) \geq M(x/u) > e^R$ and $M(y/u) \geq M(u/y)$ so that $y \in U_R$. Therefore, U_R is an upper Borel set. Moreover,

$$d(x,u) = \log M(x/u) \leq \log M(y/u) = d(y,u).$$

Hence it follows that $x \in C \mapsto 1_{U_R}(x) d(x,u)^p$ is a monotone Borel function. When $x \in V_R$ and $x \geq y \in C$, since $M(x/u) \geq M(y/u)$ and $M(u/x) \leq M(u/y)$, $M(u/y) \geq M(u/x) > e^R$ and $M(y/u) < M(u/y)$ so that $y \in V_R$. Therefore, V_R is a lower open set and

$$d(x,u) = \log M(u/x) \leq \log M(u/y) = d(y,u).$$

Hence we see that $x \in C \mapsto 1_{V_R}(x) d(x,u)^p$ is an antitone Borel function. If $\mu \in [\nu_1, \nu_2]$, then by Proposition 3.6 and Corollary 3.7 applied to the right-hand side of (6.2) we obtain

$$\begin{aligned} \int_{d(x,u) > R} d(x,u)^p d\mu(x) &\leq \int_C 1_{U_R}(x) d(x,u)^p d\nu_2(x) + \int_C 1_{V_R}(x) d(x,u)^p d\nu_1(x) \\ &\leq \int_{d(x,u) > R} d(x,u)^p d(\nu_1 + \nu_2)(x) \longrightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$, since $\int_C d(x,u)^p d(\nu_1 + \nu_2)(x) < \infty$. Hence (6.1) has been proved, which in particular implies that $[\nu_1, \nu_2] \subseteq \mathcal{P}^p(C)$. Moreover, from a basic fact on the convergence in Wasserstein spaces [15, Theorem 7.12], we see that $[\nu_1, \nu_2]$ is compact in the d_p^W -topology. Indeed, for every sequence $\{\mu_n\}$ in $[\nu_1, \nu_2]$, from the assertion (1) one can choose a subsequence $\{\mu_{n(m)}\}$ such that $\mu_{n(m)} \rightarrow \mu$ weakly for some $\mu \in \mathcal{P}(C)$. Hence, it follows from [15, Theorem 7.12] that $\mu \in \mathcal{P}^p(C)$ and $d_p^W(\mu_{n(m)}, \mu) \rightarrow 0$. Note that the limit μ is in $[\nu_1, \nu_2]$, since the d_p^W -convergence implies the weak convergence. Thus, $[\nu_1, \nu_2]$ is d_p^W -compact. \square

The next proposition gives the order-completeness (or a monotone convergence property) of the stochastic order on $\mathcal{P}(C)$ in the weak topology.

Proposition 6.3. *Let $\mu_n, \nu \in \mathcal{P}(C)$ for $n \in \mathbb{N}$.*

- (1) *If $\mu_1 \leq \mu_2 \leq \dots \leq \nu$, then there exists a $\mu \in \mathcal{P}(C)$ such that $\mu_n \leq \mu \leq \nu$ for all n and $\mu_n \rightarrow \mu$ weakly.*
- (2) *If $\mu_1 \geq \mu_2 \geq \dots \geq \nu$, then there exists a $\mu \in \mathcal{P}(C)$ such that $\mu_n \geq \mu \geq \nu$ for all n and $\mu_n \rightarrow \mu$ weakly.*

Proof. (1): Since $\{\mu_n\} \subset [\mu_1, \nu]$ and Proposition 6.2(1) says that $[\mu_1, \nu]$ is compact in the weak topology, to see that $\mu_n \rightarrow \mu$ weakly for some $\mu \in \mathcal{P}(C)$, it suffices to prove that a weak limit point of $\{\mu_n\}$ is unique. Now, let $\mu, \mu' \in \mathcal{P}(C)$ be weak limit points of $\{\mu_n\}$, so there are subsequences $\{\mu_{n(l)}\}$ and $\{\mu_{n(m)}\}$ such that $\mu_{n(l)} \rightarrow \mu$ and $\mu_{n(m)} \rightarrow \mu'$ weakly. Let $f : C \rightarrow \mathbb{R}^+$ be any continuous bounded and monotone function. Since $\int_C f d\mu_n$ is increasing in n by Proposition 5.2, we have

$$\int_C f d\mu = \lim_l \int_C f d\mu_{n(l)} = \lim_m \int_C f d\mu_{n(m)} = \int_C f d\mu'.$$

This implies by Proposition 5.2 again that $\mu \leq \mu'$ and $\mu' \leq \mu$ so that $\mu = \mu'$ by Theorem 4.3. Therefore $\mu_n \rightarrow \mu \in \mathcal{P}(C)$ weakly. Moreover, since $\int_C f d\mu_n \leq \int_C f d\mu \leq \int_C f d\nu$ for every continuous bounded and monotone function $f \geq 0$ on C , we have $\mu_n \leq \mu \leq \nu$ for all n .

(2): The proof is similar to the above with a slight modification. \square

The next proposition gives the order-completeness of the stochastic order restricted on $\mathcal{P}^p(C)$ in the d_p^W -convergence.

Proposition 6.4. *Let $1 \leq p < \infty$ and $\mu_n, \nu \in \mathcal{P}^p(C)$ for $n \in \mathbb{N}$.*

- (1) *If $\mu_1 \leq \mu_2 \leq \dots \leq \nu$, then there exists a $\mu \in \mathcal{P}^p(C)$ such that $\mu_n \leq \mu \leq \nu$ for all n and $d_p^W(\mu_n, \mu) \rightarrow 0$.*
- (2) *If $\mu_1 \geq \mu_2 \geq \dots \geq \nu$, then there exists a $\mu \in \mathcal{P}^p(C)$ such that $\mu_n \geq \mu \geq \nu$ for all n and $d_p^W(\mu_n, \mu) \rightarrow 0$.*

Proof. For both assertions (1) and (2), by Proposition 6.2(2) it suffices to prove that a d_p^W -limit point of $\{\mu_n\}$ is unique. Since the d_p^W -convergence implies the weak convergence, this is immediate from the proof of Proposition 6.3. \square

Corollary 6.5. *Let $\mu, \mu_n \in \mathcal{P}(C)$, $n \in \mathbb{N}$. Then μ_n weakly converges to μ increasingly (resp. decreasingly) in the stochastic order if and only if $\int_C f d\mu_n$ increases (resp. decreases) to $\int_C f d\mu$ for every continuous bounded and monotone $f : C \rightarrow \mathbb{R}^+$. Moreover, if $\mu, \mu_n \in \mathcal{P}^p(C)$ where $1 \leq p < \infty$, then the above conditions are also equivalent to: μ_n converges to μ in the metric d_p^W increasingly (resp. decreasingly) in the stochastic order.*

Proof. Assume that for any $f : C \rightarrow \mathbb{R}^+$ as stated above, $\int_C f d\mu_n$ increases (resp. decreases) to $\int_C f d\mu$. Then by Proposition 5.2, $\mu_1 \leq \mu_2 \leq \dots \leq \mu$ ($\mu_1 \geq \mu_2 \geq \dots \geq \mu$). By Proposition 6.3 there exists a $\mu_0 \in \mathcal{P}(C)$ such that $\mu_n \rightarrow \mu_0$ weakly. By assumption, $\int_C f d\mu = \int_C f d\mu_0$ for any f as above, which implies that $\mu = \mu_0$ by Theorem 4.3 and Proposition 5.2. Hence $\mu_n \rightarrow \mu$ weakly. Since the converse implication is obvious, the first assertion has been shown. The second follows similarly from Proposition 6.4. \square

Remark 6.6. It is straightforward to see that $x \mapsto \delta_x$ is a homeomorphism from (C, d_T) into $\mathcal{P}(C)$ with the weak topology and also an isometry from (C, d_T) into $(\mathcal{P}^1(C), d_1^W)$. Hence each conclusion of (1) and (2) of Proposition 6.2 implies that the interval $[x_1, x_2]$ in C is compact for any $x_1, x_2 \in C$ with $x_1 \leq x_2$. Since $(2^{-1}u + C) \cap (2u - C)$ is a non-empty open subset of $[2^{-1}u, 2u]$ for any $u \in C$, this forces E to be finite-dimensional. Thus, the finite dimensionality of E is essential in Proposition 6.2. But, there might be a possibility for Propositions 6.3 and 6.4 to hold true beyond the finite-dimensional case.

7. AGH mean inequalities

In this section we adopt a more specialized setting where $E = \mathcal{B}(H)$ is the Banach space of bounded operators on a (general) Hilbert space H with the operator norm, and $C = \mathbb{P}$ is the open cone consisting of positive invertible operators on H . Note that \mathbb{P} is a complete metric space with the Thompson metric d_T . Let Λ be the *Karcher barycenter* on $\mathcal{P}^1(\mathbb{P})$; in particular, for a finitely and uniformly supported measure $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{A_j}$,

$$\Lambda_n(A_1, \dots, A_n) := \Lambda \left(\frac{1}{n} \sum_{j=1}^n \delta_{A_j} \right)$$

is the *Karcher* or *least squares mean* of $(A_1, \dots, A_n) \in \mathbb{P}^n$, which is uniquely determined by the *Karcher equation*

$$\sum_{j=1}^n \log(X^{-1/2} A_j X^{-1/2}) = 0.$$

Moreover, the map $\Lambda : \mathcal{P}^1(\mathbb{P}) \rightarrow \mathbb{P}$ is contractive

$$d_T(\Lambda(\mu), \Lambda(\nu)) \leq d_1^W(\mu, \nu), \quad \mu, \nu \in \mathcal{P}^1(\mathbb{P}).$$

See, e.g., [9–11] for the Karcher equation and the Karcher (or Cartan) barycenter.

We consider the complete metric d_n on the product space \mathbb{P}^n

$$d_n((A_1, \dots, A_n), (B_1, \dots, B_n)) := \frac{1}{n} \sum_{j=1}^n d_T(A_j, B_j). \quad (7.1)$$

The contraction property of the Karcher barycenter implies that the map

$$\Lambda_n : \mathbb{P}^n \rightarrow \mathbb{P}, \quad (A_1, \dots, A_n) \mapsto \Lambda_n(A_1, \dots, A_n)$$

is a Lipschitz map with Lipschitz constant 1.

The arithmetic and harmonic means

$$\mathcal{A}_n(A_1, \dots, A_n) = \frac{1}{n} \sum_{j=1}^n A_j, \quad \mathcal{H}_n(A_1, \dots, A_n) = \left[\frac{1}{n} \sum_{j=1}^n A_j^{-1} \right]^{-1}$$

are continuous from \mathbb{P}^n to \mathbb{P} and are also Lipschitz with Lipschitz constant 1 for the sup-metric on \mathbb{P}^n

$$d_n^\infty((A_1, \dots, A_n), (B_1, \dots, B_n)) := \max_{1 \leq j \leq n} d_T(A_j, B_j). \quad (7.2)$$

Definition 7.1. For each $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathbb{P})$, note that the product measure $\mu_1 \times \dots \times \mu_n$ is in $\mathcal{P}(\mathbb{P}^n)$. This is easily verified since the support of the product measure is the product of the supports of μ_i 's having the measure 1. As seen from Proposition 2.1, note also that the push-forward of a τ -additive measure by a continuous map is τ -additive. Hence one can define the following three measures in $\mathcal{P}(\mathbb{P})$, regarded as the geometric, arithmetic and harmonic means of μ_1, \dots, μ_n :

$$\Lambda(\mu_1, \dots, \mu_n) := (\Lambda_n)_*(\mu_1 \times \cdots \times \mu_n), \quad (7.3)$$

$$\mathcal{A}(\mu_1, \dots, \mu_n) := (\mathcal{A}_n)_*(\mu_1 \times \cdots \times \mu_n), \quad (7.4)$$

$$\mathcal{H}(\mu_1, \dots, \mu_n) := (\mathcal{H}_n)_*(\mu_1 \times \cdots \times \mu_n). \quad (7.5)$$

Example 7.2. For $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{A_j}$ and $X \in \mathbb{P}$,

$$\Lambda(\delta_X, \mu) = \frac{1}{n} \sum_{j=1}^n \delta_{X \# A_j},$$

$$\mathcal{A}(\delta_X, \mu) = \frac{1}{n} \sum_{j=1}^n \delta_{(X+A_j)/2},$$

$$\mathcal{H}(\delta_X, \mu) = \frac{1}{n} \sum_{j=1}^n \delta_{2(X^{-1}+A_j^{-1})^{-1}}.$$

Proposition 7.3. For every $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathbb{P})$,

$$\mathcal{H}(\mu_1, \dots, \mu_n) = [\mathcal{A}(\mu_1^{-1}, \dots, \mu_n^{-1})]^{-1},$$

where μ^{-1} is the push-forward of μ by operator inversion $A \mapsto A^{-1}$.

Proof. For every bounded continuous function $f : \mathbb{P} \rightarrow \mathbb{R}$ we have

$$\begin{aligned} & \int_{\mathbb{P}} f(A) d[\mathcal{H}(\mu_1, \dots, \mu_n)]^{-1}(A) \\ &= \int_{\mathbb{P}} f(A^{-1}) d\mathcal{H}(\mu_1, \dots, \mu_n)(A) \\ &= \int_{\mathbb{P}^n} f\left(\frac{1}{n} \sum_{j=1}^n A_j^{-1}\right) d(\mu_1 \times \cdots \times \mu_n)(A_1, \dots, A_n) \\ &= \int_{\mathbb{P}^n} f\left(\frac{1}{n} \sum_{j=1}^n A_j\right) d(\mu_1^{-1} \times \cdots \times \mu_n^{-1})(A_1, \dots, A_n) \\ &= \int_{\mathbb{P}} f(A) d\mathcal{A}(\mu_1^{-1}, \dots, \mu_n^{-1})(A), \end{aligned}$$

which shows that $[\mathcal{H}(\mu_1, \dots, \mu_n)]^{-1} = \mathcal{A}(\mu_1^{-1}, \dots, \mu_n^{-1})$. \square

For a complete metric space (X, d) , in addition to $\mathcal{P}^p(X)$ with the p -Wasserstein metric d_p^W in (5.1) for $1 \leq p < \infty$, we also consider the set $\mathcal{P}^\infty(X)$ of $\mu \in \mathcal{P}(X)$ whose support is a bounded set of X , equipped with the ∞ -Wasserstein metric

$$d_\infty^W(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \sup\{d(x, y) : (x, y) \in \text{supp}(\pi)\},$$

where $\Pi(\mu, \nu)$ is the set of all couplings for μ, ν .

Proposition 7.4. For every $p \in [1, \infty]$ and $M = \Lambda, \mathcal{A}, \mathcal{H}$ in (7.3)–(7.5), if $\mu_1, \dots, \mu_n \in \mathcal{P}^p(\mathbb{P})$ then $M(\mu_1, \dots, \mu_n) \in \mathcal{P}^p(\mathbb{P})$. Moreover,

$$(\mu_1, \dots, \mu_n) \in (\mathcal{P}^p(\mathbb{P}))^n \mapsto M(\mu_1, \dots, \mu_n) \in \mathcal{P}^p(\mathbb{P})$$

is Lipschitz continuous with respect to the Wasserstein metric d_p^W .

Proof. Since $\Lambda_n : \mathbb{P}^n \rightarrow \mathbb{P}$ is a Lipschitz map with Lipschitz constant 1 with respect to d_n in (7.1), we can use [11, Lemma 1.3] to see that for each $p \in [1, \infty]$ the push-forward map $(\Lambda_n)_* : \mathcal{P}^p(\mathbb{P}^n) \rightarrow \mathcal{P}^p(\mathbb{P})$ is Lipschitz with Lipschitz constant 1 with respect to the metric d_p^W , where d_p^W on $\mathcal{P}^p(\mathbb{P}^n)$ is defined in terms of d_n . Let $\mu_1, \dots, \mu_n; \nu_1, \dots, \nu_n \in \mathcal{P}^p(\mathbb{P})$. Then it is clear that $\mu_1 \times \dots \times \mu_n \in \mathcal{P}^p(\mathbb{P}^n)$ and hence $\Lambda(\mu_1, \dots, \mu_n) = (\Lambda_n)_*(\mu_1 \times \dots \times \mu_n)$ is in $\mathcal{P}^p(\mathbb{P})$. To show the Lipschitz continuity, we may prove more precisely that

$$\begin{aligned} d_p^W(\Lambda(\mu_1, \dots, \mu_n), \Lambda(\nu_1, \dots, \nu_n)) &\leq \left[\frac{1}{n} \sum_{j=1}^n \left(d_p^W(\mu_j, \nu_j) \right)^p \right]^{1/p} \quad \text{when } 1 \leq p < \infty, \\ d_\infty^W(\Lambda(\mu_1, \dots, \mu_n), \Lambda(\nu_1, \dots, \nu_n)) &\leq \max_{1 \leq j \leq n} d_\infty^W(\mu_j, \nu_j) \quad \text{when } p = \infty. \end{aligned}$$

To prove this, let $\pi_j \in \Pi(\mu_j, \nu_j)$, $1 \leq j \leq n$. Since $\pi_1 \times \dots \times \pi_n \in \Pi(\mu_1 \times \dots \times \mu_n, \nu_1 \times \dots \times \nu_n)$, we have, for the case $1 \leq p < \infty$,

$$\begin{aligned} &d_p^W((\Lambda_n)_*(\mu_1 \times \dots \times \mu_n), (\Lambda_n)_*(\nu_1 \times \dots \times \nu_n)) \\ &\leq d_p^W(\mu_1 \times \dots \times \mu_n, \nu_1 \times \dots \times \nu_n) \\ &\leq \left[\int_{\mathbb{P}^n \times \mathbb{P}^n} d_p^p((A_1, \dots, A_n), (B_1, \dots, B_n)) d(\pi_1 \times \dots \times \pi_n) \right]^{1/p} \\ &= \left[\int_{\mathbb{P}^n \times \mathbb{P}^n} \left(\frac{1}{n} \sum_{j=1}^n d_T(A_j, B_j) \right)^p d(\pi_1 \times \dots \times \pi_n) \right]^{1/p} \\ &\leq \left[\int_{\mathbb{P}^n \times \mathbb{P}^n} \frac{1}{n} \sum_{j=1}^n d_T^p(A_j, B_j) d(\pi_1 \times \dots \times \pi_n) \right]^{1/p} \\ &= \left[\frac{1}{n} \sum_{j=1}^n \int_{\mathbb{P} \times \mathbb{P}} d_T^p(A_j, B_j) d\pi_j(A_j, B_j) \right]^{1/p}. \end{aligned}$$

By taking the infima over π_j , $1 \leq j \leq n$, in the last expression, we have the desired d_p^W -inequality when $1 \leq p < \infty$. The proof when $p = \infty$ is similar, so we omit the details.

Since $\mathcal{A}_n, \mathcal{H}_n : \mathbb{P}^n \rightarrow \mathbb{P}$ is Lipschitz with Lipschitz constant 1 with respect to d_n^∞ in (7.2), we can use [11, Lemma 1.3] again with the metric d_p^W in terms of d_n^∞ (in place of d_n in the above). For the Lipschitz continuity of $\mathcal{A}(\mu_1, \dots, \mu_n)$ we have, for $1 \leq p < \infty$,

$$\begin{aligned} &d_p^W((\mathcal{A}_n)_*(\mu_1 \times \dots \times \mu_n), (\mathcal{A}_n)_*(\nu_1 \times \dots \times \nu_n)) \\ &\leq d_p^W(\mu_1 \times \dots \times \mu_n, \nu_1 \times \dots \times \nu_n) \\ &\leq \left[\int_{\mathbb{P}^n \times \mathbb{P}^n} \max_{1 \leq j \leq n} d_T^p(A_j, B_j) d(\pi_1 \times \dots \times \pi_n) \right]^{1/p} \end{aligned}$$

$$\leq \left[\sum_{j=1}^n \int_{\mathbb{P} \times \mathbb{P}} d_T^p(A_j, B_j) d\pi_j(A_j, B_j) \right]^{1/p},$$

which implies that

$$d_p^W(\mathcal{A}(\mu_1, \dots, \mu_n), \mathcal{A}(\nu_1, \dots, \nu_n)) \leq \left[\sum_{j=1}^n \left(d_p^W(\mu_j, \nu_j) \right)^p \right]^{1/p}.$$

For $p = \infty$, we similarly have

$$d_\infty^W(\mathcal{A}(\mu_1, \dots, \mu_n), \mathcal{A}(\nu_1, \dots, \nu_n)) \leq \max_{1 \leq j \leq n} d_\infty^W(\mu_j, \nu_j).$$

The proof for $\mathcal{H}(\mu_1, \dots, \mu_n)$ is analogous, or we may use Proposition 7.3. \square

The next theorem is the AGH mean inequalities in the stochastic order for probability measures.

Theorem 7.5. *For any $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathbb{P})$,*

$$\mathcal{H}(\mu_1, \dots, \mu_n) \leq \Lambda(\mu_1, \dots, \mu_n) \leq \mathcal{A}(\mu_1, \dots, \mu_n).$$

Proof. The assertion immediately follows from Proposition 3.12 if applied to the maps $\mathcal{H}_n, \Lambda_n, \mathcal{A}_n : \mathbb{P}^n \rightarrow \mathbb{P}$ in view of the AGH mean inequalities for operators. \square

Lemma 7.6. *Let X_1, \dots, X_n be metric spaces equipped with a closed partial order. Let $X_1 \times \dots \times X_n$ be the product metric space endowed with the product order. Then the stochastic order on $\mathcal{P}(X_1) \times \dots \times \mathcal{P}(X_n)$ as a subset of $\mathcal{P}(X_1 \times \dots \times X_n)$ is the product of the stochastic orders, that is, if $\mu_j, \nu_j \in \mathcal{P}(X_j)$ for $1 \leq j \leq n$, then $\mu_1 \times \dots \times \mu_n \leq \nu_1 \times \dots \times \nu_n$ if and only if $\mu_j \leq \nu_j$ for all $j = 1, \dots, n$.*

Proof. Let $\mu_j, \nu_j \in \mathcal{P}(X_j)$ for $1 \leq j \leq n$. If $\mu_1 \times \dots \times \mu_n \leq \nu_1 \times \dots \times \nu_n$, then for every upper Borel set B_j in X_j ,

$$\mu_j(B_j) = (\mu_1 \times \dots \times \mu_n)(p_j^{-1}(B_j)) \leq (\nu_1 \times \dots \times \nu_n)(p_j^{-1}(B_j)) = \nu_j(B_j),$$

where p_i is the projection of $X_1 \times \dots \times X_n$ onto X_i . Hence $\mu_j \leq \nu_j$, $1 \leq j \leq n$. Conversely, assume that $\mu_j \leq \nu_j$ for all j . For every monotone bounded Borel function $f : X_1 \times \dots \times X_n \rightarrow \mathbb{R}^+$, by using Fubini's theorem we have

$$\begin{aligned} \int f d(\mu_1 \times \dots \times \mu_n) &= \int \left(\int f(x_1, x_2, \dots, x_n) d\mu_1(x_1) \right) d(\mu_2 \times \dots \times \mu_n) \\ &\leq \int \left(\int f(x_1, x_2, \dots, x_n) d\nu_1(x_1) \right) d(\mu_2 \times \dots \times \mu_n) \\ &= \int f d(\nu_1 \times \mu_2 \times \dots \times \mu_n), \end{aligned}$$

which implies by Proposition 3.6 that $\mu_1 \times \mu_2 \times \dots \times \mu_n \leq \nu_1 \times \mu_2 \times \dots \times \mu_n$. Repeating the argument shows that $\nu_1 \times \mu_2 \times \dots \times \mu_n \leq \nu_1 \times \nu_2 \times \mu_3 \times \dots \times \mu_n$ and so on. Hence $\mu_1 \times \dots \times \mu_n \leq \nu_1 \times \dots \times \nu_n$. \square

Theorem 7.7. *The maps $\Lambda, \mathcal{A}, \mathcal{H} : (\mathcal{P}(\mathbb{P}))^n \rightarrow \mathcal{P}(\mathbb{P})$ are monotonically increasing in the sense that if $\mu_j, \nu_j \in \mathcal{P}(\mathbb{P})$ and $\mu_j \leq \nu_j$ for $1 \leq j \leq n$, then $M(\mu_1, \dots, \mu_n) \leq M(\nu_1, \dots, \nu_n)$ for $M = \Lambda, \mathcal{A}, \mathcal{H}$.*

Proof. Since $\Lambda_n, \mathcal{A}_n, \mathcal{H}_n : \mathbb{P}^n \rightarrow \mathbb{P}$ is monotone, the theorem immediately follows from Lemma 7.6 and Proposition 3.13. \square

Remark 7.8. One can apply the arguments in this section to other multivariate operator means of $(A_1, \dots, A_n) \in \mathbb{P}^n$ having the monotonicity property. For instance, let $P_t(A_1, \dots, A_n)$ for $t \in [-1, 1]$ be the one-parameter family of multivariate power means interpolating $\mathcal{H}_n, \Lambda_n, \mathcal{A}_n$ as $P_{-1} = \mathcal{H}_n, P_0 = \Lambda_n$ and $P_1 = \mathcal{A}_n$. The power mean $P_t(A_1, \dots, A_n)$ for $t \in (0, 1]$ is defined by the unique positive definite solution of $X = \frac{1}{n} \sum_{j=1}^n X \#_t A_j$, where $A \#_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}$ denotes the t -weighted geometric mean of A and B . It is monotonic and Lipschitz

$$d_T(P_t(A_1, \dots, A_n), P_t(B_1, \dots, B_n)) \leq \max_{1 \leq j \leq n} d_T(A_j, B_j).$$

Moreover, $P_t(A_1, \dots, A_n)$ is monotonically increasing in $t \in [-1, 1]$ and

$$\lim_{t \rightarrow 0} P_t(A_1, \dots, A_n) = \Lambda_n(A_1, \dots, A_n). \quad (7.6)$$

For power means, see [12] for positive definite matrices and [9,10] for positive operators on an infinite-dimensional Hilbert space. Then one has the one-parameter family of $P_t(\mu_1, \dots, \mu_n)$ for $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathbb{P})$ so that each $P_t(\mu_1, \dots, \mu_n)$ is monotonically increasing in μ_1, \dots, μ_n as in Theorem 7.7 and $P_t(\mu_1, \dots, \mu_n)$ is monotonically increasing in t , extending the AGH mean inequalities in Theorem 7.5. In particular

$$P_s(\mu_1, \dots, \mu_n) \leq \Lambda(\mu_1, \dots, \mu_n) \leq P_t(\mu_1, \dots, \mu_n)$$

for $-1 \leq s < 0 < t \leq 1$.

Now assume that \mathbb{P} is the cone of positive definite matrices of some fixed dimension, and let $\mu_j \in \mathcal{P}^1(\mathbb{P})$, $1 \leq j \leq n$. For any continuous bounded and monotone $f : \mathbb{P} \rightarrow \mathbb{R}^+$ we see by (7.6) that

$$\int_{\mathbb{P}} f dP_t(\mu_1, \dots, \mu_n) = \int_{\mathbb{P}^n} (f \circ P_t)(A_1, \dots, A_n) d(\mu_1 \times \dots \times \mu_n)(A_1, \dots, A_n)$$

increases as $t \nearrow 0$ and decreases as $t \searrow 0$ to

$$\int_{\mathbb{P}^n} (f \circ \Lambda_n)(A_1, \dots, A_n) d(\mu_1 \times \dots \times \mu_n)(A_1, \dots, A_n) = \int_{\mathbb{P}} f d\Lambda(\mu_1, \dots, \mu_n).$$

Hence by Corollary 6.5,

$$\lim_{t \rightarrow 0} d_1^W(P_t(\mu_1, \dots, \mu_n), \Lambda(\mu_1, \dots, \mu_n)) = 0.$$

It would be interesting to know whether this convergence holds true in the infinite-dimensional case as well.

Remark 7.9. Several issues arise related to $\Lambda(\mu_1, \dots, \mu_n)$. For example, it is interesting to consider existence and uniqueness for the least squares mean on $\mathcal{P}^1(\mathbb{P})$;

$$\arg \min_{\mu \in \mathcal{P}^1(\mathbb{P})} \sum_{j=1}^n d_1^W(\mu, \mu_j)^2$$

and a connection with the probability measure $\Lambda(\mu_1, \dots, \mu_n)$. Moreover, the Borel probability measure equation

$$\mu = \mathcal{A}(\mu \#_t \mu_1, \dots, \mu \#_t \mu_n), \quad \mu_j \in \mathcal{P}_{cp}(\mathbb{P}), \quad t \in (0, 1],$$

where $\mu \#_t \nu = f_*(\mu \times \nu)$ is the push-forward by the t -weighted geometric mean map $f(A, B) = A \#_t B$, seems to have a unique solution in $\mathcal{P}_{cp}(\mathbb{P})$, the set of probability measures with compact support. See [6] for $n = 1$.

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