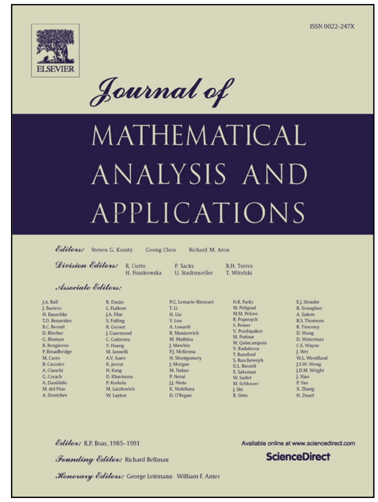

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Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity

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Abstract

In this paper, we study the blow-up of solutions for semilinear wave equations with scale-invariant dissipation and mass in the case in which the model is somehow "wave-like". A Strauss type critical exponent is determined as the upper bound for the exponent in the nonlinearity in the main theorems. Two blow-up results are obtained for the subcritical case and for the critical case, respectively. In both cases, an upper bound lifespan estimate is given.

Keywords: Semilinear wave equation; Strauss exponent; Blow-up; Lifespan.
2010 MSC: Primary 35L71; Secondary 35B44, 35L05

1. Introduction and main results

In present paper, we consider the following strictly hyperbolic model

$$\begin{aligned} u_{tt}-\Delta u+\frac{\mu_1}{1+t}u_t+\frac{\mu_2^2}{(1+t)^2}u &= |u|^p \qquad && (x,t)\ \in\ \mathbb{R}^n\times[0,\infty), \\ u(0,x) &= \varepsilon\, f(x) && x\ \in\ \mathbb{R}^n, \\ u_t(0,x) &= \varepsilon\, g(x) && x\ \in\ \mathbb{R}^n. \end{aligned} \tag{1}$$

where $\mu_1,\ \mu_2^2\geqslant 0$ and $\varepsilon>0$ is a parameter that describes the smallness of initial data. The time-dependent coefficients for the damping and for the mass term are chosen in order to have for the corresponding linear equation

$$u_{tt}-\Delta u+\frac{\mu_1}{1+t}u_t+\frac{\mu_2^2}{(1+t)^2}u=0 \tag{2}$$

a scaling property. More precisely, (2) is invariant with respect to the so-called hyperbolic transformation

$$\widetilde{u}(t,x)=u(\lambda(1+t)-1,\lambda x)\quad \text{with } \lambda>0.$$

In the last years, (1) has been studied in [16, 18, 21, 22, 4, 19, 20].

It turns out that the quantity

$$\delta:=(\mu_1-1)^2-4\mu_2^2$$

describes the interplay between the damping and the mass term in (1). For further considerations on this interplay cf. [16, 22, 4].

Combining the results from [16, 18], it follows that the shift of the Fujita exponent $p_F\left(n+\frac{\mu_1-1-\sqrt{\delta}}{2}\right)$ is the critical exponent for (1) in the case $\delta\geqslant(n+1)^2$, where $p_F(n):=1+\frac{2}{n}$. Therefore, (1) is "parabolic-like"

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from the point of view of the critical exponent for "large" δ. On the other hand, in [22] it has been proved a blow-up result for $\delta \in (0,1]$ provided that

$$1 < p \leqslant \max\left\{p_S(n+\mu_1), p_F\left(n+\tfrac{\mu_1-1-\sqrt{\delta}}{2}\right)\right\}$$

with the exception of the critical case $p = p_S(n+\mu_1)$ in dimension $n=1$. In the preceding condition $p_S(r)$ denotes the so-called Strauss exponent, that is, the positive root of the quadratic equation

$$\gamma(p,r) := 2 + (r+1)p - (r-1)p^2 = 0 \quad \text{for } r > 1. \tag{3}$$

Briefly, in [22] a suitable change of variables allows transforming (1) in a semilinear wave equation with time-dependent speed of propagation. Hence, a suitable test function, involving the modified Bessel function of the second kind, and Kato's lemma are used. Consequently, we see that for small and positive δ, using the same jargon as before, (1) seems to be "wave-like", at least concerning blow-up results.

The goal of this paper is to enlarge the range of δ for which a blow-up result can be proved for $1 < p \leqslant p_S(n+\mu_1)$. Furthermore, upper bound estimates for the lifespan of the local (in time) solution of (1) are derived.

In the subcritical case we combine the approach from [29], in order to determine a lower bound for the integral with respect to spatial variables of the nonlinearity, and an iteration method introduced in [10] for the semilinear free wave equation in dimension $n=3$ and very recently applied to several different models (see [11, 12, 13, 25], for example).

In the critical case, we adapt the approach of [9], which is based in turn on that one of [32], in order to include the scale-invariant mass term.

We briefly recall some related background concerning model (1). When $\mu_1 = \mu_2 = 0$, this model reduces to the classic semilinear wave equation. In this case, the Strauss exponent $p_S(n)$ is known to be the critical exponent. We refer to the classical works [10, 7, 30, 15, 6] for small data global existence results when $p > p_S(n)$, and [10, 8, 24, 23, 29, 31] for the blow-up results when $1 < p \leqslant p_S(n)$.

When $\mu_2 = 0$, model (1) is reduced to the scale invariant damping wave equation which has drawn more and more attention recently. As mentioned in [28], such type damping is a threshold betweeen "effective" and "non-effective" dampings. Moreover, the size of μ_1 plays an important role in determining the solution behavior type. In [1, 27] it is proved that $p_F(n)$ is critical for sufficiently large μ_1, while for $\mu_1 < \mu^* := \frac{n^2+n+2}{n+2}$ in [3, 14, 9, 25, 26] several blow-up results are given for $p \leqslant p_S(n+\mu_1)$. We note that μ^* satisfies the identity $p_F(n) = p_S(n+\mu^*)$. In particular, in [26] a different test function from that of [9] is used in the critical case. Finally, some global (in time) existence results of small data solutions are proved for $\mu_1 = 2$ in [3, 2].

We state now the main results of this paper. According to [14], we introduce a notion of energy solution in the following way.

Definition 1.1. *Let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. We say that u is an energy solution of (1) on $[0,T)$ if*

$$u \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)) \cap L^p_{loc}(\mathbb{R}^n \times [0,T))$$

satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(t,x)\phi(t,x)\,dx - \int_{\mathbb{R}^n} u_t(0,x)\phi(0,x)\,dx - \int_0^t \int_{\mathbb{R}^n} u_t(s,x)\phi_t(s,x)\,dx\,ds \\
&+ \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x)\cdot\nabla\phi(s,x)\,dx\,ds + \int_0^t \int_{\mathbb{R}^n} \left(\frac{\mu_1}{1+s}u_t(s,x) + \frac{\mu_2^2}{(1+s)^2}u(s,x)\right)\phi(s,x)\,dx\,ds \\
&= \int_0^t \int_{\mathbb{R}^n} |u(s,x)|^p \phi(s,x)\,dx\,ds
\end{aligned} \tag{4}$$

for any $\phi \in C_0^\infty([0,T) \times \mathbb{R}^n)$ and any $t \in [0,T)$.

After a further integration by parts in (4), letting $t \to T$, we find that u fulfills the definition of weak solution of (1).

Our main results are the following two theorems, where we study the subcritical case and the critical case, respectively.

Theorem 1.2. *Let $n \geqslant 1$ and let μ_1, μ_2^2 be nonnegative constants such that $\delta \geqslant 0$. Let us consider p satisfying $1 < p < p_S(n+\mu_1)$.*

Assume that $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are compactly supported in $B_R := \{x \in \mathbb{R}^n : |x| \leqslant R\}$ and

$$f(x) \geqslant 0 \text{ and } g(x) + \tfrac{\mu_1 - 1 - \sqrt{\delta}}{2} f(x) \geqslant 0. \tag{5}$$

Let u be an energy solution of (1) with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, \mu_1, \mu_2^2, R)$ such that $T(\varepsilon)$ fulfills

$$T(\varepsilon) \leqslant C\varepsilon^{-2p(p-1)/\gamma(p,n+\mu_1)}$$

for any $0 < \varepsilon \leqslant \varepsilon_0$, where C is a positive constant independent of ε.

Theorem 1.3. *Let $n \geqslant 1$ and let μ_1, μ_2^2 be nonnegative constants such that $0 \leqslant \delta < n^2$. Let us consider $p = p_S(n+\mu_1)$. Furthermore, we assume $p > \frac{2}{n-\sqrt{\delta}}$.*

Let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ be nonnegative, not identically zero and compactly supported in B_R for some $R < 1$.

Let us consider an energy solution u of (1) with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, \mu_1, \mu_2^2, R)$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$ the solution u blows up in finite time. Furthermore, it holds the following upper bound estimate for the lifespan $T = T(\varepsilon)$ of u:

$$T(\varepsilon) \leqslant \exp(C\varepsilon^{-p(p-1)}) \tag{6}$$

for some constant C which is independent of ε.

The remaining part of the paper is arranged as follows. In Section 2, we construct the test function that will be employed in the proof of Theorem 1.2. Furthermore, a lower bound for the p norm of the solution of (1) is derived. This lower bound will play in turn a fundamental role in the derivation of the lower bound for the time-dependent functional that we will consider in the proof of Theorem 1.2. Similarly, in Section 4 we deal with the construction of a different test function, involving Gauss hypergeometric function, and we prove some preliminary results to the proof of Theorem 1.3. In Sections 3 and 5, we provide the proofs of Theorems 1.2 and 1.3, respectively.

2. Test function and preliminaries: subcritical case

Before starting with the construction of the test functions, we recall the definition of the modified Bessel function of the second kind of order ν

$$K_\nu(t) = \int_0^\infty \exp(-t\cosh z)\cosh(\nu z)dz, \quad \nu \in \mathbb{R}$$

which is a solution of the equation

$$\left(t^2 \frac{d^2}{dt^2} + t\frac{d}{dt} - (t^2 + \nu^2)\right)K_\nu(t) = 0, \quad t > 0.$$

We collect some important properties concerning $K_\nu(t)$ in the case in which ν is a real parameter. Interested reader may refer to [5].

- The limiting behavior of $K_\nu(t)$:

$$K_\nu(t) = \sqrt{\frac{\pi}{2t}}\, e^{-t}[1 + O(t^{-1})] \quad \text{as} \quad t \to \infty. \tag{7}$$

- The derivative identity:

$$\frac{d}{dt}K_\nu(t) = -K_{\nu+1}(t) + \frac{\nu}{t}K_\nu(t). \tag{8}$$

Firstly, we set the auxiliary function with respect to the time variable

$$\lambda(t) := (1+t)^{\frac{\mu_1+1}{2}} K_{\frac{\sqrt{\delta}}{2}}(1+t), \quad t \geqslant 0.$$

It is clear by direct computation that $\lambda(t)$ satisfies

$$\begin{cases} \left((1+t)^2\dfrac{d^2}{dt^2} - \mu_1(1+t)\dfrac{d}{dt} + (\mu_1 + \mu_2^2 - (1+t)^2)\right)\lambda(t) = 0, & t > 0. \\ \lambda(0) = K_{\frac{\sqrt{\delta}}{2}}(1), \quad \lambda(\infty) = 0. \end{cases} \tag{9}$$

Following [29], let us introduce the function

$$\varphi(x) := \begin{cases} \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega} d\omega & \text{for} \quad n \geqslant 2, \\ e^x + e^{-x} & \text{for} \quad n = 1. \end{cases}$$

The function φ satisfies

$$\Delta\varphi(x) = \varphi(x)$$

and the asymptotic estimate

$$\varphi(x) \sim C_n|x|^{-\frac{n-1}{2}} e^{|x|} \quad \text{as} \quad |x| \to \infty. \tag{10}$$

We define the test function for the subcritical case

$$\psi(t,x) := \lambda(t)\varphi(x).$$

In the following lemma, we derive a lower bound for $\int_{\mathbb{R}^n} |u(x,t)|^p dx$.

Lemma 2.1. *Let us assume f, g such that* $\operatorname{supp} f$, $\operatorname{supp} g \subset B_R$ *for some $R > 0$ and* (5) *is fulfilled. Then, a local energy solution u satisfies*

$$\operatorname{supp} u \ \subset \{(t,x) \in [0,T) \times \mathbb{R}^n : |x| \leqslant t + R\}$$

and there exists a large T_0, which is independent of f, g and ε, such that for any $t > T_0$ and $p > 1$, it holds

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geqslant C_1\varepsilon^p(1+t)^{n-1-\frac{n+\mu_1-1}{2}p}, \tag{11}$$

where $C_1 = C_1(f,g,\varphi,p,R) > 0$.

Proof. Define the functional

$$F(t) := \int_{\mathbb{R}^n} u(t,x)\psi(t,x)dx$$

with $\psi(t,x) = \lambda(t)\varphi(x)$ defined as above. Then, by Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geqslant |F(t)|^p \left(\int_{|x|\leqslant t+R} \psi^{p'}(t,x)dx\right)^{-(p-1)}. \tag{12}$$

The next step is to determine a lower bound for $|F(t)|$ and an upper bound for the integral $\int_{|x|\leqslant t+R} \psi^{p'}(t,x)dx$, respectively. From the definition of energy solution, we have

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^n} u_{tt}\psi \, dxds - \int_0^t \int_{\mathbb{R}^n} u\Delta\psi \, dxds \\ &+ \int_0^t \int_{\mathbb{R}^n} \Big(\partial_s\Big(\frac{\mu_1}{1+s}\psi u\Big) - \partial_s\Big(\frac{\mu_1}{1+s}\psi\Big)u + \frac{\mu_2^2}{(1+s)^2}\psi u\Big)dxds = \int_0^t \int_{\mathbb{R}^n} |u|^p\psi \, dxds.\end{aligned}$$

Applying integration by parts and $\Delta\varphi = \varphi$, we obtain:

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^n} u_{tt}\psi \, dxds + \int_0^t \int_{\mathbb{R}^n} u\varphi\Big(-\lambda + \frac{\mu_1+\mu_2^2}{(1+s)^2}\lambda - \frac{\mu_1}{1+s}\lambda'\Big)dxds \\ &+ \int_{\mathbb{R}^n} \frac{\mu_1}{1+s}\psi \, udx\Big|_0^t = \int_0^t \int_{\mathbb{R}^n} |u|^p\psi \, dxds.\end{aligned}$$

Simplifying the above equation by plugging (9) gives

$$\int_0^t \int_{\mathbb{R}^n} u_{tt}\psi \, dxds - \int_0^t \int_{\mathbb{R}^n} u\varphi\lambda'' dxds + \int_{\mathbb{R}^n} \frac{\mu_1}{1+s}\psi u \, dx\Big|_0^t = \int_0^t \int_{\mathbb{R}^n} |u|^p\psi \, dxds.$$

Hence, a further integration by parts leads to

$$\int_{\mathbb{R}^n} \Big(u_t\psi - u\psi_t + \frac{\mu_1}{1+s}u\psi\Big)dx\Big|_0^t = \int_0^t \int_{\mathbb{R}^n} |u|^p\psi \, dxds.$$

As the righthand side integral is positive, we obtain

$$F'(t) + \left(\frac{\mu_1}{1+t} - 2\frac{\lambda'(t)}{\lambda(t)}\right)F(t) \geqslant \varepsilon \int_{\mathbb{R}^n} \Big(g(x)\lambda(0) + (\mu_1\lambda(0) - \lambda'(0))f(x)\Big)\varphi(x)\, dx.$$

Using (8), we have

$$\begin{aligned}\lambda'(t) &= \tfrac{\mu_1+1}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\frac{\sqrt{\delta}}{2}}(1+t) + (1+t)^{\frac{\mu_1+1}{2}} K'_{\frac{\sqrt{\delta}}{2}}(1+t) \\ &= \tfrac{\mu_1+1}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\frac{\sqrt{\delta}}{2}}(1+t) + (1+t)^{\frac{\mu_1+1}{2}} \left(-K_{\frac{\sqrt{\delta}}{2}+1}(1+t) + \tfrac{\sqrt{\delta}}{2(1+t)} K_{\frac{\sqrt{\delta}}{2}}(1+t)\right) \\ &= \tfrac{\mu_1+1+\sqrt{\delta}}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\frac{\sqrt{\delta}}{2}}(1+t) - (1+t)^{\frac{\mu_1+1}{2}} K_{\frac{\sqrt{\delta}}{2}+1}(1+t),\end{aligned}$$

Also,

$$\begin{aligned}\lambda'(0) &= \tfrac{\mu_1+1+\sqrt{\delta}}{2} K_{\frac{\sqrt{\delta}}{2}}(1) - K_{\frac{\sqrt{\delta}}{2}+1}(1), \\ \mu_1\lambda(0) - \lambda'(0) &= \tfrac{\mu_1-1-\sqrt{\delta}}{2} K_{\frac{\sqrt{\delta}}{2}}(1) + K_{\frac{\sqrt{\delta}}{2}+1}(1).\end{aligned}$$

Consequently,

$$g(x)\lambda(0) + (\mu_1\lambda(0) - \lambda'(0))f(x) = K_{\frac{\sqrt{\delta}}{2}}(1)\left(g(x) + \tfrac{\mu_1-1-\sqrt{\delta}}{2}f(x)\right) + K_{\frac{\sqrt{\delta}}{2}+1}(1)f(x).$$

Denote

$$C_{f,g} := \int_{\mathbb{R}^n} \Big(g(x)\lambda(0) + (\mu_1\lambda(0) - \lambda'(0))f(x)\Big)\varphi(x)\, dx,$$

then, since we assume compactly supported and satisfying (5) f and g , $C_{f,g}$ is finite and positive. We thus conclude that F satisfies the differential inequality

$$F'(t) + \left(\frac{\mu_1}{1+t} - 2\frac{\lambda'(t)}{\lambda(t)}\right)F(t) \geqslant \varepsilon\, C_{f,g}.$$

Multiplying $\frac{(1+t)^{\mu_1}}{\lambda^2(t)}$ on two sides and then integrating over $[0,t]$, we derive

$$F(t) \geqslant \varepsilon\, C_{f,g} \frac{\lambda^2(t)}{(1+t)^{\mu_1}} \int_0^t \frac{(1+s)^{\mu_1}}{\lambda^2(s)}\, ds.$$

Inserting $\lambda(t) = (1+t)^{\frac{\mu_1+1}{2}} K_{\frac{\sqrt{\delta}}{2}}(1+t)$, we obtain the lower bound for F

$$F(t) \geqslant \varepsilon\, C_{f,g} \int_0^t \frac{(1+t)K^2_{\frac{\sqrt{\delta}}{2}}(1+t)}{(1+s)K^2_{\frac{\sqrt{\delta}}{2}}(1+s)}\, ds \geqslant 0. \tag{13}$$

The second factor in the right-hand side of (12) can be estimated in standard way (cf. [29, estimate (2.5)])

$$\begin{aligned} \int_{|x|\leqslant t+R} \psi^{p'}(t,x)\, dx &\leqslant \lambda^{\frac{p}{p-1}}(t) \int_{|x|\leqslant t+R} \varphi^{p'}(x)\, dx \\ &\leqslant C_{\varphi,R}(1+t)^{n-1+\left(\frac{\mu_1+1}{2}-\frac{n-1}{2}\right)\frac{p}{p-1}} e^{\frac{p}{p-1}(t+R)} K^{\frac{p}{p-1}}_{\frac{\sqrt{\delta}}{2}}(1+t), \end{aligned} \tag{14}$$

where $C_{\varphi,R}$ is a suitable positive constant.

Combing the estimate (13), (14) and (12), we now have

$$\begin{aligned} &\int_{\mathbb{R}^n} |u(t,x)|^p dx \\ &\geqslant C^p_{f,g} C^{1-p}_{\varphi,R} \varepsilon^p (1+t)^{p-(n-1)(p-1)-\left(\frac{\mu_1+1}{2}-\frac{n-1}{2}\right)p} e^{-p(t+R)} K^p_{\frac{\sqrt{\delta}}{2}}(1+t) \left(\int_0^t \frac{ds}{(1+s)K^2_{\frac{\sqrt{\delta}}{2}}(1+s)}\right)^p \\ &\geqslant C^p_{f,g} C^{1-p}_{\varphi,R} e^{p(1-R)} \varepsilon^p (1+t)^{(2-n-\mu_1)\frac{p}{2}+(n-1)} e^{-p(1+t)} K^p_{\frac{\sqrt{\delta}}{2}}(1+t) \left(\int_0^t \frac{ds}{(1+s)K^2_{\frac{\sqrt{\delta}}{2}}(1+s)}\right)^p. \end{aligned}$$

Since (7), then for sufficient large T_0 (which is independent of $f,\ g,\ \varepsilon$) and $t > T_0$, we have

$$K^p_{\frac{\sqrt{\delta}}{2}}(1+t) \sim \left(\frac{\pi}{2(1+t)}\right)^{\frac{p}{2}} e^{-p(t+1)}$$

and

$$\int_0^t \frac{1}{(1+s)K^2_{\frac{\sqrt{\delta}}{2}}(1+s)} ds \geqslant \int_{\frac{t}{2}}^t \frac{2}{\pi} e^{2(1+s)} ds = \frac{1}{\pi}\left(e^{2(1+t)} - e^{2+t}\right) \geqslant \frac{1}{2\pi} e^{2(1+t)}.$$

Consequently,

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geqslant C_1 \varepsilon^p (1+t)^{\frac{p}{2}(1-n-\mu_1)+(n-1)} \quad \text{for} \quad t > T_0,$$

where $C_1 := 2^{-3p/2} C^p_{f,g} C^{1-p}_{\varphi,R} e^{p(1-R)} \pi^{-p/2}$. This concludes the proof. □

3. Proof of Theorem 1.2

Let u be an energy solution of (1) on $[0,T)$ and define

$$G(t) := \int_{\mathbb{R}^n} u(t,x)\, dx.$$

Choosing a $\phi = \phi(s, x)$ in (4) that satisfies $\phi \equiv 1$ in $\{(x, s) \in [0, t] \times \mathbb{R}^n : |x| \leqslant s + R\}$, we obtain

$$\int_{\mathbb{R}^n} u_t(t,x)\,dx - \int_{\mathbb{R}^n} u_t(0,x)\,dx + \int_0^t ds \int_{\mathbb{R}^n} \left(\frac{\mu_1 u_t(s,x)}{1+s} + \frac{\mu_2^2 u(s,x)}{(1+s)^2}\right) dx$$
$$= \int_0^t ds \int_{\mathbb{R}^n} |u(s,x)|^p dx$$

which means that

$$G'(t) - G'(0) + \int_0^t \frac{\mu_1 G'(s)}{1+s}\,ds + \int_0^t \frac{\mu_2^2 G(s)}{(1+s)^2}\,ds = \int_0^t ds \int_{\mathbb{R}^n} |u(s,x)|^p dx.$$

Since all functions in this equation aside from $G'(t)$ are differentiable in t, $G'(t)$ is differentiable in t as well. Hence, we have

$$G''(t) + \frac{\mu_1}{1+t} G'(t) + \frac{\mu_2^2}{(1+t)^2} G(t) = \int_{\mathbb{R}^n} |u(t,x)|^p dx. \tag{15}$$

Consider the quadratic equation

$$r^2 - (\mu_1 - 1)r + \mu_2^2 = 0.$$

As $\delta \geqslant 0$, there exit two real roots,

$$r_1 = \frac{\mu_1 - 1 - \sqrt{\delta}}{2}, \quad r_2 = \frac{\mu_1 - 1 + \sqrt{\delta}}{2}.$$

Clearly, if $\mu_1 > 1$ then both r_1 and r_2 are positive. Else, if $0 \leqslant \mu_1 < 1$, both r_1 and r_2 are negative. When $\mu_1 = 1$ then $\mu_2 = 0$ as $\delta \geqslant 0$, and hence $r_1 = r_2 = 0$. Moreover, in whatever situation

$$r_{1,2} + 1 > 0.$$

We may rewrite (15) as

$$\left(G'(t) + \frac{r_1}{1+t} G(t)\right)' + \frac{r_2+1}{1+t}\left(G'(t) + \frac{r_1}{1+t} G(t)\right) = \int_{\mathbb{R}^n} |u(t,x)|^p dx.$$

Multiplying by $(1+t)^{r_2+1}$ and integrating over $[0, t]$, we obtain

$$(1+t)^{r_2+1}\left(G'(t) + \frac{r_1}{1+t} G(t)\right) - \left(G'(0) + r_1 G(0)\right) = \int_0^t (1+s)^{r_2+1} ds \int_{\mathbb{R}^n} |u|^p dx.$$

Using (5), we have

$$G'(t) + \frac{r_1}{1+t} G(t) > (1+t)^{-r_2-1} \int_0^t (1+s)^{r_2+1} ds \int_{\mathbb{R}^n} |u|^p dx. \tag{16}$$

Multiplying the above inequality by $(1+t)^{r_1}$ and integrating over $[0, t]$ gives

$$(1+t)^{r_1} G(t) - G(0) > \int_0^t (1+\tau)^{r_1 - r_2 - 1} d\tau \int_0^\tau (1+s)^{r_2+1} ds \int_{\mathbb{R}^n} |u(s,x)|^p dx.$$

By the positivity assumption on f, we have

$$G(t) \geqslant \int_0^t \left(\frac{1+\tau}{1+t}\right)^{r_1} d\tau \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{r_2+1} ds \int_{\mathbb{R}^n} |u(s,x)|^p dx. \tag{17}$$

Furthermore, using Hölder inequality and the compactness of the support of solution with respect to x, we get from (17)

$$G(t) \geqslant C_0 \int_0^t \left(\frac{1+\tau}{1+t}\right)^{r_1} d\tau \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{r_2+1} (1+s)^{n(1-p)} |G(s)|^p ds \tag{18}$$

where

$$C_0 := (\mathrm{meas}(B_1))^{1-p} R^{-n(p-1)} > 0.$$

At this moment, we are ready to prove Theorem 1.2. We shall apply an iteration method based on lower bound estimates (11), (17) and (18).

Proof of Theorem 1.2. Plugging (11) into (17), we have for $t > T_0$,

$$\begin{aligned} G(t) &\geqslant \int_{T_0}^t \left(\frac{1+\tau}{1+t}\right)^{r_1} d\tau \int_{T_0}^\tau \left(\frac{1+s}{1+\tau}\right)^{r_2+1} C_1 \varepsilon^p (1+s)^{n-1-\frac{n+\mu_1-1}{2}p} ds \\ &\geqslant C_1 \varepsilon^p (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1} d\tau \int_{T_0}^\tau (1+s)^{n+r_2-(n+\mu_1-1)\frac{p}{2}} ds \\ &\geqslant C_1 \varepsilon^p (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1-(n+\mu_1-1)\frac{p}{2}} d\tau \int_{T_0}^\tau (1+s)^{n+r_2} ds \\ &\geqslant C_1 \varepsilon^p (1+t)^{-r_2-1-(n+\mu_1-1)\frac{p}{2}} \int_{T_0}^t d\tau \int_{T_0}^\tau (s-T_0)^{n+r_2} ds. \end{aligned} \tag{19}$$

That is,

$$G(t) \geqslant C_2\, \varepsilon^p (1+t)^{-r_2-1-(n+\mu_1-1)\frac{p}{2}} (t-T_0)^{n+r_2+2} \quad \text{for} \quad t > T_0, \tag{20}$$

where $C_2 = \frac{C_1}{(n+r_2+1)(n+r_2+2)}$. Notice that, in (19) we may simply use the property

$$r_1 - r_2 - 1 - (n+\mu_1-1)\frac{p}{2} \leqslant 0.$$

Now we begin our iteration argument. Assume that

$$G(t) \geqslant D_j (1+t)^{-a_j} (t-T_0)^{b_j} \quad \text{for} \quad t > T_0, \quad j = 1, 2, 3 \dots \tag{21}$$

with positive constants D_j, a_j and b_j to be determined later. From (20) it follows that (21) is true for $j = 1$ with

$$D_1 = C_2 \varepsilon^p, \quad a_1 = r_2 + 1 + (n+\mu_1-1)\frac{p}{2}, \quad b_1 = n + r_2 + 2. \tag{22}$$

Plugging (21) into (18), we have for $t > T_0$

$$\begin{aligned} G(t) &\geqslant C_0\, (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1} d\tau \int_{T_0}^\tau (1+s)^{r_2+1+n(1-p)} D_j^p (1+s)^{-pa_j} (s-T_0)^{pb_j} ds \\ &\geqslant C_0\, D_j^p (1+t)^{-r_2-1-n(p-1)-pa_j} \int_{T_0}^t \int_{T_0}^\tau (s-T_0)^{r_2+1+pb_j} ds d\tau \\ &\geqslant \frac{C_0 D_j^p}{(r_2+pb_j+2)(r_2+pb_j+3)} (1+t)^{-r_2-1-n(p-1)-pa_j} (t-T_0)^{r_2+pb_j+3}. \end{aligned} \tag{23}$$

where in (23), we utilize

$$r_1 - r_2 - 1 - n(p-1) - pa_j \leqslant 0.$$

So (21) is true if the sequences $\{D_j\}_{j\geqslant 1}$, $\{a_j\}_{j\geqslant 1}$, $\{b_j\}_{j\geqslant 1}$ fulfill

$$D_{j+1} \geqslant \frac{C_0}{(r_2+pb_j+3)^2} D_j^p, \tag{24}$$

$$a_{j+1} = r_2 + 1 + n(p-1) + pa_j, \quad b_{j+1} = r_2 + 3 + pb_j. \tag{25}$$

It follows from (22) and (25) that for $j = 1, 2, 3\dots$

$$a_j = \left(a_1 + n + \frac{r_2+1}{p-1}\right)p^{j-1} - \left(n + \frac{r_2+1}{p-1}\right) = \alpha p^{j-1} - \left(n + \frac{r_2+1}{p-1}\right), \tag{26}$$

$$b_j = \left(b_1 + \frac{r_2+3}{p-1}\right)p^{j-1} - \frac{r_2+3}{p-1} = \beta p^{j-1} - \frac{r_2+3}{p-1}, \tag{27}$$

where we denote the positive constants

$$\alpha = r_2 + 1 + (n + \mu_1 - 1)\frac{p}{2} + n + \frac{r_2+1}{p-1}, \quad \beta = n + r_2 + 2 + \frac{r_2+3}{p-1}.$$

Using (25) and (27), we get

$$b_{j+1} = r_2 + 3 + pb_j < p^j\beta.$$

Therefore, we obtain from the previous inequality and (24)

$$D_{j+1} \geqslant C_3 \frac{D_j^p}{p^{2j}},$$

where

$$C_3 = \frac{C_0}{\beta^2} = \frac{C_0}{\left(n + r_2 + 2 + \frac{r_2+3}{p-1}\right)^2}.$$

Hence,

$$\begin{aligned}
\log D_j &\geqslant p\log D_{j-1} - 2(j-1)\log p + \log C_3 \\
&\geqslant p^2 \log D_{j-2} - 2(p(j-2) + (j-1))\log p + (p+1)\log C_3 \\
&\geqslant \cdots \\
&\geqslant p^{j-1}\log D_1 - 2\log p \sum_{k=1}^{j-1} kp^{j-1-k} + \log C_3 \sum_{k=1}^{j-1} p^k.
\end{aligned}$$

Using an inductive argument, the following formulas can be shown:

$$\sum_{k=1}^{j-1} kp^{j-1-k} = \frac{1}{p-1}\left(\frac{p^j-1}{p-1} - j\right)$$

and

$$\sum_{k=1}^{j-1} p^k = \frac{p-p^j}{1-p},$$

which yield

$$\begin{aligned}
\log D_j &\geqslant p^{j-1}\log D_1 - \frac{2\log p}{p-1}\left(\frac{p^j-1}{p-1} - j\right) + \log C_3 \frac{p-p^j}{1-p} \\
&= p^{j-1}\left(\log D_1 - \frac{2p\log p}{(p-1)^2} + \frac{p\log C_3}{p-1}\right) + \frac{2\log p}{p-1} j + \frac{2\log p}{(p-1)^2} + \frac{p\log C_3}{1-p}.
\end{aligned}$$

Consequently, for $j > \left[\frac{p\log C_3}{2\log p} - \frac{1}{p-1}\right] + 1$ it holds

$$D_j \geqslant \exp\{p^{j-1}(\log D_1 - S_p(\infty))\} \tag{28}$$

with

$$S_p(\infty) := \frac{2p\log p}{(p-1)^2} - \frac{p\log C_3}{p-1}.$$

Inserting (26), (27) and (28) into (21) gives

$$\begin{aligned} G(t) &\geqslant \exp\left(p^{j-1}(\log D_1 - S_p(\infty))\right)(1+t)^{-\alpha p^{j-1}+n+\frac{r_2+1}{p-1}}(t-T_0)^{\beta p^{j-1}-\frac{r_2+3}{p-1}} \\ &\geqslant \exp\left(p^{j-1}J(t)\right)(1+t)^{n+\frac{r_2+1}{p-1}}(t-T_0)^{-\frac{r_2+3}{p-1}}, \end{aligned} \tag{29}$$

where

$$J(t) := \log D_1 - S_p(\infty) - \alpha\log(1+t) + \beta\log(t-T_0).$$

For $t > 2T_0 + 1$, we have

$$\begin{aligned} J(t) &\geqslant \log D_1 - S_p(\infty) - \alpha\log(2t-2T_0) + \beta\log(t-T_0) \\ &\geqslant \log D_1 - S_p(\infty) + (\beta-\alpha)\log(t-T_0) - \alpha\log 2 \\ &= \log(D_1\cdot(t-T_0)^{\beta-\alpha}) - S_p(\infty) - \alpha\log 2. \end{aligned}$$

Note that

$$\beta - \alpha = b_1 - a_1 - n + \frac{2}{p-1} = \frac{p+1}{p-1} - (n+\mu_1-1)\frac{p}{2} = \frac{\gamma(p,n+\mu_1)}{2(p-1)}.$$

Thus, if

$$t > \max\left\{T_0 + \left(\frac{e^{(S_p(\infty)+\alpha\log 2)+1}}{C_2\varepsilon^p}\right)^{2(p-1)/\gamma(p,n+\mu_1)}, 2T_0+1\right\},$$

then, we get $J(t) > 1$, and this in turn gives $G(t) \to \infty$ by taking $j \to \infty$ in (29). Therefore, there exists a sufficiently small $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ we obtain the desired upper bound,

$$T \leqslant C_4\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu_1)}}$$

with

$$C_4 := \left(\frac{e^{(S_p(\infty)+\alpha\log 2)+1}}{C_2}\right)^{2(p-1)/\gamma(p,n+\mu_1)}.$$

This completes our proof of Theorem 1.2. □

4. Test function and preliminaries: critical case

In this section and in the next one, we adapt the approach from [9], with the purpose to include the scale-invariant mass term.

Firstly, let us construct a suitable solution of the adjoint equation of (2) in $Q_1 := \{(t,x) \in [0,\infty)\times\mathbb{R}^n : |x| < 1+t\}$. In other terms, we look for a function $\Phi = \Phi(t,x)$ which solves

$$\partial_t^2\Phi - \Delta\Phi - \partial_t\Big(\frac{\mu_1}{1+t}\,\Phi\Big) + \frac{\mu_2^2}{(1+t)^2}\,\Phi = 0 \quad \text{for any} \quad (t,x)\in Q_1. \tag{30}$$

Proposition 4.1. *Let β be a real parameter. Let us make the following ansatz:*

$$\Phi_\beta(t,x) := (1+t)^{-\beta+1}\psi_\beta\Big(\frac{|x|^2}{(1+t)^2}\Big), \tag{31}$$

where $\psi_\beta \in \mathcal{C}^2([0,1))$. Then, Φ_β solves (30) *if and only if ψ_β solves*

$$z(1-z)\psi_\beta''(z) + \left(\tfrac{n}{2} - \left(\beta + \tfrac{1}{2} + \tfrac{\mu_1}{2}\right)z\right)\psi_\beta'(z) - \left(\tfrac{\beta(\beta+\mu_1-1)+\mu_2^2}{4}\right)\psi_\beta(z) = 0. \tag{32}$$

Proof. For the sake of brevity we introduce the notation $z := \frac{|x|^2}{(1+t)^2}$. By straightforward computations, it follows

$$\begin{aligned}\partial_t \Phi_\beta(t,x) &= (-\beta+1)(1+t)^{-\beta}\psi_\beta(z) - 2(1+t)^{-\beta} z\,\psi'_\beta(z),\\ \partial_t^2 \Phi_\beta(t,x) &= (\beta-1)\beta(1+t)^{-\beta-1}\psi_\beta(z) + 4(\beta-1)(1+t)^{-\beta-1} z\,\psi'_\beta(z)\\ &\quad + 4(1+t)^{-\beta-1} z^2\,\psi''_\beta(z) + 6(1+t)^{-\beta-1} z\,\psi'_\beta(z),\end{aligned}$$

and

$$\Delta\Phi_\beta(t,x) = 2n(1+t)^{-\beta-1}\psi'_\beta(z) + 4(1+t)^{-\beta-1} z\,\psi''_\beta(z).$$

Plugging the previous relations, we obtain the following identity:

$$\begin{aligned}\partial_t^2\Phi_\beta - &\Delta\Phi_\beta - \partial_t\Big(\frac{\mu_1}{1+t}\,\Phi_\beta\Big) + \frac{\mu_2^2}{(1+t)^2}\,\Phi_\beta\\ &= (1+t)^{-\beta-1}\Big(4z(z-1)\psi''_\beta(z) + ((4(\beta-1)+6+2\mu_1)z - 2n)\psi'_\beta(z)\\ &\qquad + (\beta(\beta-1) - \mu_1(-\beta+1) + \mu_1 + \mu_2^2)\psi_\beta(z)\Big).\end{aligned}$$

Also, Φ_β solves (30) if and only if ψ_β is a solution to (32). $\square$

If we find a, b such that

$$a+b+1 = \beta + \tfrac{1}{2} + \tfrac{\mu_1}{2}\,, \quad ab = \tfrac{\beta(\beta+\mu_1-1)+\mu_2^2}{4}\,, \tag{33}$$

then, (32) coincides with the hypergeometric equation with parameters $(a, b\,; \frac{n}{2})$, namely,

$$z(1-z)\psi''_\beta(z) + \big(\tfrac{n}{2} - (a+b+1)z\big)\psi'_\beta(z) - ab\psi_\beta(z) = 0.$$

Hence, whether a, b fulfill (33), we can choose the Gauss hypergeometric function with parameters $(a, b\,; \frac{n}{2})$ as solution to the above equation, i.e.,

$$\psi_\beta(z) := F(a, b\,; \tfrac{n}{2}\,; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(n/2)_k}\,\frac{z^k}{k!}, \tag{34}$$

provided that $|z| < 1$ or, equivalently, $(t,x) \in Q_1$. In (34) the so-called Pochhammer's symbol $(m)_k$ is defined by

$$(m)_k = \begin{cases} 1 & \text{if } k = 0,\\ \prod_{j=1}^{k}(m+j-1) & \text{if } k \geqslant 0.\end{cases}$$

It is actually possible to choose a, b satisfying (33). Indeed, the quadratic equations

$$r^2 - \big(\beta + \tfrac{\mu_1}{2} - \tfrac{1}{2}\big)r + \tfrac{\beta(\beta+\mu_1-1)+\mu_2^2}{4} = 0$$

has an independent of β and nonnegative discriminant due to the assumption $\delta \geqslant 0$. Let us introduce

$$a := \tfrac{\beta}{2} + \tfrac{\mu_1-1}{4} + \tfrac{\sqrt{\delta}}{4}, \tag{35}$$

$$b := \tfrac{\beta}{2} + \tfrac{\mu_1-1}{4} - \tfrac{\sqrt{\delta}}{4}. \tag{36}$$

Then, a and b fulfill (33).

Definition 4.2. *Let a, b be defined by* (35) *and* (36)*, respectively. We introduce the following function*

$$\begin{aligned}\Phi_\beta = \Phi_\beta(t,x\,;\mu_1,\mu_2) &= (1+t)^{-\beta+1}\psi_\beta\Big(\tfrac{|x|^2}{(1+t)^2}\Big)\\ &:= (1+t)^{-\beta+1}F\Big(a,b\,;\tfrac{n}{2}\,;\tfrac{|x|^2}{(1+t)^2}\Big)\quad \textit{for } (t,x)\in Q_1.\end{aligned} \tag{37}$$

According to Proposition 4.1 Φ_β solves (30) in Q_1. The next step is to provide the asymptotic behavior of ψ_β and ψ'_β.

Lemma 4.3. *The following estimates are satisfied:*

(i) *if* $\frac{\sqrt{\delta}-\mu_1+1}{2} < \beta < \frac{n-\mu_1+1}{2}$*, then, there exists* $C' = C'(\beta, n, \mu_1, \mu_2) > 1$ *such that for any* $z \in [0,1)$ *it holds*

$$1 \leqslant \psi_\beta(z) \leqslant C'\,; \tag{38}$$

(ii) *if* $\beta > \frac{n-\mu_1-1}{2}$*, then, there exists* $C'' = C''(\beta, n, \mu_1, \mu_2) > 1$ *such that for any* $z \in [0,1)$ *it holds*

$$\tfrac{1}{C''}(1-\sqrt{z})^{\frac{n-\mu_1-1}{2}-\beta} \leqslant \big|\psi'_\beta(z)\big| \leqslant C''(1-\sqrt{z})^{\frac{n-\mu_1-1}{2}-\beta}. \tag{39}$$

Proof. (i) The assumption on β implies that $a, b > 0$ and $a + b < \frac{n}{2}$. Since $\psi_\beta = F(a, b, ; \frac{n}{2}\,; \cdot)$, (38) follows immediately by [17, Section 15.4 (ii), formula 15.4.20].
(ii) Because of $\psi'_\beta = \frac{2ab}{n}F(a+1, b+1\,; \frac{n}{2}+1\,; \cdot)$, the assumption on β and [17, Section 15.4 (ii), formula 15.4.23] imply (39). □

Before proving Theorem 1.3, we derive some preliminary lemmas. First of all, we introduce the following functionals

$$\mathcal{G}_\beta(t) := \int_{\mathbb{R}^n} |u(t,x)|^p\, \Phi_\beta(t,x\,;\mu_1,\mu_2)\, dx\,, \tag{40}$$

$$\mathcal{H}_\beta(t) := \int_0^t (t-s)(1+s)\,\mathcal{G}_\beta(s)\, ds\,, \tag{41}$$

$$\mathcal{J}_\beta(t) := \int_0^t (2+s)^{-3}\,\mathcal{H}_\beta(s)\, ds\,, \tag{42}$$

where $\beta \in \Big(\frac{\sqrt{\delta}-\mu_1+1}{2}, \frac{n-\mu_1+1}{2}\Big)$ and $t \geqslant 0$. We remark that δ should be smaller than n^2 in order to get a nonempty range for β.

Remark 4.4. *From* (38) *it follows that* $\mathcal{G}_\beta(t) \approx (1+t)^{1-\beta}\|u(t,\cdot)\|^p_{L^p(\mathbb{R}^n)}$*. Hence, if we prove that* $\mathcal{J}_\beta$ *blows up in finite time, then, in turn,* $\mathcal{H}_\beta$ *blows up in finite time and* $\mathcal{G}_\beta(t)$ *as well. Due to the previous relation, we get hence that the lifespan of* $\mathcal{J}_\beta$ *is an upper bound for the lifespan* T *of the energy solution solution* u *of* (1).

Lemma 4.5. *For any* $\beta \in \Big(\frac{\sqrt{\delta}-\mu_1+1}{2}, \frac{n-\mu_1+1}{2}\Big)$ *and* $t \geqslant 0$ *it holds*

$$(1+t)^2\mathcal{J}_\beta(t) \leqslant \frac{1}{2}\int_0^t (t-s)^2\mathcal{G}_\beta(s)\, ds.$$

Proof. Differentiating twice (41), we have

$$\mathcal{H}'_\beta(t) = \int_0^t (1+s)\,\mathcal{G}_\beta(s)\, ds\,, \quad \mathcal{H}''_\beta(t) = (1+t)\,\mathcal{G}_\beta(t). \tag{43}$$

Then, by using integration by parts, since $\mathcal{H}_\beta(0)=\mathcal{H}'_\beta(0)=0$, we get

$$\begin{aligned}\int_0^t (t-s)^2\mathcal{G}_\beta(s)\,ds &= \int_0^t (t-s)^2(1+s)^{-1}\mathcal{H}''_\beta(s)\,ds = \int_0^t \partial_s^2[(t-s)^2(1+s)^{-1}]\mathcal{H}_\beta(s)\,ds\\ &= 2\int_0^t (1+s)^{-3}(1+t)^2\mathcal{H}_\beta(s)\,ds \geqslant 2\,(1+t)^2\mathcal{J}_\beta(t),\end{aligned}$$

which is exactly the desired inequality. □

Lemma 4.6. *Let us assume $(f,g)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ nonnegative, not identically zero, compactly supported such that*

$$\operatorname{supp}(f),\ \operatorname{supp}(g)\subset B_R \ \textit{and} \quad R<1.$$

Let u be a solution of (1). Then, for every $\beta\in\left(\frac{\sqrt{\delta}-\mu_1+1}{2},\frac{n-\mu_1+1}{2}\right)$ such that $\beta\geqslant 1-\mu_1$ and $t\geqslant 0$ the following identity holds

$$\begin{aligned}&\varepsilon E_{0,\beta}(f)+\varepsilon E_{1,\beta}(f,g)\,t+\int_0^t (t-s)\mathcal{G}_\beta(s)\,ds\\ &\quad= \int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx+\int_0^t (1+s)^{-\beta}\int_{\mathbb{R}^n} u(s,x)\,\widetilde{\psi}_\beta\Big(\tfrac{|x|^2}{(1+s)^2}\Big)\,dx\,ds,\end{aligned}\tag{44}$$

where

$$E_{1,\beta}(f,g):=\int_{\mathbb{R}^n}\Big(g(x)\psi_\beta(|x|^2)+f(x)\big((\beta-1+\mu_1)\psi_\beta(|x|^2)+2|x|^2\psi'_\beta(|x|^2)\big)\Big)\,dx,\tag{45}$$

$$E_{0,\beta}(f):=\int_{\mathbb{R}^n} f(x)\psi_\beta(|x|^2)\,dx\tag{46}$$

are positive quantities and

$$\widetilde{\psi}_\beta(z):=(2\beta+\mu_1-2)\psi_\beta(z)+4z\,\psi'_\beta(z).\tag{47}$$

Proof. Due to the property of finite speed of propagation for solutions of strictly hyperbolic equations, for the solution u of the semilinear Cauchy problem (1) we have $\operatorname{supp} u(t,\cdot)\subset B_{R+t}$ for any $t\geqslant 0$, which implies $\operatorname{supp} u\subset Q_1$, as $R<1$.

For the sake of brevity, we will denote simply $\Phi_\beta(t,x\,;\mu_1,\mu_2)\equiv\Phi_\beta(t,x)$. Then, using (30), we have

$$\begin{aligned}\mathcal{G}_\beta(t) &= \int_{\mathbb{R}^n}\Big(u_{tt}(t,x)-\Delta u(t,x)+\tfrac{\mu_1}{1+t}u_t(t,x)+\tfrac{\mu_2^2}{(1+t)^2}u(t,x)\Big)\Phi_\beta(t,x)\,dx\\ &\quad-\int_{\mathbb{R}^n} u(t,x)\Big(\partial_t^2\Phi_\beta(t,x)-\Delta\Phi_\beta(t,x)-\partial_t\Big(\tfrac{\mu_1}{1+t}\Phi_\beta(t,x)\Big)+\tfrac{\mu_2^2}{(1+t)^2}\Phi_\beta(t,x)\Big)\,dx\\ &= \int_{\mathbb{R}^n}\Big(u_{tt}(t,x)+\tfrac{\mu_1}{1+t}u_t(t,x)\Big)\Phi_\beta(t,x)\,dx\\ &\quad-\int_{\mathbb{R}^n} u(t,x)\Big(\partial_t^2\Phi_\beta(t,x)-\partial_t\Big(\tfrac{\mu_1}{1+t}\Phi_\beta(t,x)\Big)\Big)\,dx\\ &= \frac{d}{dt}\bigg(\int_{\mathbb{R}^n}\big(u_t(t,x)\Phi_\beta(t,x)-u(t,x)\partial_t\Phi_\beta(t,x)\big)\,dx+\tfrac{\mu_1}{1+t}\int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx\bigg),\end{aligned}\tag{48}$$

where in the second equality we used Green's second identity (the boundary integrals with respect to x disappear due to the support property of u).

Since $\partial_t\Phi_\beta(t,x) = (1+t)^{-\beta}\big((-\beta+1)\psi_\beta(z) - 2z\psi'_\beta(z)\big)$, then,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\big(u_t(0,x)\Phi_\beta(0,x) - u(0,x)\partial_t\Phi_\beta(0,x)\big)\,dx + \mu_1\int_{\mathbb{R}^n} u(0,x)\Phi_\beta(0,x)\,dx\\
&\quad= \varepsilon\int_{\mathbb{R}^n}\Big(g(x)\psi_\beta(|x|^2) - f(x)\big((-\beta+1)\psi_\beta(|x|^2) - 2|x|^2\psi'_\beta(|x|^2)\big) + \mu_1 f(x)\psi_\beta(|x|^2)\Big)\,dx\\
&\quad= \varepsilon\int_{\mathbb{R}^n}\Big(g(x)\psi_\beta(|x|^2) + f(x)\big((\beta-1+\mu_1)\psi_\beta(|x|^2) + 2|x|^2\psi'_\beta(|x|^2)\big)\Big)\,dx = \varepsilon E_{\beta,1}(f,g).
\end{aligned}$$

Since $\psi_\beta(|x|^2) = F(a,b\,;\frac{n}{2}\,;|x|^2) \geqslant 1$ and

$$\psi'_\beta(|x|^2) = F'\big(a,b\,;\tfrac{n}{2}\,;|x|^2\big) = \tfrac{2ab}{n}F\big(a+1,b+1\,;\tfrac{n}{2}+1\,;|x|^2\big) > 0$$

for $|x|<1$ and we required $\beta \geqslant 1-\mu_1$ in the assumptions, then, it results $E_{\beta,1}(f,g)>0$, as f and g are nonnegative.

Integrating (48) over $[0,t]$, we obtain

$$\begin{aligned}
&\varepsilon E_{\beta,1}(f,g) + \int_0^t \mathcal{G}_\beta(s)\,ds\\
&\quad= \int_{\mathbb{R}^n}\big(u_t(t,x)\Phi_\beta(t,x) - u(t,x)\partial_t\Phi_\beta(t,x)\big)\,dx + \tfrac{\mu_1}{1+t}\int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx\\
&\quad= \frac{d}{dt}\bigg(\int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx\bigg) - 2\int_{\mathbb{R}^n} u(t,x)\partial_t\Phi_\beta(t,x)\,dx + \tfrac{\mu_1}{1+t}\int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx\\
&\quad= \frac{d}{dt}\bigg(\int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx\bigg) + (1+t)^{-\beta}\int_{\mathbb{R}^n} u(t,x)\,\widetilde{\psi}_\beta\Big(\tfrac{|x|^2}{(1+t)^2}\Big)\,dx,
\end{aligned}$$

where $\widetilde{\psi}_\beta$ is given by (47).

A further integration over $[0,t]$ and Fubini's theorem provide

$$\begin{aligned}
\varepsilon E_{\beta,1}(f,g)\,t + \int_0^t\int_0^\tau \mathcal{G}_\beta(s)\,ds &= \varepsilon E_{\beta,1}(f,g)\,t + \int_0^t (t-s)\,\mathcal{G}_\beta(s)\,ds\,d\tau\\
&= \int_{\mathbb{R}^n} u(t,x)\Phi_\beta(t,x)\,dx - \varepsilon\int_{\mathbb{R}^n} f(x)\psi_\beta(|x|^2)\,dx\\
&\quad+ \int_0^t (1+s)^{-\beta}\int_{\mathbb{R}^n} u(s,x)\,\widetilde{\psi}_\beta\Big(\tfrac{|x|^2}{(1+s)^2}\Big)\,dx\,ds,
\end{aligned}$$

that is, (44). □

Lemma 4.7. *Let us assume $(f,g)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ nonnegative, not identically zero, compactly supported such that*

$$\operatorname{supp}(f),\ \operatorname{supp}(g) \subset B_R \text{ and } \quad R<1.$$

Let $\mu_1,\ \mu_2$ be nonnegative constants such that $0\leqslant\delta<n^2$ and let $p = p_S(n+\mu_1)$ be such that $p > \frac{2}{n-\sqrt{\delta}}$.

(i) *Let $q>p$ and let us consider*

$$\beta_q = \tfrac{n-\mu_1+1}{2} - \tfrac{1}{q}. \tag{49}$$

Then,

$$\begin{aligned}
&\varepsilon E_{0,\beta_q}(f) + \varepsilon E_{1,\beta_q}(f,g)\,t + \int_0^t (t-s)\,\mathcal{G}_{\beta_q}(s)\,ds\\
&\quad\leqslant C_1\bigg((1+t)^{\frac{n}{p'}-\beta_q+1}\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\,ds\bigg).
\end{aligned}$$

(ii) *Let $p = q$. If β_p is defined by (49), then,*

$$\int_0^t (t-s)\,\mathcal{G}_{\beta_p}(s)\,ds \leqslant C_1\left((1+t)^{\frac{n}{p'}-\beta_p+1}\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\big(\log(2+s)\big)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\,ds\right).$$

Here the constant $C_1 > 0$ does not depend on (t,x) and u.

Remark 4.8. *Let us point out that the condition $p > \frac{2}{n-\sqrt{\delta}}$ implies*

$$\beta_q \geqslant \beta_p > \tfrac{\sqrt{\delta}-\mu_1+1}{2}.$$

for $q \geqslant p$. Moreover, the condition $\beta_p \geqslant 1-\mu_1$ is always true for $p = p_S(n+\mu_1)$. Indeed, $\beta_p \geqslant 1-\mu_1$ is equivalent to require

$$\tfrac{1}{p} \leqslant \tfrac{n+\mu_1-1}{2}.$$

Besides, p solves the quadratic equation $\gamma(p, n+\mu_1) = 0$. Therefore,

$$\tfrac{1}{p} = \tfrac{n+\mu_1-1}{2}p - \tfrac{n+\mu_1+1}{2} \leqslant \tfrac{n+\mu_1-1}{2} \quad \text{if and only if} \quad p \leqslant \tfrac{2(n+\mu_1)}{n+\mu_1-1}.$$

As $p = p_S(n+\mu_1)$, a straightforward calculation shows that the last inequality is always fulfilled by nonnegative parameters μ_1.

Proof. By (44), using again the finite speed of propagation property, we may write

$$\varepsilon E_{0,\beta_q}(f) + \varepsilon E_{1,\beta_q}(f,g)\,t + \int_0^t (t-s)\,\mathcal{G}_{\beta_q}(s)\,ds = \mathcal{I}_{\beta_q,1}(t) + \int_0^t \mathcal{I}_{\beta_q,2}(s)\,ds, \tag{50}$$

where

$$\mathcal{I}_{\beta_q,1}(t) := \int_{B_{R+t}} u(t,x)\Phi_{\beta_q}(t,x)\,dx,$$
$$\mathcal{I}_{\beta_q,2}(t) := (1+t)^{-\beta_q}\int_{B_{R+t}} u(t,x)\,\widetilde{\psi}_{\beta_q}\Big(\tfrac{|x|^2}{(1+t)^2}\Big)\,dx.$$

Let us point out explicitly that, according to Remark 4.8 we have that the assumptions on p imply $\beta_q \in \Big(\frac{\sqrt{\delta}-\mu_1+1}{2}, \frac{n-\mu_1+1}{2}\Big)$ and $\beta_q \geqslant 1-\mu_1$. For this reason, we may use Lemma 4.6 in order to derive (50). For $\beta_q \in \Big(\frac{\sqrt{\delta}-\mu_1+1}{2}, \frac{n-\mu_1+1}{2}\Big)$, as the hypergeometric function in (37) is uniformly bounded, we can estimate $\Phi_{\beta_q}(t,x) \approx (1+t)^{-\beta_q+1}$ according to (38). Therefore, if we denote by p' the conjugate exponent of p, Hölder inequality implies

$$\mathcal{I}_{\beta_q,1}(t) \leqslant \left(\int_{B_{R+t}} |u(t,x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{R+t}} \Phi_{\beta_q}(t,x)^{p'}\,dx\right)^{\frac{1}{p'}} \leqslant C_1(1+t)^{\frac{n}{p'}-\beta_q+1}\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)}, \tag{51}$$

where throughout this proof $C_1 = C_1(n,p,\mu_1,\mu_2,\beta,R) > 0$ is a suitable constant that may change from line to line.

Let us estimate now the term $\mathcal{I}_{\beta_q,2}(s)$. We remark that for β_q as in (49), then, $\beta_q > \frac{n-\mu_1-1}{2}$, since it is $q > 1$. Therefore, in order to estimate ψ'_{β_q} we may use (39). As we underlined in the previous case, due to the assumption on β_q, the function ψ_{β_q} is uniformly bounded. Thus, in (47) the dominant term as $z \to 1^-$ is the derivative. Hence,

$$|\widetilde{\psi}_{\beta_q}(z)| \leqslant C_1(1-\sqrt{z})^{\frac{n-\mu_1-1}{2}-\beta_q} \qquad \text{for } z \in [0,1).$$

Consequently, by using Hölder inequality, for $\mathcal{I}_{\beta,2}(s)$ we get

$$\begin{aligned}
\mathcal{I}_{\beta_q,2}(s) &\leqslant (1+s)^{-\beta_q}\left(\int_{B_{R+s}} |u(s,x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B_{R+s}} |\widetilde{\psi}_{\beta_q}(s,x)|^{p'}\,dx\right)^{\frac{1}{p'}} \\
&\leqslant C_1(1+s)^{-\beta_q}\left(\int_{B_{R+s}}\left(1-\tfrac{|x|}{1+s}\right)^{(\frac{n-\mu_1-1}{2}-\beta_q)p'}dx\right)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&= C_1(1+s)^{-\beta_q}\left(\int_{B_{R+s}}\left(1-\tfrac{|x|}{1+s}\right)^{(\frac{1}{q}-1)p'}dx\right)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&= C_1(1+s)^{-\beta_q}\left(\int_{B_{R+s}}\left(1-\tfrac{|x|}{1+s}\right)^{-\frac{p'}{q'}}dx\right)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where q' denotes the conjugate exponent of q.

Using polar coordinates, we get

$$\begin{aligned}
\int_{B_{R+s}}\left(1-\tfrac{|x|}{1+s}\right)^{-\frac{p'}{q'}}dx &= \omega_{n-1}\int_0^{R+s}\left(1-\tfrac{r}{1+s}\right)^{-\frac{p'}{q'}}r^{n-1}\,dr \\
&= \omega_{n-1}(1+s)^n\int_0^{\frac{R+s}{1+s}}(1-\rho)^{-\frac{p'}{q'}}\rho^{n-1}\,d\rho,
\end{aligned}$$

where ω_{n-1} is the measure of the unitary sphere ∂B_1. Also,

$$\begin{aligned}
\mathcal{I}_{\beta_q,2}(s) &\leqslant C_1(1+s)^{-\beta_q+\frac{n}{p'}}\left(\int_0^{\frac{R+s}{1+s}}(1-\rho)^{-\frac{p'}{q'}}\rho^{n-1}\,d\rho\right)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C_1(1+s)^{-\beta_q+\frac{n}{p'}}\left(\int_0^{\frac{R+s}{1+s}}(1-\rho)^{-\frac{p'}{q'}}\,d\rho\right)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C_1(1+s)^{-\beta_q+\frac{n}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\begin{cases}(1-\frac{R+s}{1+s})^{-\frac{1}{q'}+\frac{1}{p'}} & \text{if } q>p,\\ \big(-\log(1-\frac{R+s}{1+s})\big)^{\frac{1}{p'}} & \text{if } q=p,\end{cases} \\
&\leqslant \begin{cases}C_1(1+s)^{-\beta_q+\frac{n}{p'}+\frac{1}{q'}-\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} & \text{if } q>p,\\ C_1(1+s)^{-\beta_p+\frac{n}{p'}}\big(\log(2+s)\big)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)} & \text{if } q=p.\end{cases}
\end{aligned}$$

Since $-\beta_q+\frac{n}{p'}+\frac{1}{q'}-\frac{1}{p'}=\frac{n}{p'}+1-\frac{n-\mu_1+1}{2}-\frac{1}{p'}=\frac{n}{p'}-\beta_p$, integrating $\mathcal{I}_{\beta,2}(s)$ over $[0,t]$, we find

$$\int_0^t \mathcal{I}_{\beta_q,2}(s)\,ds \leqslant C_1\begin{cases}\displaystyle\int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\,ds & \text{if } q>p,\\ \displaystyle\int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\big(\log(2+s)\big)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\,ds & \text{if } q=p.\end{cases} \tag{52}$$

Due to the assumptions on (f,g), we have $E_{0,\beta_q}(f)>0$ and $E_{1,\beta_q}(f,g)>0$. Then, combining (51) and (52), from (50) we get the desired estimates in the cases $q>p$ and $q=p$. □

5. Proof of Theorem 1.3

Proof. Let us consider $\beta_{p+\sigma} = \frac{n+\mu_1-1}{2} - \frac{1}{p+\sigma}$ for $p = p_S(n+\mu_1)$, where σ is a positive constant. Since β_q is increasing with respect to q, if we assume $p > \frac{2}{n-\sqrt{\delta}}$, then, $\beta_{p+\sigma} > \beta_p > \frac{\sqrt{\delta}-\mu_1+1}{2}$ and we can apply Lemma 4.7. Also, from Lemma 4.7 (i) it follows

$$\begin{aligned}
&\varepsilon E_{0,\beta_{p+\sigma}}(f) + \varepsilon E_{1,\beta_{p+\sigma}}(f,g)\, t \\
&\quad \leqslant C_1\bigg((1+t)^{\frac{n}{p'}-\beta_{p+\sigma}+1}\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\, ds\bigg) \\
&\quad \leqslant C_1\bigg((1+t)^{\frac{n+1-\beta_p}{p'}+\beta_p-\beta_{p+\sigma}}(\mathcal{G}_{\beta_p}(t))^{\frac{1}{p}} + \int_0^t (1+s)^{\frac{n+1-\beta_p}{p'}-1}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\, ds\bigg).
\end{aligned}$$

Let us underline that $p = p(n+\mu_1)$ implies $\frac{n+1-\beta_p}{p'} = 1 + \frac{1}{p}$. Indeed,

$$\begin{aligned}
\tfrac{n+1-\beta_p}{p'} &= \Big(n+1-\Big(\tfrac{n-\mu_1+1}{2} - \tfrac{1}{p}\Big)\Big)\Big(1-\tfrac{1}{p}\Big) = \tfrac{1}{p^2}\Big(\tfrac{n+\mu_1+1}{2}p+1\Big)(p-1) \\
&= \tfrac{1}{p^2}\Big(\tfrac{n+\mu_1+1}{2}p^2 - \tfrac{n+\mu_1+1}{2}p + p - 1\Big) = \tfrac{1}{p^2}\Big(p^2 + p - \underbrace{\tfrac{\gamma(p,n+\mu_1)}{2}}_{=0}\Big) = \tfrac{p+1}{p}.
\end{aligned}$$

Then, integrating the preceding inequality over $[0,t]$ and applying Fubini's theorem and Hölder inequality, we arrive at

$$\begin{aligned}
&\varepsilon E_{0,\beta_{p+\sigma}}(f)\, t + \tfrac{\varepsilon}{2} E_{1,\beta_{p+\sigma}}(f,g)\, t^2 \\
&\quad \leqslant C_1\bigg(\int_0^t (1+s)^{1+\frac{1}{p}+\beta_p-\beta_{p+\sigma}}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\, ds + \int_0^t (t-s)(1+s)^{\frac{1}{p}}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\, ds\bigg) \\
&\quad \leqslant C_1\bigg(\int_0^t (1+s)\mathcal{G}_{\beta_p}(s)\, ds\bigg)^{\frac{1}{p}}\bigg[\bigg(\int_0^t (1+s)^{p'+(\beta_p-\beta_{p+\sigma})p'}\, ds\bigg)^{\frac{1}{p'}} + \bigg(\int_0^t (t-s)^{p'}\, ds\bigg)^{\frac{1}{p'}}\bigg] \\
&\quad \leqslant C_2\bigg(\int_0^t (1+s)\mathcal{G}_{\beta_p}(s)\, ds\bigg)^{\frac{1}{p}}(1+t)^{1+\frac{1}{p'}}.
\end{aligned}$$

From (43), we get

$$\varepsilon E_{0,\beta_{p+\sigma}}(f)\, t + \tfrac{\varepsilon}{2} E_{1,\beta_{p+\sigma}}(f,g)\, t^2 \leqslant C_2 \mathcal{H}'_{\beta_p}(t)^{\frac{1}{p}}(1+t)^{1+\frac{1}{p'}},$$

which implies for $t \geqslant 1$

$$\begin{aligned}
\mathcal{H}'_{\beta_p}(t) &\geqslant C_2^{-p}\varepsilon^p\Big(E_{0,\beta_{p+\sigma}}(f)\, t + \tfrac{1}{2}E_{1,\beta_{p+\sigma}}(f,g)\, t^2\Big)^p(1+t)^{1-2p} \\
&\geqslant C_3\varepsilon^p(1+t).
\end{aligned}$$

As the functional H_{β_p} is nonnegative, from the previous inequality we get for $t \geqslant 2$

$$\mathcal{H}_{\beta_p}(t) \geqslant \int_1^t \mathcal{H}'_{\beta_p}(s)\, ds \geqslant C_3\varepsilon^p \int_1^t (1+s)\, ds \geqslant C_4\varepsilon^p(1+t)^2. \tag{53}$$

By using Lemma 4.7 (ii), due to $E_{0,\beta_p}(f), E_{1,\beta_p}(f,g) \geqslant 0$ we have

$$\begin{aligned}
&\int_0^t (t-s)\, \mathcal{G}_{\beta_p}(s)\, ds \\
&\quad \leqslant C_1\bigg((1+t)^{\frac{n}{p'}-\beta_p+1}\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} + \int_0^t (1+s)^{\frac{n}{p'}-\beta_p}\big(\log(2+s)\big)^{\frac{1}{p'}}\|u(s,\cdot)\|_{L^p(\mathbb{R}^n)}\, ds\bigg) \\
&\quad \leqslant C_1\bigg((1+t)^{\frac{n+1-\beta_p}{p'}}(\mathcal{G}_{\beta_p}(t))^{\frac{1}{p}} + \int_0^t (1+s)^{\frac{n+1-\beta_p}{p'}-1}\big(\log(2+s)\big)^{\frac{1}{p'}}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\, ds\bigg).
\end{aligned}$$

Integrating over $[0,t]$ and using again the equality $\frac{n+1-\beta_p}{p'} = 1+\frac{1}{p}$, Hölder inequality and (43), we find

$$\begin{aligned}
&\frac{1}{2}\int_0^t (t-s)^2\,\mathcal{G}_{\beta_p}(s)\,ds \\
&\leqslant C_1\bigg(\int_0^t (1+s)^{1+\frac{1}{p}}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\,ds + \int_0^t (t-s)(1+s)^{\frac{1}{p}}\big(\log(2+s)\big)^{\frac{1}{p'}}(\mathcal{G}_{\beta_p}(s))^{\frac{1}{p}}\,ds\bigg) \\
&\leqslant C_2'\bigg(\int_0^t (1+s)\mathcal{G}_{\beta_p}(s)\,ds\bigg)^{\frac{1}{p}}\Bigg[\bigg(\int_0^t (1+s)^{p'}\,ds\bigg)^{\frac{1}{p'}} + \bigg(\int_0^t (t-s)^{p'}\log(2+s)\,ds\bigg)^{\frac{1}{p'}}\Bigg] \\
&\leqslant C_3'\bigg(\int_0^t (1+s)\mathcal{G}_{\beta_p}(s)\,ds\bigg)^{\frac{1}{p}}(1+t)^{1+\frac{1}{p'}}\big(\log(2+t)\big)^{\frac{1}{p'}} \\
&\leqslant C_3'\big(\mathcal{H}_{\beta_p}'(t)\big)^{\frac{1}{p}}(2+t)^{\frac{2p-1}{p}}\big(\log(2+t)\big)^{\frac{1}{p'}}.
\end{aligned}$$

From Lemma 4.5, we get

$$(1+t)^2\mathcal{J}_{\beta_p}(t) \leqslant C_3'\big(\mathcal{H}_{\beta_p}'(t)\big)^{\frac{1}{p}}(2+t)^{\frac{2p-1}{p}}\big(\log(2+t)\big)^{\frac{1}{p'}},$$

and, hence, for $t \geqslant 2$ we have

$$C_4'\big(\log(2+t)\big)^{1-p}\big(\mathcal{J}_{\beta_p}(t)\big)^p \leqslant \mathcal{H}_{\beta_p}'(t)(2+t)^{-1}. \tag{54}$$

By the definition of $\mathcal{J}_{\beta_p}$, it follows immediately $(2+t)^3\mathcal{J}_{\beta_p}'(t) = \mathcal{H}_{\beta_p}(t)$ which implies

$$(2+t)^3\mathcal{J}_{\beta_p}''(t) + 3(2+t)^2\mathcal{J}_{\beta_p}'(t) = \mathcal{H}_{\beta_p}'(t).$$

Combining the previous identity with (54), we have

$$(2+t)^2\mathcal{J}_{\beta_p}''(t) + 3(2+t)\mathcal{J}_{\beta_p}'(t) \geqslant C_4'\big(\log(2+t)\big)^{1-p}\big(\mathcal{J}_{\beta_p}(t)\big)^p. \tag{55}$$

Moreover, from (53), we get for $t \geqslant 2$ and for a suitable constant $c_0 > 0$

$$\mathcal{J}_{\beta_p}(t) \geqslant C_4'\varepsilon^p\int_0^t (2+s)^{-3}(1+s)^2\,ds \geqslant c_0\,\varepsilon^p\log(2+t), \tag{56}$$

$$\mathcal{J}_{\beta_p}'(t) \geqslant C_4'\varepsilon^p(2+t)^{-3}(1+t)^2 \geqslant c_0\,\varepsilon^p(2+t)^{-1}. \tag{57}$$

Let us set $2+t = \exp(\tau)$. Let $\mathcal{J}_0(\tau)$ denote the functional $\mathcal{J}_{\beta_p}(t)$ with respect to the new variable, that is, $\mathcal{J}_0(\tau) = \mathcal{J}_{\beta_p}(\exp(\tau)-2) = \mathcal{J}_{\beta_p}(t)$. Then,

$$\begin{aligned}
\mathcal{J}_0'(\tau) &= (2+t)\mathcal{J}_{\beta_p}'(t), \\
\mathcal{J}_0''(\tau) &= (2+t)^2\mathcal{J}_{\beta_p}''(t) + (2+t)\mathcal{J}_{\beta_p}'(t).
\end{aligned}$$

So, by using (55), (56) and (57), we find that $\mathcal{J}_0(\tau)$ satisfies for $\tau \geqslant \log 4$

$$\begin{cases} \mathcal{J}_0''(\tau) + 2\mathcal{J}_0'(\tau) > C_4'\tau^{1-p}\mathcal{J}_0^p(\tau), \\ \mathcal{J}_0(\tau) \geqslant c_0\varepsilon^p\tau, \\ \mathcal{J}_0'(\tau) \geqslant c_0\varepsilon^p. \end{cases} \tag{58}$$

Employing [9, Lemma 3.1 (ii)] (see also [32], where this comparison principle for ordinary differential inequalities is originally stated and proved), we get that the function $\mathcal{J}_0(\tau)$ blows up in finite time before $\tau = C\varepsilon^{-p(p-1)}$ for some constant $C > 0$. Also, $\mathcal{J}_{\beta_p}(t)$ blows up before $t = \exp(C\varepsilon^{-p(p-1)}) - 2$. According to what we have said in Remark 4.4, we have found for the lifespan T of u the upper bound (6). This concludes the proof of Theorem 1.3. $\square$

Remark 5.1. *Let us explain the restriction* $p > \frac{2}{n-\sqrt{\delta}}$ *in Theorem 1.3. Although it turns out as a technical condition coming from the inequality* $\beta_p > \frac{\sqrt{\delta}-\mu_1+1}{2}$*, in the massless case* ($\mu_2^2 = 0$) *it is equivalent to require* $\mu < \frac{n^2+n+2}{n+2}$*, which is exactly the restriction on* μ_1 *in [9]. Furthermore, for* $n \geqslant 3$ *and* $\delta < (n-2)^2$ *this condition is always fulfilled. In particular, for high dimensions, namely for* $n \geqslant 4$*, we have an improvement in the range for* δ *for which we can prove a blow-up result in the critical case with respect to [22], where the restriction* $\delta \in (0, 1]$ *is required. Finally, we remind that* (1) *is "parabolic-like" for* $\delta \geqslant (n+1)^2$*. Therefore, the restriction* $\delta < (n-2)^2$ *when* $n \geqslant 3$ *is compatible with the conjecture for* (1) *to be "wave-like" for "small" and nonnegative* δ*. Similarly, in the subcritical case, even though in Theorem 1.2 we assume* $\delta \geqslant 0$*, it is clear that the result can be sharp only for suitably "small" and nonnegative* δ*.*

Remark 5.2. *Regarding the necessity part, in the special case* $\delta = 1$ *the exponent* $p_S(n+\mu_1)$ *is proved to be really critical for* $n \geqslant 3$ *in the radially symmetric case in [19, 20]. This shows the optimality of the range for* p *which is obtained in this paper for suitably "small" and nonnegative* δ*.*

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