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The automorphism group and the non-self-duality of p-cones

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ABSTRACT

In this paper, we determine the automorphism group of the p-cones ($p \neq 2$) in dimension greater than two. In particular, we show that the automorphism group of those p-cones are the positive scalar multiples of the generalized permutation matrices that fix the main axis of the cone. Next, we take a look at a problem related to the duality theory of the p-cones. Under the Euclidean inner product it is well-known that a p-cone is self-dual only when $p = 2$. However, it was not known whether it is possible to construct an inner product depending on p which makes the p-cone self-dual. Our results show that no matter which inner product is considered, a p-cone will never become self-dual unless $p = 2$ or the dimension is less than three.

1. Introduction

In this work, we prove two results on the structure of the p-cones

$$\mathcal{L}_p^{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid t \geq \|x\|_p\}.$$

First, we describe the automorphism group of the p-cones $\mathcal{L}_p^{n+1}$ for $n \geq 2$ and $p \neq 2$, $1 < p < \infty$. We show that every automorphism of $\mathcal{L}_p^{n+1}$ must have the format

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}, \tag{1}$$

where $\alpha > 0$ and P is an $n \times n$ generalized permutation matrix. The second result is that, for $n \geq 2$ and $p \neq 2$, it is not possible to construct an inner product on $\mathbb{R}^{n+1}$ for which $\mathcal{L}_p^{n+1}$ becomes self-dual. In fact, the

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second result is derived as a corollary of a stronger result that $\mathcal{L}_p^{n+1}$ and $\mathcal{L}_q^{n+1}$ cannot be linearly isomorphic if $p < q$ and $n \geq 2$, except when $(p, q, n) = (1, \infty, 2)$.

The motivation for this research is partly due to the work by Gowda and Trott [5], where they determined the automorphism group of $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$. However, they left open the problem of determining the automorphisms of the other p-cones, for $p \neq 2$. Here, we recall that the case $p = 2$ correspond to the *second order cones* and they are *symmetric*, i.e., *self-dual* and *homogeneous*. The structure of second-order cones and their automorphisms follow from the more general theory of Jordan Algebras [4], see also [8].

In [5], Gowda and Trott also proved that $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ are *not* homogeneous cones and they posed the problem of proving/disproving that $\mathcal{L}_p^{n+1}$ is not homogeneous for $p \neq 2$, $n \geq 2$. Recall that a cone is said to be *homogeneous* if its group of automorphisms acts transitively on the interior of the cone. In [6], using the theory of T-algebras [11], we gave a proof that $\mathcal{L}_p^{n+1}$ is not homogeneous for $p \neq 2$, $n \geq 2$. However, there are two unsatisfactory aspects of our previous result. The first is that we were not able to compute the automorphism group of $\mathcal{L}_p^{n+1}$. The second is that although we showed that $\mathcal{L}_p^{n+1}$ is not homogeneous, we were unable to obtain two elements x, y in the interior of $\mathcal{L}_p^{n+1}$ such that no automorphism of $\mathcal{L}_p^{n+1}$ maps x to y. That is, we were unable to show concretely how homogeneity breaks down on $\mathcal{L}_p^{n+1}$. The results discussed here remedy those flaws and provide an alternative proof that $\mathcal{L}_p^{n+1}$ is not homogeneous.

Another motivation for this work is the general problem of determining when a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ is self-dual. If $\mathbb{R}^n$ is equipped with some inner product $\langle \cdot, \cdot \rangle$, the dual cone of $\mathcal{K}$ is defined as

$$\mathcal{K}^* = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall x \in \mathcal{K}\}.$$

As discussed in Section 1 of [6], an often overlooked point is that $\mathcal{K}^*$ *depends on* $\langle \cdot, \cdot \rangle$. Accordingly, it is entirely plausible that a cone that is not self-dual under the Euclidean inner product might become self-dual if the inner product is chosen appropriately.

This detail is quite important because sometimes we see articles claiming that a certain cone is not a symmetric cone because it is not self-dual under the Euclidean inner product. This is, of course, not enough. As long as a cone is homogeneous and there exists some inner product that makes it self-dual, the cone can be investigated under the theory of Jordan Algebras.

This state of affairs brings us to the case of the p-cones. Up until the recent articles [5,6], there was no rigorous proof that the p-cones $\mathcal{L}_p^{n+1}$ were *not* symmetric when $p \neq 2$ and $n \geq 2$. Now, although we know that $\mathcal{L}_p^{n+1}$ is not homogeneous for $p \neq 2$ and $n \geq 2$, it still remains to investigate whether $\mathcal{L}_p^{n+1}$ could become self-dual under an appropriate inner product. This question was partly discussed by Miao, Lin and Chen in [9], where they showed that a p-cone (again, $p \neq 2$, $n \geq 2$) is not self-dual under an inner product induced by a diagonal matrix. The results described here show, in particular, that no inner product can make $\mathcal{L}_p^{n+1}$ self-dual, for $p \neq 2$, $n \geq 2$.

We now explain some of the intuition behind our proof techniques. Let $n \geq 2$ and let $f_p : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the function that maps x to $\|x\|_p$. When $p \in (1, 2)$, we have that f_p is twice differentiable only at points x for which $x_i \neq 0$, for all i. In contrast, if $p \in (2, \infty)$, f_p is twice differentiable throughout $\mathbb{R}^n \setminus \{0\}$. Now, we let M_p be the boundary without the zero of the cone $\mathcal{L}_p^{n+1}$. With that, M_p is exactly the graph of the function f_p. Furthermore, M_p is a C^1-embedded smooth manifold if $p \in (1, 2)$. If $p \in (2, \infty)$, M_p is a C^2-embedded smooth manifold. Now, any linear bijection between $\mathcal{L}_p^{n+1}$ and $\mathcal{L}_q^{n+1}$ must map the boundary of $\mathcal{L}_p^{n+1}$ to the boundary of $\mathcal{L}_q^{n+1}$, thus producing a map between M_p and M_q. Then, if $p \in (1, 2)$ and $q \in (2, \infty)$, there can be no linear bijection between $\mathcal{L}_p^{n+1}$ and $\mathcal{L}_q^{n+1}$ because this would establish a diffeomorphism between submanifolds that are embedded with different levels of smoothness.

Now suppose that p, q are both in $(1, 2)$ and that there exists some linear bijection A between $\mathcal{L}_p^{n+1}$ and $\mathcal{L}_q^{n+1}$. If $(f_p(x), x) \in M_p$ is such that f_p is *not* twice differentiable at x, then A must map $(f_p(x), x)$ to a point $(f_q(y), y)$ for which f_q is *not* twice differentiable at y. This idea is made precise in Proposition 4.

In particular, this fact imposes severe restrictions on how $\mathrm{Aut}(\mathcal{L}_p^{n+1})$ acts on $\mathcal{L}_p^{n+1}$ and this is the key observation necessary for showing that the matrices in $\mathrm{Aut}(\mathcal{L}_p^{n+1})$ can be written as in (1).

This work is divided as follows. In Section 2 we present the notation used in this paper and review some facts about cones, self-duality and p-cones. In Section 3, we discuss the tools from manifold theory necessary for our discussion. Finally, in Section 4 we prove our main results.

2. Preliminaries

A *convex cone* is a subset $\mathcal{K}$ of some real vector space $\mathbb{R}^n$ such that $\alpha x + \beta y \in \mathcal{K}$ holds whenever $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$. A cone $\mathcal{K}$ is said to be *pointed* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$. For a subset S of $\mathbb{R}^n$, the (closed) *conical hull* of S, denoted by $\mathrm{cone}(S)$, is the smallest closed convex cone in $\mathbb{R}^n$ containing S. If $v \in \mathbb{R}^n$, we write $\mathbb{R}_+(v)$ for the half-line generated by v and $\mathbb{R}_{++}$ for $\mathbb{R}_+(v) \setminus \{0\}$, i.e.,

$$\mathbb{R}_+(v) = \{\alpha v \mid \alpha \geq 0\},$$
$$\mathbb{R}_{++}(v) = \{\alpha v \mid \alpha > 0\}.$$

A convex subset $\mathcal{F}$ of $\mathcal{K}$ is said to be a *face* of $\mathcal{K}$ if the following condition holds: If $x, y \in \mathcal{K}$ satisfies $\alpha x + (1-\alpha)y \in \mathcal{F}$ for some $\alpha \in (0,1)$ then $x, y \in \mathcal{F}$ holds. A one dimensional face is called an *extreme ray*. A *polyhedral convex cone* is a convex cone that can be expressed as the solution set of finitely many linear inequalities.

If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$, we can define the *dual cone* of $\mathcal{K}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ by

$$\mathcal{K}^* = \{x \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \ \forall y \in \mathcal{K}\}.$$

A convex cone $\mathcal{K}$ is *self-dual* if there exists an inner product on $\mathbb{R}^n$ for which the dual cone coincides with $\mathcal{K}$ itself.

Two convex cones $\mathcal{K}_1$ and $\mathcal{K}_2$ in $\mathbb{R}^n$ are said to be *isomorphic* if there exists a linear bijection $A \in GL_n(\mathbb{R})$, called an *isomorphism*, such that $A\mathcal{K}_1 = \mathcal{K}_2$. An *automorphism* of a convex cone $\mathcal{K}$ in $\mathbb{R}^n$ is a map $A \in GL_n(\mathbb{R})$ such that $A\mathcal{K} = \mathcal{K}$. The group of all automorphisms of $\mathcal{K}$ is written by $\mathrm{Aut}(\mathcal{K})$ and called the *automorphism group of* $\mathcal{K}$.

A convex cone $\mathcal{K}$ is said to be *homogeneous* if $\mathrm{Aut}(\mathcal{K})$ acts transitively on the interior of $\mathcal{K}$, that is, for all elements x and y of the interior of $\mathcal{K}$, there exists $A \in \mathrm{Aut}(\mathcal{K})$ such that $y = Ax$.

2.1. On self-duality

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone. As we emphasized in Section 1, self-duality is a relative concept and depends on what inner product we are considering. Let $\langle \cdot, \cdot \rangle_E$ denote the Euclidean inner product and consider the dual of $\mathcal{K}$ with respect $\langle \cdot, \cdot \rangle_E$.

$$\mathcal{K}^* = \{y \in \mathbb{R}^n \mid \langle x, y \rangle_E \geq 0, \forall x \in \mathcal{K}\}.$$

We have the following proposition.

Proposition 1. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone and let $\mathcal{K}^*$ be the dual of $\mathcal{K}$ with the respect to the Euclidean inner product $\langle \cdot, \cdot \rangle_E$. Then, there exists an inner product on $\mathbb{R}^n$ that turns $\mathcal{K}$ into a self-dual cone if and only if there exists a symmetric positive definite matrix A such that $A\mathcal{K} = \mathcal{K}^*$.*

Proof. First, suppose that there exist some inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ for which $\mathcal{K}$ becomes self-dual. Then, there is a symmetric positive definite matrix A such that

$$\langle x, y \rangle_{\mathcal{K}} = \langle x, Ay \rangle_E,$$

for all $x, y \in \mathbb{R}^n$. In fact, $A_{ij} = \langle e_i, e_j \rangle_{\mathcal{K}}$, where e_i is the i-th standard unit vector in $\mathbb{R}^n$. By assumption, we have

$$\begin{aligned}\mathcal{K} &= \{x \in \mathbb{R}^n \mid \langle x, Ay \rangle_E \geq 0, \forall y \in \mathcal{K}\} \\ &= \{x \in \mathbb{R}^n \mid \langle Ax, y \rangle_E \geq 0, \forall y \in \mathcal{K}\} \\ &= A^{-1}\{z \in \mathbb{R}^n \mid \langle z, y \rangle_E \geq 0, \forall y \in \mathcal{K}\} \\ &= A^{-1}\mathcal{K}^*.\end{aligned}$$

This shows that $A\mathcal{K} = \mathcal{K}^*$.

Reciprocally, if $A\mathcal{K} = \mathcal{K}^*$, we define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ such that

$$\langle x, y \rangle_{\mathcal{K}} \coloneqq \langle x, Ay \rangle_E,$$

for all $x, y \in \mathbb{R}^n$. Then, a straightforward calculation shows that the dual of $\mathcal{K}$ with respect $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ is indeed $\mathcal{K}$. □

Therefore, determining whether $\mathcal{K}$ is self-dual for some inner product boils down to determining the existence of a positive definite linear isomorphism between cones, which is a difficult problem in general.

2.2. p-cones

Here we present some basic facts on p-cones. The *p-cone* is the closed convex cone in $\mathbb{R}^{n+1}$ defined by

$$\mathcal{L}_p^{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid t \geq \|x\|_p\}$$

where $\|x\|_p$ is the p-norm on $\mathbb{R}^n$:

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p} \text{ for } p \in [1, \infty) \text{ and } \|x\|_\infty = \max(|x_1|, \ldots, |x_n|).$$

The dual cone of the p-cone with respect to the Euclidean inner product is given by $(\mathcal{L}_p^{n+1})^* = \mathcal{L}_q^{n+1}$ where q is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. The cones $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ are polyhedral. Note that $\mathcal{L}_1^{n+1}$ has $2n$ extreme rays

$$\mathbb{R}_+(1, \sigma e_i^n), \quad i = 1, \ldots, n, \quad \sigma \in \{-1, 1\},$$

where e_i^n denotes the i-th standard unit vector in $\mathbb{R}^n$. Moreover, $\mathcal{L}_\infty^{n+1}$ has 2^n extreme rays

$$\mathbb{R}_+(1, \sigma_1, \ldots, \sigma_n), \quad \sigma_1, \ldots, \sigma_n \in \{-1, 1\}.$$

The difference in the number of extreme rays shows that $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ are not isomorphic if $n \geq 3$. However, for $n = 2$, they are indeed isomorphic as

$$A\mathcal{L}_1^3 = \mathcal{L}_\infty^3, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}\cos(\pi/4) & -\sqrt{2}\sin(\pi/4) \\ 0 & \sqrt{2}\sin(\pi/4) & \sqrt{2}\cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2}$$

The second order cone $\mathcal{L}_2^{n+1}$ is known to be a *symmetric cone*, that is, it is both self-dual and homogeneous, admitting a Jordan algebraic structure [4]. The automorphism group of the second order cone can be identified by the result of Loewy and Schneider [8]: $A\mathcal{L}_2^{n+1} = \mathcal{L}_2^{n+1}$ or $A\mathcal{L}_2^{n+1} = -\mathcal{L}_2^{n+1}$ holds if and only if $A^T J_{n+1} A = \mu J_{n+1}$ for some $\mu > 0$ where $J_{n+1} = \mathrm{diag}(1, -1, \dots, -1)$.

Gowda and Trott determined the structure of the automorphism group of the p-cones in the case $p = 1, \infty$:

Proposition 2 *(Gowda and Trott, Theorem 7 in [5]). For $n \geq 2$, A belongs to $\mathrm{Aut}(\mathcal{L}_1^{n+1})$ if and only if A has the form*

$$A = \alpha \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where $\alpha > 0$ and P is an $n \times n$ generalized permutation matrix*, that is, a permutation matrix multiplied by a diagonal matrix whose diagonal elements are ± 1. Moreover, $\mathrm{Aut}(\mathcal{L}_\infty^{n+1}) = \mathrm{Aut}(\mathcal{L}_1^{n+1})$ holds.*

In particular, Proposition 2 yields the following consequences.

- $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ are not homogeneous for $n \geq 2$ because any $A \in \mathrm{Aut}(\mathcal{L}_1^{n+1}) = \mathrm{Aut}(\mathcal{L}_\infty^{n+1})$ fixes the "main axis" $\mathbb{R}_+(1, 0, \dots, 0)$ of these cones.
- $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ are never self-dual for $n \geq 2$. This is a known fact, but we will also obtain this result as a consequence of Corollary 14 where Proposition 2 will be helpful to prove the case $n = 2$. At this point, we should remark that Barker and Foran proved in Theorem 3 of [1] that a self-dual polyhedral cone in $\mathbb{R}^3$ must have an odd number of extreme rays. Since $\mathcal{L}_1^3$ and $\mathcal{L}_\infty^3$ have four extreme rays, Barker and Foran's result implies that they are never self-dual.

3. Manifolds, tangent spaces and the Gauss map

In this subsection, we will provide a brief overview of the tools we will use from manifold theory, more details can be seen in Lee's book [7] or the initial chapters of do Carmo's book [3]. First, we recall that an *n-dimensional smooth manifold M* is a second countable Haussdorf topological space equipped with a collection $\mathcal{A}$ of maps $\varphi : U \to \mathbb{R}^n$ with the following properties.

(i) each map $\varphi \in \mathcal{A}$ is such that $\varphi(U)$ is an open set of $\mathbb{R}^n$. Furthermore, φ is an homeomorphism between U and $\varphi(U)$, i.e., φ is a continuous bijection with continuous inverse.

(ii) if $\varphi : U \to \mathbb{R}^n, \psi : V \to \mathbb{R}^n$ both belong to $\mathcal{A}$ and $U \cap V \neq \emptyset$, then $\psi \circ \varphi^{-1} : \varphi^{-1}(U \cap V) \to \psi(U \cap V)$ is a C^∞ diffeomorphism, i.e., $\psi \circ \varphi^{-1}$ is a bijective function such that $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ have continuous derivatives of all orders.

(iii) for every $x \in M$, we can find a map $\varphi \in \mathcal{A}$ for which x belongs to the domain of φ.

(iv) if ψ is another map defined on a subset of M satisfying (i) and (ii), then $\psi \in \mathcal{A}$. That is, $\mathcal{A}$ is maximal.

The set $\mathcal{A}$ is called a *maximal smooth atlas* and the maps in $\mathcal{A}$ are called *charts*. If $\varphi : U \to \mathbb{R}^n$ is a chart and $x \in U$, we say that φ is a *chart around x*.

Let M_1, M_2 be smooth manifolds and $f : M_1 \to M_2$ be a function. The function f is said to be *differentiable at $x \in M_1$* if there is a chart φ of M_1 around x and a chart ψ of M_2 around $f(x)$ such that

$$\psi \circ f \circ \varphi^{-1}$$

is differentiable at $\varphi(x)$. Then, f is said to be *differentiable*, if it is differentiable throughout M_1. Similarly, we say that f is differentiable of class C^k if $\psi \circ f \circ \varphi^{-1}$ is of class C^k, for every pair of charts of M_1 and

M_2 such that the image of φ^{-1} and the domain of ψ intersect. Whether a function is differentiable at some point or is of class C^k does not depend on the particular choice of charts. The function $\psi \circ f \circ \varphi^{-1}$ is also said to be a *local representation of f*. If f is a bijection such that it is C^k everywhere and whose inverse f^{-1} is also C^k everywhere, then f is said to be a C^k *diffeomorphism*.

Let M be an n-dimensional smooth manifold. Let $C^\infty(M)$ denote the ring of C^∞ real functions $g : M \to \mathbb{R}$. A derivation of M at x is a function $v : C^\infty(M) \to \mathbb{R}$ such that for every $g, h \in C^\infty(M)$, we have

$$v(gh) = (v(g))h(x) + g(x)v(h).$$

Given an n-dimensional smooth manifold M and $x \in M$, we write T_xM for the tangent space of M at x, which is the subspace of derivations of M at x. It is a basic fact that the dimension of T_xM as a vector space coincides with the dimension of M as a smooth manifold.

Let $f : M_1 \to M_2$ be a C^1 map between smooth manifolds. Then, at each $x \in M_1$, f induces a linear map between $df_x : T_xM_1 \to T_{f(x)}M_2$ such that given $v \in T_xM_1$, $df_x(v)$ is the derivation of M_2 at $f(x)$ satisfying

$$(df_x(v))(g) = v(g \circ f),$$

for every $g \in C^\infty(N)$. The map df_x is the *differential map of f at x*. If the linear map df_x is injective everywhere, then f is said to be an *immersion*. Furthermore, if f is a C^k diffeomorphism with $k \geq 1$, then df_x is a linear bijection for every x. Recall that in order to check whether f is an immersion, it is enough to check that the local representations of f are immersions.

Now, suppose that $\alpha : (-\epsilon, \epsilon) \to M$ is a C^∞ curve with $\alpha(0) = x$. Then $d\alpha_0(0) \in T_xM$. Furthermore, T_xM coincides with the set of velocity vectors of smooth curves passing through x. With a slight abuse of notation, let us write $\alpha'(t) = d\alpha_0(t)$. With that, we have

$$T_xM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M, \alpha(0) = x, \alpha \text{ is } C^1\}, \tag{3}$$

see more details in Proposition 3.23 and pages 68–71 in [7]. With this, we can compute a differential $df_x(v)$ by first selecting a C^1 curve α contained in M with $\alpha(0) = x$, $\alpha'(0) = v$. Then, we have $df_x(v) = (f \circ \alpha)'(0)$, see Proposition 3.24 in [7].

A map $\iota : M_1 \to M_2$ is said to be a C^k*-embedding* if it is a C^k immersion and a homeomorphism on its image (here, $\iota(M_1)$ has the subspace topology induced from M_2). Now, suppose that, in fact, $M_1 \subseteq M_2$ and let $\iota : M_1 \to M_2$ denote the inclusion map, i.e., $\iota(x) = x$, for all $x \in M_1$. If ι is a C^k embedding, we say that M_1 is a C^k*-embedded submanifold of N*.

We remark that when M is an m-dimensional C^k-embedded submanifold of $\mathbb{R}^n$, the requirement that ι be a C^k embedding has the following consequences. First, the topology of M has to be the subspace topology of $\mathbb{R}^n$, i.e., the open sets of M are open sets of $\mathbb{R}^n$ intersected with M. Now, let $\varphi : U \to \mathbb{R}^m$ be a chart of M. Then, $\iota \circ \varphi^{-1} : \varphi(U) \to U$ is a C^k diffeomorphism. That is, although φ^{-1} is C^∞ when saw as a map between $\varphi(U)$ and M, its class of differentiability might decrease[1] when seen as a map between U and $\mathbb{R}^m$. For embedded manifolds of $\mathbb{R}^n$, as a matter of convention, we will always see the inverse of a chart φ as a function whose codomain is $\mathbb{R}^n$ and we will omit the embedding ι.

Furthermore, whenever M is a C^k-embedded submanifold of $\mathbb{R}^n$, we will define tangent spaces in a more geometric way. Given $x \in M$, we will define T_xM as the space of tangent vectors of C^1 curves that pass through x:

[1] Here is an example of what can happen. Let M be graph of the function $f(x) = |x|$. M is a differentiable manifold and to create a maximal smooth atlas for M we first start with a set $\mathcal{A}$ containing only the map $\varphi : M \to \mathbb{R}$ that takes $(|x|, x)$ to x. At this point, conditions $(i), (ii), (iii)$ of the definition of atlas are satisfied. Then, we add to $\mathcal{A}$ every map ψ such that $\mathcal{A} \cup \{\psi\}$ still satisfies $(i), (ii), (iii)$. The resulting set must be a maximal smooth atlas. Following the definition of differentiability between manifolds, the map φ^{-1} is C^∞ if we see it as a map between $\mathbb{R} \to M$, since $\varphi \circ \varphi^{-1}(x) = x$. However, $\iota \circ \varphi^{-1}$ is not even a C^1 map, because $|x|$ is not differentiable at 0.

$$T_xM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n, \alpha(0) = x, \alpha \subseteq M, \alpha \text{ is } C^1\}, \tag{4}$$

where $\alpha \subseteq M$ means that $\alpha(t) \in M$, for every $t \in (-\epsilon, \epsilon)$. Here, since we have an ambient space, $\alpha'(0)$ is the derivative of α at 0 in the usual sense.

Both definitions of tangent spaces presented so far are equivalent in the following sense. Let $\tilde{T}_xM$ denote the space of derivations of M at x and let $\iota : M \to \mathbb{R}^n$ denote the inclusion map. Then, $d\iota_x$ is a map between $\tilde{T}_xM$ and $T_x\mathbb{R}^n$. Then, identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, it holds that $d\iota_x(\tilde{T}_xM) = T_xM$. In particular, $\tilde{T}_xM$ and T_xM have the same dimension.

Finally, we recall that for smooth manifolds, the topological notion of *connectedness* is equivalent to the notion of *path-connectedness*, see Proposition 1.11 in [7]. Therefore, a manifold M is connected if and only if for every $x, y \in M$ there is a continuous curve $\alpha : [0, 1] \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

3.1. Graphs of differentiable maps

For a real valued function $f : U \to \mathbb{R}$ defined on $U \subseteq \mathbb{R}^n$, the graph of f is defined by

$$\operatorname{graph} f := \{(y, x) \in \mathbb{R} \times U \mid y = f(x)\} \subseteq \mathbb{R}^{n+1}.$$

In item (i) of the next proposition, for the sake of completeness, we give a proof of the well-known fact that if f is a C^k function, then $\operatorname{graph} f$ must be a C^k-embedded manifold. In item (ii) we observe the fact, also known but perhaps less well-known, that the converse also holds. This is important for us because if we know that f is C^1 but not C^2, then this creates an obstruction to the existence of certain maps between $\operatorname{graph} f$ and C^2 manifolds.

Proposition 3. *For $k \geq 1$, let $f : U \to \mathbb{R}$ be a C^1 function defined on an open subset U of $\mathbb{R}^n$.*

- (i) *If f is C^k on an open subset V of U, then $\operatorname{graph} f|_V$ is an n-dimensional C^k-embedded submanifold of $\mathbb{R}^{n+1}$.*
- (ii) *Suppose that a subset M of $\operatorname{graph} f$ is an n-dimensional C^k-embedded submanifold of $\mathbb{R}^{n+1}$, with $k \geq 1$. Then f is C^k on the open set $\pi_U(M)$, where $\pi_U : \mathbb{R} \times U \to U$ is the projection onto U.*

Proof. (i) The proof here is essentially the one contained Example 1.30 and Proposition 5.4 of [7], except that here we take into account the level of smoothness of the embedding.

First, let $M = \operatorname{graph}(f|_V)$ and consider the subspace topology inherited from $\mathbb{R}^{n+1}$ (again, see Examples 1.3 and 1.30 in [7] for more details). With the subspace topology, the map $\varphi : V \to M$, given by

$$\varphi(x) = (f(x), x)$$

is a homeomorphism between V and M, whose inverse is the projection restricted to M, that is $\varphi^{-1}(f(x), x) = x$. Furthermore, φ^{-1} induces a maximal smooth atlas of M making $\varphi^{-1} : M \to V$ a chart.[2] We now check that the inclusion $\iota : M \to \mathbb{R}^{n+1}$ is a C^k embedding. A local representation for ι is obtained by considering $\iota \circ \varphi : V \to \mathbb{R}^{n+1}$, which shows that ι is a C^k differentiable map. The inverse $\iota^{-1} : \iota(M) \to M$ is given by restricting the identity map in $\mathbb{R}^{n+1}$ to M. Since the topology on M is the subspace topology, this establishes that ι is an homeomorphism.

Furthermore, since the $(n + 1) \times n$ Jacobian matrix $J_{\iota \circ \varphi}$ of the representation of ι has rank n, we see that ι is an immersion. Hence, M is a C^k-embedded submanifold of $\mathbb{R}^{n+1}$.

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[2] The idea is the same as in Footnote 1, we start with $\mathcal{A} = \{\varphi^{-1}\}$ and add every map ψ for which $\mathcal{A} \cup \{\psi\}$ still satisfies properties (i), (ii), (iii) of the definition of atlas.

(*ii*) Take $x_0 \in \pi_U(M)$. Let $\Phi : V \to \mathbb{R}^n$ be a chart of M around $(f(x_0), x_0)$. We can write the map Φ^{-1} as

$$\Phi^{-1}(z) = (\psi(z), \varphi(z)) \in \mathbb{R} \times U \quad \text{for} \quad z \in \Phi(V),$$

for functions $\psi : \Phi(V) \to \mathbb{R}$, $\varphi : \Phi(V) \to U$. Since $\operatorname{Im} \Phi^{-1} \subseteq M \subseteq \operatorname{graph} f$, we have $\psi(z) = f(\varphi(z))$ for all $z \in \Phi(V)$. Then we obtain a local representation $\tilde{\iota} : \Phi(V) \subseteq \mathbb{R}^n \to \mathbb{R}^{n+1}$ of the inclusion map $\iota : M \to \mathbb{R}^{n+1}$ as follows:

$$\tilde{\iota}(z) := \iota \circ \Phi^{-1} = (\psi(z), \varphi(z)) = (f \circ \varphi(z), \varphi(z)).$$

Since M is C^k-embedded, φ and ψ are C^k when seen as maps $\Phi(V) \to \mathbb{R}$ and $\Phi(V) \to \mathbb{R}^n$, respectively. Let $z_0 = \Phi((f(x_0), x_0))$. Then $\varphi(z_0) = x_0$ since

$$(f(x_0), x_0) = \Phi^{-1}(z_0) = (\psi(z_0), \varphi(z_0)).$$

Note that $\operatorname{rank}(J_{\tilde{\iota}}(z_0)) = n$ holds because ι is an immersion. On the other hand, since f is C^1 by the assumption, it follows by the chain rule for the function $\psi = f \circ \varphi$ that

$$J_\psi(z_0) = J_f(\varphi(z_0))J_\varphi(z_0) = J_f(x_0)J_\varphi(z_0).$$

This means that each row of $J_\psi(z_0)$ is a linear combination of rows of $J_\varphi(z_0)$. Therefore, we conclude that

$$n = \operatorname{rank} J_{\tilde{\iota}}(z_0) = \operatorname{rank}(J_\psi(z_0)^T, J_\varphi(z_0)^T)^T = \operatorname{rank} J_\varphi(z_0).$$

Namely, the $n \times n$ matrix $J_\varphi(z_0)$ is nonsingular. Since φ is C^k, the inverse function theorem states that there exists a C^k inverse $\varphi^{-1} : W \to \mathbb{R}^n$ defined on a neighborhood W of $\varphi(z_0) = x_0$. Then, we conclude that the function

$$\psi \circ \varphi^{-1} = f \circ \varphi \circ \varphi^{-1} = f$$

is C^k on W.

To conclude, we will show that $\pi_U(M)$ is open. Since $\varphi^{-1}(W)$ is contained in the domain $\Phi(V)$ of the map φ, it follows that $W = \varphi \circ \varphi^{-1}(W) \subseteq \varphi(\Phi(V))$. Now, let $z \in \Phi(V)$. By definition, we have

$$(\psi(z), \varphi(z)) = \Phi^{-1}(z) \in V,$$

which shows that $\varphi(z) \in \pi_U(V)$. Therefore, $\varphi(\Phi(V)) \subseteq \pi_U(V) \subseteq \pi_U(M)$. Hence, we have $W \subseteq \pi_U(M)$ and so $\pi_U(M)$ is open in $\mathbb{R}^n$, since x_0 was arbitrary. $\square$

Given a diffeomorphism A between two graphs of C^1 maps $f, g : U \to \mathbb{R}$, the next proposition shows a relation of the categories of differentiability of f and g through the diffeomorphism $B : U \to U$ defined by

$$B(x) = \pi_U(A(f(x), x))$$

where $\pi_U : \mathbb{R} \times U \to U$ is the projection onto U. The map B will play a key role in the proof of our main result applied with $U = \mathbb{R}^n \setminus \{0\}$, $f(x) = \|x\|_p$ and $g(x) = \|x\|_q$. We give an illustration of the map B in Fig. 1.

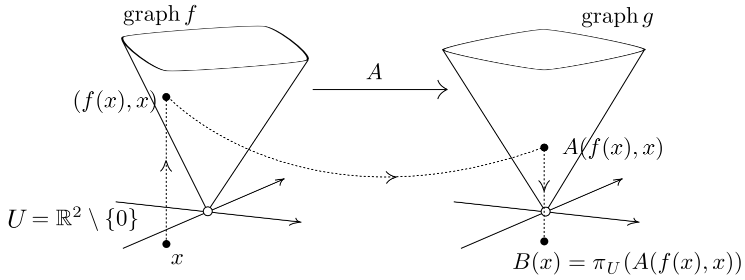

Fig. 1. Illustration of the map $B(x) = \pi_U(A(f(x), x))$.

Proposition 4. *Let $f, g : U \to \mathbb{R}$ be C^1 maps defined on an open subset U of $\mathbb{R}^n$. Suppose that $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a C^∞ diffeomorphism such that $A(\operatorname{graph} f) = \operatorname{graph} g$.*

(i) The map $B : U \to U$, $B(x) := \pi_U(A(f(x), x))$ is a C^1 diffeomorphism, where $\pi_U : \mathbb{R} \times U \to U$ satisfies $\pi_U(y, x) = x$.
(ii) For $k \geq 1$, f is C^k on a neighborhood of x if and only if g is C^k on a neighborhood of $B(x)$.

Proof. *(i)* Since f is C^1 while π_U and A are C^∞ maps, it must be the case that $B(x) = \pi_U(A(f(x), x))$ is C^1.

Let us check that the inverse of B is the map $B^{-1}(y) = \pi_U(A^{-1}(g(y), y))$. Denote

$$B'(y) = \pi_U(A^{-1}(g(y), y)).$$

For any $x \in U$, the relation $A(\operatorname{graph} f) = \operatorname{graph} g$ implies the existence of $y \in U$ such that $A(f(x), x) = (g(y), y)$. Then we have

$$B(x) = \pi_U(A(f(x), x)) = \pi_U(g(y), y) = y,$$

and, therefore,

$$B'(B(x)) = B'(y) = \pi_U(A^{-1}(g(y), y)) = \pi_U(f(x), x) = x.$$

Similarly, we obtain $B(B'(y)) = y$. Hence, $B^{-1}(y) = B'(y)$ holds.

Since $B^{-1}(y) = \pi_U(A^{-1}(g(y), y))$ is also C^1, we conclude that B is a C^1 diffeomorphism.

(ii) If f is C^k on a neighborhood V of x, then $\operatorname{graph}(f|_V)$ is an n-dimensional C^k-embedded submanifold of $\mathbb{R}^{n+1}$ by Proposition 3 *(i)*. Then, by the assumption on A, the set $M := A(\operatorname{graph} f|_V)$ is also an n-dimensional C^k-embedded submanifold of $\mathbb{R}^{n+1}$ which satisfies $M \subseteq \operatorname{graph} g$. Therefore Proposition 3 *(ii)* implies that g is C^k on the open set $\pi_U(M) = \pi_U(A(\operatorname{graph} f|_V))$ which contains the point $\pi_U(A(f(x), x))$.

The converse of the assertion follows by applying the same argument to the diffeomorphism A^{-1} because $A^{-1}(\operatorname{graph} g) = \operatorname{graph} f$ and $\pi_U(A^{-1}(g(y), y)) = x$ holds for $y = B(x) = \pi_U(A(f(x), x))$. $\square$

3.2. The Gauss map

In this subsection, let M be a C^k-embedded submanifold of $\mathbb{R}^n$ with dimension $n - 1$ and $k \geq 1$. In this case, M is sometimes called a *hypersurface* and when $n = 3$, M is called a *surface*. The differential geometry of surfaces is, of course, a classical subject discussed in many books, e.g., [2].

In the theory of surfaces, a *Gauss map* is a continuous function that associates to $x \in M$ a unit vector which is orthogonal to T_xM. Unless M is an orientable surface, it is not possible to construct a Gauss map that is defined globally over M. However, given any $x \in M$, it is always possible to construct a Gauss map in a neighborhood of x. For the sake of self-containment, we will give a brief account of the construction of the Gauss map for hypersurfaces.

For what follows, we suppose that $\mathbb{R}^n$ is equipped with some inner product $\langle \cdot, \cdot \rangle$ and the norm is given by $\|x\| = \sqrt{\langle x, x \rangle}$, for all $x \in \mathbb{R}^n$. Recalling (4), $T_x M$ is seen as a subspace of $\mathbb{R}^n$ and we will equip $T_x M$ with the same inner product $\langle \cdot, \cdot \rangle$.

Definition 5. Let M be a C^k-embedded submanifold of $\mathbb{R}^n$ and let $x \in M$. A C^r *Gauss map around* x is a C^r function $N : U \to \mathbb{R}^n$ such that $U \subseteq M$ is a neighborhood of x in M and

$$N(x) \in (T_x M)^\perp \quad \text{and} \quad \|N(x)\| = 1,$$

for all $x \in U$, where $(T_x M)^\perp$ is the orthogonal complement to $T_x M$.

For what follows, let $x^1, \ldots, x^n \in \mathbb{R}^n$ and let $\det(x^1, \ldots, x^n)$ denote the determinant of the matrix such that its i-th column is given by x^i. Since the determinant is a multilinear function, if we fix the first $n-1$ elements, we obtain a linear functional f such that

$$f(x) = \det(x^1, \ldots, x^{n-1}, x).$$

Since f is a linear functional, there is a unique vector $\Lambda(x^1, \ldots, x^{n-1}) \in \mathbb{R}^n$ satisfying

$$\langle \Lambda(x^1, \ldots, x^{n-1}), x \rangle = f(x),$$

for all $x \in \mathbb{R}^n$. Furthermore, $\Lambda(x^1, \ldots, x^{n-1}) = 0$ is zero if and only if the x^i are linearly dependent.

Proposition 6. *Let $M \subseteq \mathbb{R}^n$ be an $(n-1)$ dimensional C^k-embedded manifold, with $k \geq 1$. Then, for every chart $\varphi : U \to \mathbb{R}^{n-1}$, there exists a C^{k-1} local Gauss map of M defined over U.*

Proof. Let $\varphi : U \to \mathbb{R}^{n-1}$ be a chart of M. Then, φ^{-1} is a function with domain $\varphi(U)$ (which is an open set of $\mathbb{R}^{n-1}$) and codomain $\mathbb{R}^n$. Let $u \in U$. It is well-known that the partial derivatives of φ^{-1} at $\varphi(u)$ are a basis for $T_u M$, e.g., page 60 and Proposition 3.15 in [7]. Let $v^i(u)$ be the partial derivative of φ^{-1} at $\varphi(u)$ with respect the i-th variable. We define a Gauss map N over U by letting

$$N(x) = \frac{\Lambda(v^1(u), \ldots, v^{n-1}(u))}{\|\Lambda(v^1(u), \ldots, v^{n-1}(u))\|}.$$

Since the $v^i(u)$ are a basis for $T_u M$, $\Lambda(v^1(u), \ldots, v^{n-1}(u))$ is never zero. In addition, because φ^{-1} is of class C^k, N must be of class C^{k-1}. $\square$

3.3. A lemma on hyperplanes and embedded submanifolds

Let M be a connected C^1-embedded $n-1$ dimensional submanifold of $\mathbb{R}^n$ (i.e., a hypersurface) that is contained in a finite union of distinct hyperplanes $H_1, \ldots, H_r$. The goal of this section is to prove that M must be entirely contained in one of the hyperplanes. The intuition comes from the case $n = 3$: a surface in $\mathbb{R}^3$ cannot, say, be contained in $H_1 \cup H_2$ and also intersect both H_1 and H_2 because it would generate a "corner" at the intersection $M \cap H_1 \cap H_2$, thus destroying smoothness. This is illustrated in Fig. 2.

This is probably a well-known differential geometric fact but we could not find a precise reference, so we give a proof here. Nevertheless, our discussion is related to the following classical fact: a point in a surface for

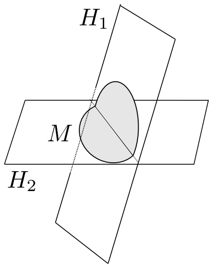

Fig. 2. A surface M cannot be smooth if it is connected, contained in $H_1 \cup H_2$, but not entirely contained in neither H_1 nor H_2.

which the derivative of the Gauss map vanishes is called a *planar point* and a connected surface in $\mathbb{R}^3$ such that all its points are planar must be a piece of a plane, see Definitions 7, 8 and the proof of Proposition 4 of Chapter 3 of [2].

In our case, the fact that M is contained in a finite number of hyperplanes hints that the image of any Gauss map of M should be confined to the directions that are orthogonal to those hyperplanes. This, by its turn, suggests that the derivative of N should vanish everywhere, i.e., all points must be planar. In fact, our proof is inspired by the proof of Proposition 4 of Chapter 3 of [2] and we will use the same compactness argument at the end.

To start, we observe that the tangent of a curve contained in $H_1, \ldots, H_r$ must also be contained in those hyperplanes.

Proposition 7. *Let $H_i = \{a_i\}^\perp$ be hyperplanes in $\mathbb{R}^n$ for $i = 1, \ldots, r$. Suppose that a C^1 curve $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is contained in $X = \bigcup_{i=1}^r H_i$. Then, $\alpha'(0) \in X$.*

Proof. Changing the order of the hyperplanes if necessary, we may assume that

$$\alpha(0) \in H_1 \cap \cdots \cap H_s$$
$$\alpha(0) \notin H_{s+1}, \ldots, H_r.$$

Since α is contained in X, we have $s \geq 1$. Furthermore, because α is continuous, there is $\hat{\epsilon} > 0$ such that

$$\alpha(\epsilon) \notin H_{s+1}, \ldots, H_r, \tag{5}$$

for $-\hat{\epsilon} < \epsilon < \hat{\epsilon}$.

Now, suppose for the sake of obtaining a contradiction that $\alpha'(0)$ does not belong to any of these hyperplanes $H_1, \ldots, H_s$. Therefore, for all $i \in \{1, \ldots, s\}$, we have

$$\langle \alpha(0), a_i \rangle = 0, \qquad \langle \alpha'(0), a_i \rangle \neq 0.$$

Since $\alpha'(\cdot)$ is continuous, we can select $0 < \tilde{\epsilon} < \hat{\epsilon}$ such that for all $i \in \{1, \ldots, s\}$ and $\epsilon \in (-\tilde{\epsilon}, \tilde{\epsilon})$, we have

$$\langle \alpha'(\epsilon), a_i \rangle \neq 0.$$

By the mean value theorem applied to $\langle \alpha(\cdot), a_i \rangle$ on the interval $[0, \tilde{\epsilon}/2]$, we obtain that $\langle \alpha(\tilde{\epsilon}/2), a_i \rangle \neq 0$, for all $i \in 1, \ldots, s$. Since $\tilde{\epsilon}/2 \in (-\hat{\epsilon}, \hat{\epsilon})$, (5) implies that

$$\langle \alpha(\tilde{\epsilon}/2), a_i \rangle \neq 0,$$

for $i \in \{s + 1, \ldots, r\}$ too. This shows that $\alpha(\tilde{\epsilon}/2) \notin X$, which is a contradiction. □

Before we prove the main lemma of this subsection, we need the following observation on finite dimensional vector spaces.

Proposition 8. *A finite dimensional real vector space V is not a countable union of subspaces of dimension strictly smaller than* $\dim V$*.*

Proof. Suppose that V is a countable union $\bigcup W_i$ of subspaces of dimension smaller than $\dim V$. Take the unit ball $B \subseteq V$. Then, $B = \bigcup W_i \cap B$. However, this is not possible since each $W_i \cap B$ has measure zero, while B has nonzero measure. □

We now have all the necessary pieces to prove the main lemma.

Lemma 9. *Let $X \subseteq \mathbb{R}^n$ be a union of finitely many hyperplanes $H_i = \{a_i\}^\perp$, $a_i \neq 0$, $i = 1, \dots, r$. Let M be an $(n-1)$ dimensional differentiable manifold that is connected, C^1-embedded in $\mathbb{R}^n$ and contained in X. Then, M must be entirely contained in one of the H_i.*

Proof. We proceed by induction on r. The case $r = 1$ is clear, so suppose that $r > 1$.

Consider a chart $\varphi : U \to \mathbb{R}^{n-1}$ such that $U \subseteq M$ is connected and construct a C^0 (i.e., continuous) Gauss map N in U, as in Proposition 6. Let $u \in U$ and let us examine the tangent space T_uM. We have

$$T_uM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M,\ \alpha(0) = u,\ \alpha \text{ is } C^1\}.$$

By Proposition 7,

$$T_uM \subseteq X.$$

Therefore,

$$T_uM = \bigcup_{i=1}^{r} H_i \cap T_uM.$$

Each $H_i \cap T_uM$ is a subspace of T_uM (an intersection of subspaces is also a subspace!). By Proposition 8, T_uM cannot be a union of subspaces of dimension less than $\dim T_uM = n-1$. Therefore, there exists some index j such that $H_j \cap T_uM = T_uM$. Since both T_uM and H_j have dimension $n-1$, we conclude that $H_j = T_uM$.

In particular, the Gauss map N satisfies $N(u) = a_j/\|a_j\|$ or $N(u) = -a_j/\|a_j\|$. Therefore, for all $u \in U$, we have

$$N(u) \in \left\{ \pm \frac{a_i}{\|a_i\|} \mid i = 1, \dots, r \right\}.$$

Since U is connected and N is continuous, we conclude that the Gauss map N is constant. Denote this constant vector by v.

Let $\psi = \langle \varphi^{-1}(\cdot), v \rangle$. Since φ is a chart, given any $w \in \varphi(U)$, the differential

$$d\varphi_w^{-1} : \mathbb{R}^{n-1} \to T_{\varphi^{-1}(w)}M$$

is a linear bijection. Since $T_{\varphi^{-1}(w)}M$ is orthogonal to v, we conclude that $\psi' = 0$. Therefore ψ must be constant and there is κ_0 such that $\langle \varphi^{-1}(w), v \rangle = \kappa_0$, for all $w \in \varphi(U)$. That is, $\langle u, v \rangle = \kappa_0$, for all $u \in U$.

Recall that, given $x \in M$, we can always obtain a chart $\varphi : U \to M$ around x such that U is connected. Therefore, the discussion so far shows that every $x \in M$ has a neighborhood U such that U is entirely contained in a hyperplane

$$\{z \mid \langle z, v_x \rangle = \kappa_x\},$$

where v_x has the same direction as one of the $a_1, \ldots, a_r$. Now, fix some $x \in M$ and let $y \in M$, $y \neq x$. Since M is connected, there is a continuous curve $\alpha : [0, 1] \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Similarly, for every $t \in [0, 1]$, we can find a neighborhood $U_t \subseteq M$ of $\alpha(t)$ such that U_t is contained in a hyperplane $\{z \mid \langle z, v_t \rangle = \kappa_t\}$ where v_t is parallel to one of $a_1, \ldots, a_r$. In particular

$$[0, 1] \subseteq \bigcup_{t \in [0,1]} \alpha^{-1}(U_t).$$

Since the U_t are open in M and α is continuous, the $\alpha^{-1}(U_t)$ form an open cover for the compact set $[0, 1]$. Therefore, the Heine–Borel theorem implies that a finite number of the $\alpha^{-1}(U_t)$ are enough to cover $[0, 1]$. As a consequence, α itself is contained in finitely many neighborhoods $U_{t_1}, \ldots U_{t_\ell}$. Now, we note the following:

- If $U_{t_i} \cap U_{t_j} \neq \emptyset$ then $U_{t_i} \cap U_{t_j}$ is a nonempty open set in M and therefore, an embedded submanifold of dimension $n - 1$, see Proposition 5.1 in [7]. Furthermore $U_{t_i} \cap U_{t_j}$ is contained in the set

$$H = \{z \in \mathbb{R}^n \mid \langle z, v_{t_i} \rangle = \kappa_{t_i},\ \langle z, v_{t_j} \rangle = \kappa_{t_j}\}.$$

  Therefore, the smooth manifold H must have at least dimension $n-1$. We conclude that "$\langle z, v_{t_i} \rangle = \kappa_{t_i}$" and "$\langle z, v_{t_j} \rangle = \kappa_{t_j}$" define the same hyperplane. So, U_{t_i} and U_{t_j} are in fact, contained in the same hyperplane.
- U_{t_1} must intersect some of the $U_{t_2}, \ldots, U_{t_\ell}$ because if it does not, then $\alpha^{-1}(U_{t_1})$ and $\alpha^{-1}(\cup_{i=2}^{n} U_{t_i})$ disconnect the connected set $[0, 1]$. Changing the order of the sets if necessary, we may therefore assume that U_{t_1} and U_{t_2} intersect and, therefore, lie in the same hyperplane. Similarly, the union $U_{t_1} \cup U_{t_2}$ must intersect one of the remaining neighborhoods $U_{t_3}, \ldots, U_{t_\ell}$, lest we disconnect the interval $[0, 1]$. By induction, we conclude that all neighborhoods lie in the same hyperplane.

In particular, x and y lie in the same hyperplane and, therefore, M is entirely contained in some hyperplane whose normal direction has the same direction as one of the $a_1, \ldots, a_r$.

So far, we have shown that M is entirely contained in a hyperplane of the form

$$\{z \in \mathbb{R}^n \mid \langle z, v \rangle = \kappa_0\}.$$

Without loss of generality, we may assume that v has the same direction as a_1. If $\kappa_0 = 0$, we are done. Otherwise, since v has the same direction as a_1, it follows that M does not intersect H_1 and

$$M \subseteq \bigcup_{i=2}^{r} H_i.$$

By the induction hypothesis, M must be contained in one of the $H_2, \ldots, H_r$. $\square$

4. Main results

In this section, we show the main results on p-cones. We begin by observing a basic fact on the differentiability of p-norms.

Lemma 10. *Let $n \geq 2$ and $p \in (1, \infty)$.*

(i) $\|\cdot\|_p$ *is* C^1 *on* $\mathbb{R}^n \setminus \{0\}$.
(ii) If $p \in (1, 2)$ *then* $\|\cdot\|_p$ *is* C^2 *on a neighborhood of* x *if and only if* $x_i \neq 0$ *for all* i.
(iii) If $p \in [2, \infty)$ *then* $\|\cdot\|_p$ *is* C^2 *on* $\mathbb{R}^n \setminus \{0\}$.

Proof. (i) $\|\cdot\|_p$ is C^1 on $\mathbb{R}^n \setminus \{0\}$ because

$$\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = \|x\|_p^{1-p} |x_i|^{p-1} \operatorname{sign}(x_i).$$

(ii) If $x_i \neq 0$ for all i, it is straightforward to see that $\|\cdot\|_p$ is C^2 on a neighborhood of x. For the converse, consider a point $x \neq 0$ with $x_i = 0$ for some i. Let us verify that $\frac{\partial^2 \|\cdot\|_p}{\partial x_i^2}(x)$ does not exist if $p \in (1, 2)$. Indeed, $\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = 0$ holds and so

$$\begin{aligned}
\lim_{h \to 0} \frac{1}{h} \left(\frac{\partial \|\cdot\|_p}{\partial x_i}(x + he_i) - \frac{\partial \|\cdot\|_p}{\partial x_i}(x) \right) &= \lim_{h \to 0} h^{-1} \frac{\partial \|\cdot\|_p}{\partial x_i}(x + he_i) \\
&= \lim_{h \to 0} h^{-1} \|x + he_i\|_p^{1-p} |h|^{p-2} h \\
&= \lim_{h \to 0} \|x + he_i\|_p^{1-p} |h|^{p-2} \\
&= \begin{cases} +\infty & (p < 2) \\ 0 & (p > 2) \end{cases}.
\end{aligned}$$

Hence, when $p \in (1, 2)$, the derivative $\frac{\partial^2 \|\cdot\|_p}{\partial x_i^2}(x)$ exists if and only if $x_i \neq 0$.

(iii) For $p > 2$ (the assertion in the case $p = 2$ is clear),

$$\frac{\partial^2 \|\cdot\|_p}{\partial x_j \partial x_i}(x) = (1 - p) \|x\|_p^{1-2p} |x_i x_j|^{p-1} \operatorname{sign}(x_i) \operatorname{sign}(x_j)$$

holds if $i \neq j$, otherwise we have

$$\frac{\partial^2 \|\cdot\|_p}{\partial x_i^2}(x) = (1 - p) \|x\|_p^{1-2p} x_i^{2(p-1)} + (p - 1) \|x\|_p^{1-p} |x_i|^{p-2}. \quad \square$$

We now move on to the main result of this paper.

Theorem 11. *Let $p, q \in [1, \infty]$, $p \leq q$, $n \geq 2$ and $(p, q, n) \neq (1, \infty, 2)$. Suppose that $\mathcal{L}_p^{n+1}$ and $\mathcal{L}_q^{n+1}$ are isomorphic, that is,*

$$A \mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1}$$

holds for some $A \in GL_{n+1}(\mathbb{R})$. Then $p = q$ must hold. Moreover, if $p \neq 2$, then we have $A \in \operatorname{Aut}(\mathcal{L}_1^{n+1})$.

Proof. The proof consists of three parts **I**, **II**, and **III**.

I. First we consider the case $p \in \{1, \infty\}$ corresponding to the case when $\mathcal{L}_p^{n+1}$ is polyhedral. Since A preserves polyhedrality, q must be 1 or ∞ too. Note that $\mathcal{L}_1^{n+1}$ and $\mathcal{L}_\infty^{n+1}$ cannot be isomorphic if $n \geq 3$ because they have different numbers of extreme rays, see Section 2.2. Therefore, $p = q = 1$ or $p = q = \infty$ must hold. Since $\mathrm{Aut}(\mathcal{L}_\infty^{n+1}) = \mathrm{Aut}(\mathcal{L}_1^{n+1})$ holds (Proposition 2), the assertion is verified in the case $p \in \{1, \infty\}$.

II. Now let $p, q \in (1, \infty)$. Then the set

$$M_p := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \mid t = \|x\|_p\}$$

becomes a C^1-embedded submanifold of $\mathbb{R}^{n+1}$ by Lemma 10 (i) and Proposition 3 (i). Note that $A\mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1}$ implies $AM_p = M_q$ since A maps the boundary of $\mathcal{L}_p^{n+1}$ onto the boundary of $\mathcal{L}_q^{n+1}$.

It suffices to consider the case $p, q \in (1, 2)$ by the following observation.

(a) The case $1 < p < 2 \leq q < \infty$ does not happen in view of Proposition 4 and Lemma 10. In fact, since $\|\cdot\|_q$ is C^2 on $\mathbb{R}^n \setminus \{0\}$ and $A^{-1}M_q = M_p$ holds, Proposition 4 implies that $\|\cdot\|_p$ is C^2 on $\mathbb{R}^n \setminus \{0\}$ but this is a contradiction.

(b) If $2 \leq p \leq q < \infty$ holds, then taking the dual of the relation $A\mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1}$ with respect to the Euclidean inner product, it follows that

$$A^{-T}\mathcal{L}_{p^*}^{n+1} = \mathcal{L}_{q^*}^{n+1}$$

where p^* and $q^* \in (1, 2]$ are the conjugates of p and q, respectively. Either $p^* = q^* = 2$ or $p^*, q^* \in (1, 2)$ must hold by (a). If $p^* = q^* = 2$, then we are done since this implies that $p = q = 2$. Now, suppose that $p^*, q^* \in (1, 2)$. If we prove that $p^* = q^*$ and $A^{-T} \in \mathrm{Aut}(\mathcal{L}_1^{n+1})$, then we conclude that $p = q$ and $A \in \mathrm{Aut}(\mathcal{L}_1^{n+1})^{-T}$. However, by Proposition 2, $\mathrm{Aut}(\mathcal{L}_1^{n+1})^{-T} = \mathrm{Aut}(\mathcal{L}_1^{n+1})$. (Note that, if P is a generalized permutation matrix, then so is P^{-T}.)

From cases (a), (b) we conclude that it is enough to consider the case $p, q \in (1, 2)$, which we will do next.

III. Let $p, q \in (1, 2)$. We show by induction on n that every $A \in GL_{n+1}(\mathbb{R})$ with $A\mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1}$ is a bijection from the set

$$E = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_i^n)$$

onto E itself, where e_i^n is the i-th standard unit vector in $\mathbb{R}^n$. First, let us check that this claim implies $A \in \mathrm{Aut}(\mathcal{L}_1^{n+1})$ and $p = q$. Taking the conical hull of the relation $AE = E$, we conclude that

$$A\mathcal{L}_1^{n+1} = A(\mathrm{cone}(E)) = \mathrm{cone}(AE) = \mathrm{cone}(E) = \mathcal{L}_1^{n+1},$$

where the relation $\mathrm{cone}(E) = \mathcal{L}_1^{n+1}$ holds because a pointed closed convex cone is the conical hull of its extreme rays (see Theorem 18.5 in [10]) and E is precisely the union of all the extreme rays of $\mathcal{L}_1^{n+1}$ with the origin removed, see Section 2.2. Therefore, we have

$$A \in \mathrm{Aut}(\mathcal{L}_1^{n+1}) \subseteq \mathrm{Aut}(\mathcal{L}_p^{n+1}),$$

where the last inclusion follows by Proposition 2 because $\|Px\|_p = \|x\|_p$ for any generalized permutation matrix P. Then $\mathcal{L}_p^{n+1} = A\mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1}$ and so $p = q$ must hold.

Now, let us show the claim that A is a bijection on E. Consider the map $\xi_p : \mathbb{R}^n \setminus \{0\} \to M_p$ defined by $\xi_p(x) = (\|x\|_p, x)$ whose inverse $\xi_p^{-1} : M_p \to \mathbb{R}^n \setminus \{0\}$ is the projection $\xi_p^{-1}(t, x) = x$. By Proposition 4, the map $B : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by

$$B(x) = \xi_q^{-1} \circ A|_{M_p} \circ \xi_p(x)$$

is a C^1 diffeomorphism. Moreover, $\|\cdot\|_p$ is C^2 on a neighborhood of x if and only if $\|\cdot\|_q$ is C^2 on a neighborhood of $B(x)$. Since $p, q \in (1, 2)$, each of the functions $\|\cdot\|_p$ and $\|\cdot\|_q$ is C^2 on a neighborhood of x if and only if $x_i \neq 0$ for all i (Lemma 10). This implies that the set

$$X = \{x \in \mathbb{R}^n \setminus \{0\} \mid x_i = 0 \text{ for some } i\}$$

satisfies

$$B(X) = X$$

because x belongs to X if and only if $\|\cdot\|_p$ and $\|\cdot\|_q$ are never C^2 on any neighborhood of x.

III.a. Consider the case $n = 2$. Then the set X can be written as

$$\begin{aligned} X &= \{x \in \mathbb{R}^2 \setminus \{0\} \mid x_1 = 0 \text{ or } x_2 = 0\} \\ &= \mathbb{R}_{++}(0,1) \cup \mathbb{R}_{++}(0,-1) \cup \mathbb{R}_{++}(1,0) \cup \mathbb{R}_{++}(-1,0) \\ &= \bigcup_{i=1}^{2} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(\sigma e_i^2). \end{aligned}$$

Then $\xi_p(X)$ and $\xi_q(X)$ coincide with E:

$$\xi_p(X) = \xi_q(X) = \bigcup_{i=1}^{2} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_i^2) = E.$$

Moreover, A is bijective on E because

$$A(\xi_p(X)) = \xi_q \circ \xi_q^{-1} \circ A|_{M_p} \circ \xi_p(X) = \xi_q \circ B(X) = \xi_q(X).$$

Thus, the claim $AE = E$ holds in the case $n = 2$.

III.b. Now let $n \geq 3$ and suppose that the claim is valid for $n - 1$. Denote

$$X_i := \{x \in \mathbb{R}^n \setminus \{0\} \mid x_i = 0\}, \quad M_p^i := \xi_p(X_i) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\} : t = \|x\|_p,\ x_i = 0\}.$$

With that, we have

$$X = \bigcup_{i=1}^{n} X_i.$$

We show that for any $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that

$$B(X_i) = X_j.$$

For any i, the set X_i is a connected $(n - 1)$ dimensional C^1-embedded submanifold of $\mathbb{R}^n$ contained in X. Since $B : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a C^1 diffeomorphism satisfying $B(X) = X$, the set $B(X_i)$ is also a connected $(n - 1)$ dimensional C^1-embedded submanifold of $\mathbb{R}^n$ contained in X. Then, since $X \cup \{0\}$ is the union of the hyperplanes $X_i \cup \{0\}$, $i = 1, \dots, n$, it follows from Proposition 9 that $B(X_i)$ is entirely contained in some hyperplane $X_j \cup \{0\}$. Then we have

$$B(X_i) \subseteq X_j.$$

By the same argument, the set $B^{-1}(X_j)$ is contained in some hyperplane $X_k \cup \{0\}$, that is, $B^{-1}(X_j) \subseteq X_k$ holds. This shows that

$$X_i = B^{-1}(B(X_i)) \subseteq B^{-1}(X_j) \subseteq X_k.$$

Since X_i cannot be a subset of X_k if $i \neq k$, it follows that $i = k$. Then, we obtain $X_i = B^{-1}(X_j)$, i.e., $B(X_i) = X_j$.

Since B is a bijection, the above argument shows that there exists a permutation τ on $\{1, \dots, n\}$ such that

$$B(X_i) = X_{\tau(i)}.$$

Then we have

$$A(M_p^i) = \xi_q \circ \xi_q^{-1} \circ A|_{M_p} \circ \xi_p(X_i) = \xi_q \circ B(X_i) = \xi_q(X_{\tau(i)}) = M_q^{\tau(i)}.$$

Taking the linear span both sides, we also have

$$A(V_i) = V_{\tau(i)} \quad \text{where} \quad V_i := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid x_i = 0\}.$$

Now we apply the induction hypothesis to the isomorphism $A|_{V_i}$ as follows. Define the isomorphism $\varphi_i : V_i \to \mathbb{R}^n$ by

$$\varphi_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = (t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

and consider the isomorphism $A_i := \varphi_{\tau(i)} \circ A|_{V_i} \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. By the above argument, we see that $A_i(\mathcal{L}_p^n) = \mathcal{L}_q^n$:

$$A_i(\mathcal{L}_p^n) = \varphi_{\tau(i)} \circ A|_{V_i} \circ \varphi_i^{-1}(\mathcal{L}_p^n) = \varphi_{\tau(i)} \circ A(\operatorname{cone} M_p^i) = \varphi_{\tau(i)}(\operatorname{cone} M_q^{\tau(i)}) = \mathcal{L}_q^n.$$

So the induction hypothesis implies that A_i is bijective on

$$\bigcup_{j=1}^{n-1} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^{n-1}).$$

Therefore, $A|_{V_i} = \varphi_{\tau(i)}^{-1} \circ A_i^{-1} \circ \varphi_i$ is a bijection from

$$\bigcup_{j \in \{1,\dots,n\} \setminus \{i\}} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^n)$$

onto

$$\bigcup_{j \in \{1,\dots,n\} \setminus \{\tau(i)\}} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^n).$$

Combining this result for each $i = 1, \dots, n$, it turns out that A is bijective on

$$E = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_i^n). \quad \square$$

Combining the latter assertion of Theorem 11 and Proposition 2, we obtain the description of the automorphism group of the p-cones.

Corollary 12. *For $p \in [1, \infty]$, $p \neq 2$ and $n \geq 2$, we have $\mathrm{Aut}(\mathcal{L}_p^{n+1}) = \mathrm{Aut}(\mathcal{L}_1^{n+1})$. In particular, any $A \in \mathrm{Aut}(\mathcal{L}_p^{n+1})$ can be written as*

$$A = \alpha \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where $\alpha > 0$ and P is an $n \times n$ generalized permutation matrix.

We can also recover our previous result on the non-homogeneity of p-cones with $p \neq 2$. In contrast to [6], here we do not require the theory of T-algebras.

Corollary 13. *For $p \in [1, \infty]$, $p \neq 2$ and $n \geq 2$, the p-cone $\mathcal{L}_p^{n+1}$ is not homogeneous.*

Proof. By Corollary 12, for any $A \in \mathrm{Aut}(\mathcal{L}_p^{n+1}) = \mathrm{Aut}(\mathcal{L}_1^{n+1})$, we have that the vector $(1, 0, \ldots, 0)$ is an eigenvector of A. So, there is no automorphism of $\mathcal{L}_p^{n+1}$ that maps $(1, 0, \ldots, 0)$ to an interior point of $\mathcal{L}_p^{n+1}$ that does not belong to

$$\{(\beta, 0, \ldots, 0) \mid \beta > 0\}.$$

Hence, $\mathcal{L}_p^{n+1}$ cannot be homogeneous. $\square$

Now the non-self-duality of p-cones $\mathcal{L}_p^{n+1}$ for $p \neq 2$ and $n \geq 2$ is an immediate consequence of Theorem 11 in view of Proposition 1, while we need an extra argument for the case $(p, q, n) = (1, \infty, 2)$.

Corollary 14. *For $p \in [1, \infty]$, $p \neq 2$ and $n \geq 2$, the p-cone $\mathcal{L}_p^{n+1}$ is not self-dual under any inner product.*

Proof. Suppose that $\mathcal{L}_p^{n+1}$ is self-dual under some inner product. Then, by Proposition 1, there exists a symmetric positive definite matrix A such that

$$A\mathcal{L}_p^{n+1} = \mathcal{L}_q^{n+1} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

If $(p, q, n) \neq (1, \infty, 2), (\infty, 1, 2)$, then $p = q = 2$ must hold by Theorem 11. Now let us consider the case $(p, q, n) = (1, \infty, 2)$, i.e., $A\mathcal{L}_1^3 = \mathcal{L}_\infty^3$. Recalling (2), we have $B\mathcal{L}_1^3 = \mathcal{L}_\infty^3$ with

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}\cos(\pi/4) & -\sqrt{2}\sin(\pi/4) \\ 0 & \sqrt{2}\sin(\pi/4) & \sqrt{2}\cos(\pi/4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore, $B^{-1}A \in \mathrm{Aut}(\mathcal{L}_1^3)$ holds. Then, by Proposition 2, the matrix A can be written as $A = BC$ where C is of the form

$$C = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix}, \quad \alpha > 0.$$

Since A is symmetric, it has one of the following forms:

$$\alpha\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & -1\\ 0 & -1 & 1\end{pmatrix},\quad \alpha\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1\end{pmatrix},\quad \alpha\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & 1\end{pmatrix},\quad \alpha\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & -1\end{pmatrix}.$$

None of them is positive definite. Therefore, we obtain a contradiction. □

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