J. Math. Anal. Appl. 478 (2019) 1133–1146

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence results of infinitely many solutions for $p(x)$-Kirchhoff type triharmonic operator with Navier boundary conditions

Belgacem Rahal

Institut Supérieur des Sciences Appliquées et de Technologie de Kairouan, Avenue Beit El Hikma, 3100 Kairouan, Tunisia

ARTICLE INFO

Article history:
Received 13 January 2019
Available online 6 June 2019
Submitted by E. Saksman

Keywords:
Variational method
$p(x)$-Triharmonic operator
Kirchhoff type equation
Nonlocal problem
Navier boundary conditions
Generalized Lebesgue-Sobolev spaces

ABSTRACT

This paper is concerned with the existence of weak solutions to a class of nonlinear elliptic Navier boundary value problem involving the **p(x)-Kirchhoff type triharmonic operator**. By means of a variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence of weak solutions for the problem.

1. Introduction

In this paper, we are concerned with the existence of weak solutions for the following nonlinear elliptic Navier boundary value problem involving the $p(x)$-Kirchhoff type triharmonic operator

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\Delta^3_{p(x)}u = \lambda\zeta(x)|u|^{\alpha(x)-2}u - \lambda\xi(x)|u|^{\beta(x)-2}u & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, with $n > 3$, is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $1 < p(x) < \frac{n}{3}$ for any $x \in \overline{\Omega}$, $\zeta, \xi, \alpha, \beta \in C(\overline{\Omega})$ are nonnegative functions, λ is a positive parameter and $\Delta^3_{p(x)}u := div\left(\Delta\left(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right)\right)$ is the so-called $p(x)$-triharmonic operator.

Equation (1.1) is called a nonlocal problem because of the presence of the term M, which implies that the equation in (1.1) is no longer a pointwise equation.

The $p(x)$-trilaplacian possesses more complicated nonlinearities than the p-trilaplacian with $p(x) \equiv p > 1$ is a constant, for example, it is inhomogeneous. The study of various mathematical problems with variable

E-mail address: rahhalbelgacem@gmail.com.

https://doi.org/10.1016/j.jmaa.2019.06.006
0022-247X/

exponent has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [33,37]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [6,5]. We refer the reader to [15,22,34,38] for an overview of references on this subject.

A typical model of an elliptic equation with $p(x)$-growth conditions is

$$-\Delta_{p(x)}u = f(x,u). \tag{1.2}$$

The operator $\Delta_{p(x)}u := div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called the $p(x)$-Laplace operator and it is a natural generalization of the p-Laplace operator, in which $p(x) \equiv p > 1$ is a constant.

Problems like (1.2) with Dirichlet boundary condition have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic. We will use the notations such as p^+ and p^- where

$$p^- := \min_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \max_{x\in\overline{\Omega}} p(x).$$

In the case $f(x,u) = \lambda|u|^{p(x)-2}u$ in [20] the authors established the existence of infinitely many eigenvalues for problem (1.2) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that Λ is discrete, $\sup\Lambda = \infty$ and pointed out $\inf\Lambda = 0$ for general $p(x)$, and only under some special conditions $\inf\Lambda > 0$. In the case $f(x,u) = \lambda|u|^{q(x)-2}u$, there are different papers, for example, in [19] the same authors proved that any $\lambda > 0$ is an eigenvalue of problem (1.2) when $p^+ < q^-$ and also when $q^+ < p^-$. In [32] the authors proved the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin when $q^- < p^-$ and $q(x)$ has subcritical growth in problem (1.2).

In the case $f(x,u) = \lambda v(x)|u|^{q(x)-2}u + \lambda w(x)|u|^{h(x)-2}u$ with q and h are continuous functions on $\overline{\Omega}$ such that $1 < q(x) < p(x) < h(x) < p^*(x) := \frac{np(x)}{n-p(x)}$ and $p(x) < n$, the authors [30] showed the existence of at least one nontrivial weak solution. There approach relies on the variable exponent theory of Lebesgue and Sobolev spaces combined with adequate variational methods and the Mountain Pass Theorem.

In recent years, elliptic problems involving $p(x)$-Kirchhoff type Laplacian operator

$$-M\left(\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u), \tag{1.3}$$

have been studied in many papers, we refer to [1,9,11,13], in which the authors have used different methods to get the existence of the solutions for (1.3). Infinitely many solutions of the problem (1.3) in the special case when $M(t) = a + bt$, has been studied by Dai and Liu in [13], by using a direct variational approach. In [11], the author considered the problem (1.3) in the case when $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying the following conditions:

(M_0) There exist $c_2 \ge c_1 > 0$, $\delta_2 \ge \delta_1 > 1$ such that

$$c_1 t^{\delta_1 - 1} \le M(t) \le c_2 t^{\delta_2 - 1}$$

for all $t \in \mathbb{R}_+$.

(M_1) For all $t \in \mathbb{R}_+$, $\widehat{M}(t) \geq M(t)t$ holds, where $\widehat{M}(t) = \int_0^t M(z)dz$, and the special case

$$f(x,u) = \lambda \left(a(x)|u|^{s_1(x)-2}u + b(x)|u|^{s_2(x)-2}u \right),$$

where $p, s_1, s_2 \in C(\overline{\Omega})$ with

$$1 < s_1^- \leq s_1^+ < \delta_1 p^- < \delta_2 p^+ < s_2^- \leq s_2^+ < \min\left\{ n, \frac{np^-}{n-p^-} \right\}.$$

Using the Mountain Pass Theorem and Ekeland variational principle, he has proved that the problem (1.3) has at least two distinct, nontrivial weak solution.

The study of problems involving $p(x)$-biharmonic operators has been widely approached. For background and recent results, we refer the reader to [2,3,7,8,36] and the references therein for details. For example, in [2] by using critical point theory, the existence of infinitely many weak solutions for a class of Navier boundary-value problem depending on two parameters and involving the $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\Omega \times \mathbb{R})$ was studied. Kong [26] using variational arguments based on Ekeland's variational principle and some recent theory on the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ studied $p(x)$-biharmonic nonlinear eigenvalue problem, while in [26] using variational arguments based on the Mountain Pass lemma and some recent theory on the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, he studied the multiplicity of weak solutions to a fourth-order nonlinear elliptic problem with a $p(x)$-biharmonic operator. In [21] considering different situations concerning the growth rates involved in a $p(x)$-biharmonic nonlinear eigenvalue problem, employing the Mountain Pass lemma and Ekeland's variational principle the existence of a continuous family of eigenvalues was proved. In [10], Cammaroto and Vilasi derived the existence of infinitely many solutions for an elliptic problem involving the $p(x)$-biharmonic under Navier boundary conditions. Their approach is of variational nature and does not require any symmetry of the nonlinearities. Instead, a crucial role is played by suitable test functions in some variable exponent Sobolev space, of which they provided the abstract structure better suited to the framework. Also, many authors have looked for multiple solutions of elliptic equations involving $p(x)$-biharmonic type operators (see, for instance, [17,24,23, 26,25,29,28]). The generalization of Kirchhoff equations to the case involving the $p(x)$-biharmonic operator

$$M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx \right) \Delta^2_{p(x)} u = f(x,u) \tag{1.5}$$

is a quite new topic, so there exists only a few papers (see [4,14]). Motivated by the above references and some ideas in [12], the authors [14] established the existence and multiplicity of solutions for problem (1.5) using variational method and the theory of the variable exponent Sobolev spaces.

In the present paper, considering different ordering cases of the functions α, β and p, which makes problem (1.1) involving a concave-convex nonlinearity, we obtain few results for problem (1.1). Since each case has specific challenges, we do not use a unique straightforward technique. In this context, the presentation of the current paper is unique. We believe that the present paper will make a contribution to the related literature because considering a number of different cases for the functions α, β and p is very important for the representation of the various physical situations described by the model equation (1.1). Motivated by the ideas introduced in [3,11,32], the goal of this article is to study the existence of weak solutions of the problem (1.1) involving concave-convex nonlinearities.

This article is organized as follows. In section 2, we recall some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. In section 3 and 4, we state and prove our main results respectively.

2. Preliminary results

In this section we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

Let $C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \text{ such that } \inf_{x\in\overline{\Omega}} p(x) > 1 \right\}$, and define

$$p^- := \min_{x\in\overline{\Omega}} p(x) \text{ and } p^+ := \max_{x\in\overline{\Omega}} p(x), \ \forall p \in C_+(\overline{\Omega}).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \text{ is measurable, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

under the norm

$$|u|_{p(x)} = \inf \left\{ \eta > 0 : \int_\Omega \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \le 1 \right\},$$

which makes $\left(L^{p(x)}(\Omega), \ |.|_{p(x)}\right)$ a separable and reflexive Banach space.

The variable exponent Sobolev space $W^{m,p(x)}(\Omega)$ is defined by

$$W^{m,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \ D^\gamma u \in L^{p(x)}(\Omega), \ |\gamma| \le m \right\},$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ is a multi-index, $|\gamma| = \sum_{i=1}^n \gamma_i$ and $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial^{\gamma_1} x_1 ... \partial^{\gamma_n} x_n}$. Then, the space $\left(W^{m,p(x)}(\Omega), \ \|.\|_{m,p(x)}\right)$ equipped with the norm

$$\|u\|_{m,p(x)} = \sum_{|\gamma|\le m} |D^\gamma u|_{p(x)}$$

is a separable and reflexive Banach space, provided $1 < p^- \le p^+ < \infty$. We denote by $W_0^{m,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$.

Throughout this paper, we let $X = W_0^{1,p(x)}(\Omega) \cap W^{3,p(x)}(\Omega)$. Define a norm $\|.\|_X$ of X by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

Moreover, it is well known that if $1 < p^- \le p^+ < \infty$, the space $(X, \|.\|_X)$ is a separable and reflexive Banach space, $\|u\|_X$ and $|\nabla\Delta u|_{p(x)}$ are two equivalent norms on X (see [18,27]).

Let

$$\|u\| = \inf \left\{ \eta > 0 : \int_\Omega \left| \frac{\nabla\Delta u}{\eta} \right|^{p(x)} dx \le 1 \right\}$$

for all $u \in X$. It is easy to see that $\|u\|$ is equivalent to the norms $\|u\|_X$ and $|\nabla\Delta u|_{p(x)}$ in X. In this paper, for the convenience, we will use the norm $\|.\|$ on the space X.

For any $x \in \overline{\Omega}$, let

$$p^*(x) = \begin{cases} \frac{np(x)}{n-3p(x)} & \text{if } p(x) < \frac{n}{3}, \\ \infty & \text{if } p(x) \geq \frac{n}{3}. \end{cases}$$

Proposition 2.1. *[17,18,27] Set $\widetilde{\Phi}(u) = \int\limits_\Omega |u|^{p(x)} dx$. For $u \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} \leq 1 \Longrightarrow |u|_{p(x)}^{p^+} \leq \widetilde{\Phi}(u) \leq |u|_{p(x)}^{p^-}$.
2. $|u|_{p(x)} \geq 1 \Longrightarrow |u|_{p(x)}^{p^-} \leq \widetilde{\Phi}(u) \leq |u|_{p(x)}^{p^+}$.

Similar to Proposition 2.1, we have

Proposition 2.2. *Set $\Phi_{p(x)}(u) = \int\limits_\Omega |\nabla\Delta u|^{p(x)} dx$ for any $u \in X$. Then, we have*

1. $\|u\| \leq 1 \Longrightarrow \|u\|^{p^+} \leq \Phi_{p(x)}(u) \leq \|u\|^{p^-}$.
2. $\|u\| \geq 1 \Longrightarrow \|u\|^{p^-} \leq \Phi_{p(x)}(u) \leq \|u\|^{p^+}$.

Proposition 2.3. *[7,18,27] Assume that $q \in C_+(\overline{\Omega})$ satisfy $q(x) < p^*(x)$ on $\overline{\Omega}$. Then, there exists a continuous and compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$.*

3. Main results

We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int\limits_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)} dx\right) \int\limits_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u \cdot \nabla\Delta v dx$$
$$- \lambda \int\limits_\Omega \left(\zeta(x)|u|^{\alpha(x)-2}uv - \xi(x)|u|^{\beta(x)-2}uv\right) dx = 0,$$

for all $v \in X$.

The energy functional $I_\lambda : X \to \mathbb{R}$ corresponding to the problem (1.1) is defined as

$$I_\lambda(u) = \widehat{M}\left(\int\limits_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)} dx\right) - \lambda \int\limits_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\right) dx,$$

where $\widehat{M}(t) := \int_0^t M(z)dz$. At this point, let us define the functionals I_λ, $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \widehat{M}\left(\int\limits_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)} dx\right),$$

$$I_\lambda(u) = \Psi(u) - \lambda \int\limits_\Omega \left(\frac{\zeta(x)}{\alpha(x)} |u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)} |u|^{\beta(x)} \right) dx.$$

In a standard way, it can be shown that Ψ is sequentially weakly lower semicontinuous, $\Psi \in C^1(X, R)$, and its Gâteaux derivative $\Psi'(u)$ at $u \in X$ is given by

$$< \Psi'(u), v >= M \left(\int\limits_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)} dx \right) \int\limits_\Omega |\nabla \Delta u|^{p(x)-2} \nabla \Delta u \cdot \nabla \Delta v dx, \quad \text{for all } v \in X.$$

Then, the functional I_λ is well-defined, $I_\lambda \in C^1(X, R)$, and its Gâteaux derivative $I'_\lambda(u)$ at $u \in X$ is given by

$$\begin{aligned} < I'_\lambda(u), v > = {} & M \left(\int\limits_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)} dx \right) \int\limits_\Omega |\nabla \Delta u|^{p(x)-2} \nabla \Delta u \cdot \nabla \Delta v dx \\ & - \lambda \int\limits_\Omega \left(\zeta(x) |u|^{\alpha(x)-2} uv - \xi(x) |u|^{\beta(x)-2} uv \right) dx, \end{aligned}$$

for all $v \in X$. Thus, we can infer that critical points of functional I_λ are exactly the weak solutions of problem (1.1). Hereafter, we introduce the following assumptions on the function $M(t)$:

($\mathbf{B}_1$) There exist $m_2 \geq m_1 > 0$ and $\tau \geq \mu > 1$ such that for all $t \in \mathbb{R}_+$,

$$m_1 t^{\mu-1} \leq M(t) \leq m_2 t^{\tau-1}.$$

($\mathbf{B}_2$) For all $t \in \mathbb{R}_+$,

$$\widehat{M}(t) \geq M(t)t,$$

where $\widehat{M}(t) = \int\limits_0^t M(z)dz$.

In this paper, we obtain different results for the problem (1.1). For each result, the functions α, β, $p \in C_+(\overline{\Omega})$ have different ordering cases. Therefore, we split up the results of the present paper into the different natural parts. Moreover, in the rest of the paper, we always assume that ζ^-, $\xi^- > 0$. Now, we state our main result as follows.

Theorem 3.1. *Assume that the conditions* ($\boldsymbol{B}_1$), $\tau p(x) < \min\left\{\frac{n}{3}, \frac{np(x)}{n-3p(x)}\right\}$ *and*

$$1 < \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < \tau p^- \quad \text{on } \overline{\Omega} \tag{3.1}$$

are satisfied. Then for all $\lambda > 0$, *problem* (1.1) *has at least one nontrivial weak solution.*

Theorem 3.2. *Assume that the conditions* ($\boldsymbol{B}_1$), $\beta(x) < \min\left\{\frac{n}{3}, \frac{np(x)}{n-3p(x)}\right\}$ *and*

$$1 < \alpha^- \leq \alpha^+ < \mu p^- \leq \tau p^+ < \beta^- \quad \text{on } \overline{\Omega} \tag{3.2}$$

are satisfied. Then there exists $\lambda_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$, *the problem* (1.1) *has at least one nontrivial weak solution.*

Theorem 3.3. *Assume that the conditions* $(\boldsymbol{B}_1)$, $(\boldsymbol{B}_2)$, $\alpha(x) < \min\left\{\frac{n}{3}, \frac{np(x)}{n-3p(x)}\right\}$ *and*

$$1 < \beta^- \leq \beta^+ < \mu p^- \leq \tau p^+ < \alpha^- \quad \text{on } \overline{\Omega} \tag{3.3}$$

are satisfied. Then for any $\lambda > 0$ *the problem* (1.1) *has at least one nontrivial weak solution.*

4. Proof of the main results

4.1. Proof of Theorem 3.1

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 4.1. *For any* a_1, $a_2 > 0$ *and* $0 < k < m$, *we have*

$$a_1 s^k - a_2 s^m \leq a_1 \left(\frac{a_1}{a_2}\right)^{\frac{k}{m-k}}, \quad \forall s \geq 0. \tag{4.1}$$

Proof. Since the function $[0, +\infty) \ni s \mapsto s^\theta$ is increasing for any $\theta > 0$ it follows that

$$a_1 - a_2 s^{m-k} < 0,\ \forall s > \left(\frac{a_1}{a_2}\right)^{\frac{1}{m-k}},$$

and

$$s^k(a_1 - a_2 s^{m-k}) \leq a_1 s^k < a_1 \left(\frac{a_1}{a_2}\right)^{\frac{k}{m-k}}, \quad \forall s \in \left[0, \left(\frac{a_1}{a_2}\right)^{\frac{1}{m-k}}\right].$$

The above inequalities show that (4.1) holds true. □

Lemma 4.2. *For any* $\lambda > 0$, *we have*

1. I_λ *is bounded from below and coercive on* X.
2. I_λ *is sequentially weakly lower semicontinuous on* X.

Proof.
Proof of 1.) Using the hypotheses $(\mathbf{B}_1)$ and (4.1), we deduce that for any $u \in X$ with $\|u\| > 1$, the following hold

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right) - \lambda \int_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\right) dx \\
&\geq \frac{m_1}{\mu}\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)^\mu - \lambda \int_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\right) dx \\
&\geq \frac{m_1}{\mu(p^+)^\mu}\left[\Phi_{p(x)}(u)\right]^\mu - \lambda \int_\Omega \left(\frac{\zeta^+}{\alpha^-}|u|^{\alpha(x)} - \frac{\xi^-}{\beta^+}|u|^{\beta(x)}\right) dx \\
&\geq \frac{m_1}{\mu(p^+)^\mu}\left[\Phi_{p(x)}(u)\right]^\mu - \frac{\lambda\zeta^+}{\alpha^-}\int_\Omega \left(\frac{\zeta^+\beta^+}{\alpha^-\xi^-}\right)^{\frac{\alpha(x)}{\beta(x)-\alpha(x)}} dx
\end{aligned}$$

$$\geq \frac{m_1}{\mu(p^+)^\mu}\left[\Phi_{p(x)}(u)\right]^\mu - |\Omega|K,$$

where $K := \frac{\lambda\zeta^+}{\alpha^-}\max\left\{\left(\frac{\zeta^+\beta^+}{\alpha^-\xi^-}\right)^{\frac{\alpha^-}{\beta^+-\alpha^-}}, \left(\frac{\zeta^+\beta^+}{\alpha^-\xi^-}\right)^{\frac{\alpha^+}{\beta^--\alpha^+}}\right\}$.

Then

$$I_\lambda(u) \geq \frac{m_1}{\mu(p^+)^\mu}\|u\|^{\mu p^-} - |\Omega|K.$$

Hence, I_λ is bounded from below and coercive, that is, **1.)** is proved.

Proof of 2.) Let $\{u_j\} \subset X$ be a sequence such that $u_j \rightharpoonup u \in X$. Since Ψ is sequentially weakly lower semicontinuous. Then,

$$\Psi(u) \leq \liminf_{j\to+\infty}\Psi(u_j). \tag{4.2}$$

Moreover, by Proposition 2.3, X is compactly embedded to $L^{\alpha(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$:

$$u_j \to u \text{ in } L^{\alpha(x)}(\Omega) \quad \text{and} \quad u_j \to u \text{ in } L^{\beta(x)}(\Omega). \tag{4.3}$$

Then, from (4.2) and (4.3) it reads

$$\begin{aligned} I_\lambda(u) &\leq \liminf_{j\to+\infty}\Psi(u_j) - \lambda\lim_{j\to+\infty}\int_\Omega\left(\frac{\zeta(x)}{\alpha(x)}|u_j|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u_j|^{\beta(x)}\right)dx \\ &\leq \liminf_{j\to+\infty}\left(\Psi(u_j) - \lambda\int_\Omega\left(\frac{\zeta(x)}{\alpha(x)}|u_j|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u_j|^{\beta(x)}\right)dx\right), \end{aligned}$$

that is $I_\lambda(u) \leq \liminf_{j\to+\infty} I_\lambda(u_j)$. Thus, I_λ is sequentially weakly lower semicontinuous. □

Lemma 4.3. *For any $\lambda > 0$ it holds*

$$\inf_{u\in X} I_\lambda(u) < 0.$$

Proof. If we consider the condition (3.1), it reads

$$\liminf_{t\to 0}\frac{\frac{\zeta^-}{\alpha^+}|t|^{\alpha(x)} - \frac{\xi^+}{\beta^-}|t|^{\beta(x)}}{|t|^{\tau p^-}} = +\infty$$

uniformly in $x \in \Omega$. Then, for any $H > 0$ there exists $\delta > 0$ such that

$$\left|\inf_{x\in\Omega}\left(\frac{\zeta^-}{\alpha^+}|t|^{\alpha(x)} - \frac{\xi^+}{\beta^-}|t|^{\beta(x)}\right)\right| > H|t|^{\tau p^-} \quad \text{for every } 0 < |t| \leq \delta.$$

Take a nonzero nonnegative function $\vartheta \in C_0^\infty(\Omega)$ with $\inf_{x\in\Omega}\vartheta(x) > 0$, $\lambda > 0$ and put

$$H > \frac{m_2\|\vartheta\|^{\tau p^-}}{\tau(p^-)^{\tau-1}\lambda\int_\Omega|\vartheta|^{\tau p^-}dx}.$$

Moreover, choose $\varepsilon > 0$ such that $\varepsilon \sup_{x\in\Omega} \vartheta(x) < \delta$, and let $u_0 = \varepsilon\vartheta$. Then, for any $\lambda > 0$ we have

$$\begin{aligned}
I_\lambda(\varepsilon\vartheta) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta\varepsilon\vartheta|^{p(x)}dx\right) - \lambda\int_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|\varepsilon\vartheta|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|\varepsilon\vartheta|^{\beta(x)}\right)dx \\
&\le \frac{m_2}{\tau(p^-)^\tau}\left[\Phi_{p(x)}(\varepsilon\vartheta)\right]^\tau - \lambda\int_\Omega \left(\frac{\zeta^-}{\alpha^+}|\varepsilon\vartheta|^{\alpha(x)} - \frac{\xi^+}{\beta^-}|\varepsilon\vartheta|^{\beta(x)}\right)dx \\
&\le \frac{m_2}{\tau(p^-)^\tau}\varepsilon^{\tau p^-}\|\vartheta\|^{\tau p^-} - \lambda H\varepsilon^{\tau p^-}\int_\Omega |\vartheta|^{\tau p^-}dx \\
&< \frac{m_2}{\tau(p^-)^{\tau-1}}\varepsilon^{\tau p^-}\left(\frac{1}{p^-} - 1\right)\|\vartheta\|^{\tau p^-}.
\end{aligned}$$

So, we get $\inf_{u\in X} I_\lambda(u) < 0$, which completes the proof. $\square$

Proof of Theorem 3.1. From Lemma 4.2, it follows that for any $\lambda > 0$, I_λ has a global minimizer $u \in X$ such that $I'_\lambda(u) = 0$ (see [31]). Then, u is a weak solution of the problem (1.1). Moreover, since $I_\lambda(0) = 0$ and $I_\lambda(u) < 0$ (Lemma 4.3), $u \neq 0$, i.e. u is a nontrivial solution. $\square$

4.2. Proof of Theorem 3.2

Under the condition (3.2), we cannot show (in a straightforward fashion) that any Palais-Smale (PS) sequence is bounded in X. Thus, we will look for a weak solution of (1.1) as a local minimizer of the functional I_λ using Ekeland's variational principle (see [16]). We need the following auxiliary results.

Lemma 4.4. *Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exist ρ, $\delta > 0$ such that $I_\lambda(u) \ge \delta$ for any $u \in X$ with $\|u\| = \rho$.*

Proof. By using the condition (3.2) and the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we have

$$|u|_{\alpha(x)} \le C\|u\|,\ C > 0. \tag{4.4}$$

Let $\|u\| = \rho < 1$. Then by (4.4) and ($\mathbf{B}_1$), we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right) - \lambda\int_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\right)dx \\
&\ge \frac{m_1}{\mu(p^+)^\mu}\left[\Phi_{p(x)}(u)\right]^\mu - \lambda\int_\Omega \left(\frac{\zeta^+}{\alpha^-}|u|^{\alpha(x)} - \frac{\xi^-}{\beta^+}|u|^{\beta(x)}\right)dx \\
&\ge \frac{m_1}{\mu(p^+)^\mu}\|u\|^{\mu p^+} - \frac{\lambda\zeta^+C^{\alpha^-}}{\alpha^-}\|u\|^{\alpha^-} \\
&\ge \left(\frac{m_1}{\mu(p^+)^\mu}\|u\|^{\mu p^+-\alpha^-} - \frac{\lambda\zeta^+C^{\alpha^-}}{\alpha^-}\right)\|u\|^{\alpha^-} \\
&= \left(\frac{m_1}{\mu(p^+)^\mu}\rho^{\mu p^+-\alpha^-} - \frac{\lambda\zeta^+C^{\alpha^-}}{\alpha^-}\right)\rho^{\alpha^-}.
\end{aligned} \tag{4.5}$$

Let $\lambda_0 = \frac{m_1\alpha^-}{2\mu(p^+)^\mu\zeta^+C^{\alpha^-p^+}}\rho^{\mu p^+-\alpha^-}$. Then for any $u \in X$ with $\|u\| = \rho$, there exists $\delta = \frac{m_1\rho^{\mu p^+}}{2\mu(p^+)^\mu}$ such that $I_\lambda(u) \geq \delta > 0$. □

Lemma 4.5. *There exists $\varphi \in X$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $I_\lambda(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, $\varphi \neq 0$ and $t \in (0,1)$. Since $\alpha^+ < \mu p^- < \beta^-$, it reads

$$\begin{aligned}
I_\lambda(t\varphi) &\leq \frac{m_2 t^{\tau p^-}}{\tau(p^-)^\tau}\left[\Phi_{p(x)}(\varphi)\right]^\tau - \frac{\lambda\zeta^- t^{\alpha^+}}{\alpha^+}\int_\Omega |\varphi|^{\alpha(x)}dx + \frac{\lambda\xi^+ t^{\beta^-}}{\beta^-}\int_\Omega |\varphi|^{\beta(x)}dx \\
&\leq t^{\tau p^-}\left[\frac{m_2}{\tau(p^-)^\tau}\left[\Phi_{p(x)}(\varphi)\right]^\tau + \frac{\lambda\xi^+}{\beta^-}\int_\Omega |\varphi|^{\beta(x)}dx\right] - \frac{\lambda\zeta^- t^{\alpha^+}}{\alpha^+}\int_\Omega |\varphi|^{\alpha(x)}dx \\
&\leq t^{\mu p^-}\left[\frac{m_2}{\tau(p^-)^\tau}\left[\Phi_{p(x)}(\varphi)\right]^\tau + \frac{\lambda\xi^+}{\beta^-}\int_\Omega |\varphi|^{\beta(x)}dx\right] - \frac{\lambda\zeta^- t^{\alpha^+}}{\alpha^+}\int_\Omega |\varphi|^{\alpha(x)}dx < 0,
\end{aligned}$$

for $t < \varepsilon^{\frac{1}{\mu p^- - \alpha^+}}$ with

$$0 < \varepsilon < \min\left\{1, \frac{\frac{\lambda\zeta^-}{\alpha^+}\int_\Omega |\varphi|^{\alpha(x)}dx}{\frac{m_2}{\tau(p^-)^\tau}\left[\Phi_{p(x)}(\varphi)\right]^\tau + \frac{\lambda\xi^+}{\beta^-}\int_\Omega |\varphi|^{\beta(x)}dx}\right\},$$

from which we conclude that $I_\lambda(t\varphi) < 0$. □

Lemma 4.6. *Let $(u_j) \subset X$ be a bounded sequence such that $I_\lambda(u_j)$ is bounded and $I'_\lambda(u_j) \to 0$ in X^{-1}. Then, (u_j) is relatively compact.*

Thus, we will look for a weak solution of (1.1) as a local minimizer of the functional I_λ using Ekeland's variational principle. We begin by proving the following auxiliary results.

Proof. By Lemma 4.4 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have $\inf\limits_{\partial B_\rho(0)} I_\lambda > 0$.

On the other hand, by Lemma 4.5 there exist $\varphi \in X$ such that $I_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, since relation (4.5) holds for all $u \in X$ with $\|u\| < 1$ small enough, i.e.

$$I_\lambda(u) \geq \frac{m_1}{\mu(p^+)^\mu}\|u\|^{\mu p^+} - \frac{\lambda\zeta^+C^{\alpha^-}}{\alpha^-}\|u\|^{\alpha^-},$$

it follows that $-\infty < \overline{c} := \inf\limits_{B_\rho(0)} I_\lambda < 0$. So, we have $0 < \varepsilon < \inf\limits_{\partial B_\rho(0)} I_\lambda - \inf\limits_{B_\rho(0)} I_\lambda$.

Applying Ekeland's variational principle to the functional $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that $u_\varepsilon \in B_\rho(0)$.

Now, let us define $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $J_\lambda(u) := I_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of J_λ and this implies that $\|I'_\lambda(u_\varepsilon)\|_{X^{-1}} \leq \varepsilon$. So, we deduce that there exists a (PS)-sequence $(u_j) \subset B_\rho(0)$ such that

$$I_\lambda(u_j) \to \overline{c} \quad \text{and} \quad I'_\lambda(u_j) \to 0 \text{ in } X^{-1}. \tag{4.6}$$

Since the sequence $(u_j) \in X$ is bounded and X is reflexive, up to a subsequence, we get $u_j \rightharpoonup \overline{u}$ in X. So, by (4.6) we have $< I'_\lambda(u_j), u_j - \overline{u} > \to 0$. Therefore, we have

$$< I'_\lambda(u_j), u_j - \overline{u} > = M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\int_\Omega |\nabla\Delta u_j|^{p(x)-2}\nabla\Delta u_j\cdot\nabla\Delta(u_j-\overline{u})dx$$
$$-\lambda\int_\Omega\left(\zeta(x)|u_j|^{\alpha(x)-2}u_j(u_j-\overline{u}) - \xi(x)|u_j|^{\beta(x)-2}u_j(u_j-\overline{u})\right)dx \to 0.$$

Since $u_j \rightharpoonup \overline{u}$ in X, by compact embedding, we have $u_j \to \overline{u}$ in $L^{\alpha(x)}(\Omega)$ and $u_j \to \overline{u}$ in $L^{\beta(x)}(\Omega)$. Therefore,

$$\int_\Omega\left(\zeta(x)|u_j|^{\alpha(x)-2}u_j(u_j-\overline{u}) - \xi(x)|u_j|^{\beta(x)-2}u_j(u_j-\overline{u})\right)dx \to 0.$$

So, we conclude that

$$< \Psi'(u_j), u_j - \overline{u} >= M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\int_\Omega |\nabla\Delta u_j|^{p(x)-2}\nabla\Delta u_j\cdot\nabla\Delta(u_j-\overline{u})dx \to 0.$$

Since the functional Ψ is of $(S+)$ type (see [[17], Proposition 2.5]), we obtain that $u_j \to \overline{u}$ in X, which completes the proof. □

Proof of Theorem 3.2. Since $I_\lambda \in C^1(X, \mathbb{R})$, by the relation (4.6) it follows that $I_\lambda(\overline{u}) = \overline{c}$ and $I'_\lambda(\overline{u}) = 0$. Thus, $\overline{u} \in X$ is a nontrivial weak solution for (1.1). □

4.3. Proof of Theorem 3.3

To prove Theorem 3.3, we will apply Mountain Pass Theorem (see, e.g. [31,35]). To this end, we need the following lemma.

Lemma 4.7.

1. *There exist $\gamma > 0$, $\delta > 0$ such that $I_\lambda(u) \geq \delta$ for any $u \in X$ with $\|u\| = \gamma$.*
2. *There exists $u \in X$ such that $\|u\| > \gamma$, $I_\lambda(u) < 0$.*

Proof.
Proof of 1.) By using the condition (3.3) and the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we have $|u|_{\alpha(x)} \leq C'\|u\|$, $C' > 0$.

Let $\|u\| = \gamma < 1$. Then we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right) - \lambda\int_\Omega\left(\frac{\zeta(x)}{\alpha(x)}|u|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u|^{\beta(x)}\right)dx \\
&\geq \frac{m_1}{\mu(p^+)^\mu}\left[\Phi_{p(x)}(u)\right]^\mu - \lambda\int_\Omega\left(\frac{\zeta^+}{\alpha^-}|u|^{\alpha(x)} - \frac{\xi^-}{\beta^+}|u|^{\beta(x)}\right)dx \\
&\geq \frac{m_1}{\mu(p^+)^\mu}\|u\|^{\mu p^+} - \frac{\lambda\zeta^+ {C'}^{\alpha^-}}{\alpha^-}\|u\|^{\alpha^-} \\
&\geq \frac{m_1}{\mu(p^+)^\mu}\|u\|^{\tau p^+} - \frac{\lambda\zeta^+ {C'}^{\alpha^-}}{\alpha^-}\|u\|^{\alpha^-}.
\end{aligned}$$

Then for any $u \in X$ with $\|u\| = \gamma < 1$ small enough, there exists $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$, for every $\lambda > 0$.

Proof of 2.) Let $u \in X$ with $\|u\| = \gamma > 1$, and $t > 1$. Then

$$I_\lambda(tu) \leq \frac{m_2 t^{\tau p^+}}{\tau (p^-)^\tau} \left[\Phi_{p(x)}(u)\right]^\tau - \frac{\lambda \zeta^+ t^{\alpha^+}}{\alpha^-} \int\limits_\Omega |u|^{\alpha(x)} dx + \frac{\lambda \xi^- t^{\beta^-}}{\beta^+} \int\limits_\Omega |u|^{\beta(x)} dx.$$

So, we conclude that $I_\lambda(tu) \to -\infty$ as $t \to +\infty$. $\square$

Finally, we will show that under the condition (3.3), Lemma 4.6 holds for functional I_λ as well for all $\lambda > 0$. To this end, using Lemma 4.7 and the Mountain Pass Theorem, we deduce that there exists a (PS)-sequence, defined as in (4.6), $(u_j) \subset X$ for I_λ. We prove that (u_j) is bounded in X. Arguing by contradiction. We assume that, passing eventually to a subsequence, still denoted by (u_j), $\|u_j\| \to +\infty$ as $j \to +\infty$. Moreover, by condition (3.3), for any real number t we have

$$\begin{aligned} \Lambda(x,t) &\geq \zeta(x) \left(\frac{1}{\alpha^-} - \frac{1}{\alpha(x)}\right) |t|^{\alpha(x)} + \xi(x) \left(\frac{1}{\beta(x)} - \frac{1}{\alpha^-}\right) |t|^{\beta(x)} \\ &\geq \zeta^- \left(\frac{1}{\alpha^-} - \frac{1}{\alpha(x)}\right) |t|^{\alpha(x)} + \xi^- \left(\frac{1}{\beta^+} - \frac{1}{\alpha^-}\right) |t|^{\beta(x)} \geq K_0 > 0, \end{aligned} \tag{4.7}$$

where $\Lambda(x,t) := \frac{1}{\alpha^-}\left(\zeta(x)|t|^{\alpha(x)} - \xi(x)|t|^{\beta(x)}\right) - \left(\frac{\zeta(x)}{\alpha(x)}|t|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|t|^{\beta(x)}\right)$.

By (4.6), $(\mathbf{B}_1)$, $(\mathbf{B}_2)$ and (4.7) for j large enough, we have

$$\begin{aligned} C(1+\|u_j\|) &\geq I_\lambda(u_j) - \frac{1}{\alpha^-} < I'_\lambda(u_j), u_j > \\ &= \widehat{M}\left(\int\limits_\Omega \frac{1}{p(x)} |\nabla \Delta u_j|^{p(x)} dx\right) - \lambda \int\limits_\Omega \left(\frac{\zeta(x)}{\alpha(x)}|u_j|^{\alpha(x)} - \frac{\xi(x)}{\beta(x)}|u_j|^{\beta(x)}\right) dx \\ &\quad - \frac{1}{\alpha^-}\left(M\left(\int\limits_\Omega \frac{1}{p(x)}|\nabla\Delta u_j|^{p(x)}dx\right)\Phi_{p(x)}(u_j) - \lambda\int\limits_\Omega \left(\zeta(x)|u_j|^{\alpha(x)} - \xi(x)|u_j|^{\beta(x)}\right)dx\right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right) M\left(\int\limits_\Omega \frac{1}{p(x)}|\nabla\Delta u_j|^{p(x)}dx\right)\Phi_{p(x)}(u_j) + \lambda\int\limits_\Omega \Lambda(x,u_j)dx \\ &\geq \frac{m_1}{(p^+)^{\mu-1}}\left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right)\left[\Phi_{p(x)}(u_j)\right]^\mu + \lambda\int\limits_\Omega \Lambda(x,u_j)dx \\ &\geq \frac{m_1}{(p^+)^{\mu-1}}\left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right)\|u_j\|^{\mu p^-} + \lambda K_0|\Omega|. \end{aligned}$$

Using (3.3), we infer that $\frac{1}{p^+} - \frac{1}{\alpha^-} > 0$. Then

$$C(1+\|u_j\|) \geq C_0\|u_j\|^{\mu p^-} + \lambda K_0|\Omega|,$$

where $C_0 = \frac{m_1}{(p^+)^{\mu-1}}\left(\frac{1}{p^+} - \frac{1}{\alpha^-}\right) > 0$.

Since $\mu p^- > 1$, we get a contradiction. So, $\|u_j\|$ must be bounded. The rest of the proof is similar to the proof of Lemma 4.6, so we omit it. Therefore we obtain that $u_j \to u$ in X.

Proof of Theorem 3.3. From Lemmas 4.6 and 4.7, and the fact that $I_\lambda(0) = 0$, I_λ satisfies the Mountain Pass Theorem. So I_λ has a nontrivial critical point, i.e. (1.1) has at least one nontrivial weak solution. □

Declaration of Competing Interest

The author declares that he has no competing interests.

References

[1] G.A. Afrouzi, M. Mirzapour, Eigenvalue problems for $p(x)$-Kirchhoff type equations, Electron. J. Differential Equations 2013 (2013) 1–10.

[2] G.A. Afrouzi, S. Shokooh, Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator, Electron. J. Differential Equations 317 (2015) 1–14.

[3] R. Ayazoglu, G. Alisoy, I. Ekincioglu, Existence of one weak solution for $p(x)$-Biharmonic equations involving a concave-covex nonlinearity, Mat. Vesnik 69 (4) (2017) 296–307.

[4] G.A. Afrouzi, M. Mirzapour, Vicenţiu D. Rădulescu, Nonlocal fourth-order Kirchhoff systems with variable growth: low and high energy solutions, Collect. Math. 67 (2016) 207–223.

[5] S.N. Antontsev, J.F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006) 19–36.

[6] S.N. Antontsev, S.I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, Nonlinear Anal. 60 (2005) 515–545.

[7] A. Ayoujil, A.R. El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent, Nonlinear Anal. 71 (2009) 4916–4926.

[8] G.M. Bisci, D. Repovš, Multiple solutions of p-biharmonic equations with Navier boundary conditions, Complex Var. Elliptic Equ. 59 (2014) 271–284.

[9] F. Cammaroto, L. Vilasi, Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator, Nonlinear Anal. 74 (2011) 1841–1852.

[10] F. Cammaroto, L. Vilasi, Sequences of weak solutions for a Navier problem driven by the $p(x)$-biharmonic operator, Minimax Theory Appl. 4 (1) (2019) 71–85.

[11] N.T. Chung, Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities, Electron. J. Qual. Theory Differ. Equ. 42 (2012) 1–13.

[12] G. Dai, R. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation, J. Math. Anal. Appl. 359 (2009) 275–284.

[13] G. Dai, D. Liu, Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation, J. Math. Anal. Appl. 359 (2009) 704–710.

[14] O. Darhouche, Existence and multiplicity results for a class of Kirchhoff type problems involving the $p(x)$-biharmonic operator, J. Soc. Parana. Mat. 372 (2019) 23–33.

[15] L. Diening, P. Hästö, A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, in: P. Drábek, J. Rákosnik (Eds.), FSDONA04, Proceedings, Milovy, Czech Republic, 2004, pp. 38–58.

[16] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974) 324–353.

[17] A.R. El Amrouss, A. Ourraoui, Existence of solutions for a boundary value problem involving $p(x)$-biharmonic operator, Bol. Soc. Parana. Mat. 31 (2013) 179–192.

[18] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001) 424–446.

[19] X.L. Fan, Q.H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal. 52 (2003) 1843–1852.

[20] X.L. Fan, Q.H. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl. 302 (2005) 306–317.

[21] B. Ge, Q.M. Zhou, Y.H. Wu, Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight, Z. Angew. Math. Phys. 66 (2015) 1007–1021.

[22] P. Harjulehto, P. Hästö, An overview of variable exponent Lebesgue and Sobolev spaces, in: D. Herron (Ed.), Future Trends in Geometric Function Theory, RNC Workshop, Jyväskylä, 2003, pp. 85–93.

[23] S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, B. Ge, Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions, Z. Angew. Math. Phys. 67 (3) (2016), 13 pp.

[24] S. Heidarkhani, M. Ferrara, A. Salari, G. Caristi, Multiplicity results for $p(x)$-biharmonic equations with Navier boundary conditions, Complex Var. Elliptic Equ. 61 (11) (2016) 1494–1516.

[25] L. Kong, On a fourth order elliptic problem with a $p(x)$-biharmonic operator, Appl. Math. Lett. 27 (2014) 21–25.

[26] L. Kong, Eigenvalues for a fourth order elliptic problem, Proc. Amer. Math. Soc. 143 (2015) 249–258.

[27] O. Kováčik, J. Răkosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (116) (1991) 592–618.

[28] L. Li, C. Tang, Existence and multiplicity of solutions for a class of $p(x)$-biharmonic equations, Acta Math. Sci. 33 (1) (2013) 155–170.

[29] L. Li, L. Ding, W. Pan, Existence of multiple solutions for a $p(x)$-biharmonic equation, Electron. J. Differential Equations 139 (2013) 1–10.

[30] R.A. Mashiyev, B. Cekic, O.M. Buhrii, Existence of solutions for $p(x)$-Laplacian equations, Electron. J. Qual. Theory Differ. Equ. 65 (2010) 1–13.

[31] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.

[32] M. Mihălescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. A 462 (2006) 2625–2641.

[33] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.
[34] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent maximal and singular operators, Integral Transforms Spec. Funct. 16 (2005) 461–482.
[35] M. Willem, Minimax Theorems, Birkhauser, Basel, 1996.
[36] H. Yin, Y. Liu, Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic, Bull. Korean Math. Soc. 6 (2013) 1817–1826.
[37] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR, Izv. 9 (1987) 33–66.
[38] V.V. Zhikov, On some variational problems, Russ. J. Math. Phys. 5 (1997) 105–116.