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Energy conservation for the weak solutions to the equations of compressible magnetohydrodynamic flows in three dimensions

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ABSTRACT

In this paper, we prove the energy conservation for the weak solutions to the three-dimensional equations of compressible magnetohydrodynamic flows (MHD) under certain conditions only about density and velocity. This work is inspired by the seminal work by Yu [24] on the energy conservation of compressible Navier-Stokes equations. Our result indicates that even the magnetic field is taken into account, we only need some regularity conditions of the density and velocity as in [24] to ensure the energy conservation.

1. Introduction and main results

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids in an electromagnetic field and has a very broad range of applications. The dynamic motion of the fluid and the magnetic field interact strongly on each other. In this paper, the fluid we consider is isentropic and compressible, namely, it is governed by the isentropic compressible Navier-Stokes equations. The equations of the magnetic field are called the induction equation. Hence the compressible MHD system for isentropic flows can be written as below [4,16,17].

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\operatorname{div} \mathbf{u}), \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}),\ \operatorname{div} \mathbf{H} = 0, \end{cases} \tag{1.1}$$

where $\rho = \rho(x,t), \mathbf{u}(x,t) = (u_1,u_2,u_3)(x,t), \mathbf{H} = (H_1,H_2,H_3)(x,t)$ denote the density of the fluid, the velocity field and the magnetic field, respectively; $P(\rho) = a\rho^\gamma$ is the pressure with constants $a > 0$, and

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$\gamma > 1$; the constants μ and λ are the shear and bulk viscosity coefficients satisfying the physical restriction $3\lambda + 2\mu \geq 0$ and $\mu > 0$; and the constant $\nu > 0$ is the magnetic diffusivity. The positive constant a does not play essential role in the following analysis. Thus for simplicity we take $a = 1$.

For the sake of simplicity we will consider the case of a bounded domain with periodic boundary conditions in $\mathbb{R}^3$, namely $\Omega = \mathbb{T}^3$, and the following initial conditions:

$$(\rho, \rho\mathbf{u}, \mathbf{H})(x, 0) = (\rho_0, \mathbf{m}_0, \mathbf{H}_0)(x), \quad x \in \Omega, \tag{1.2}$$

where we define $\mathbf{m}_0 = \mathbf{0}$, if $\rho_0 = 0$.

The global existence of weak solutions to (1.1) in a bounded domain of $\mathbb{R}^3$ was obtained by Hu and Wang [14] for $\gamma > \frac{3}{2}$. Moreover, the global weak solutions exist in the renormalized sense with arbitrarily large initial data as well, satisfying the energy inequality

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho\mathbf{u}^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}|^2\right) \mathrm{d}x + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla \times \mathbf{H}|^2\right) \mathrm{d}x\, \mathrm{d}t \\
&\leq \int_\Omega \left(\frac{1}{2}\rho_0\mathbf{u}_0^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}_0|^2\right) \mathrm{d}x
\end{aligned} \tag{1.3}$$

for $t \in (0, \infty)$. In fact, when the solutions are smooth enough such as strong solutions or classical solutions, the energy inequality (1.3) can be written as an equality, namely,

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho\mathbf{u}^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}|^2\right) \mathrm{d}x + \int_0^t \int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla \times \mathbf{H}|^2\right) \mathrm{d}x\, \mathrm{d}t \\
&= \int_\Omega \left(\frac{1}{2}\rho_0\mathbf{u}_0^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{1}{2}|\mathbf{H}_0|^2\right) \mathrm{d}x
\end{aligned} \tag{1.4}$$

for $t \in (0, \infty)$. For example, see [6,7,15,23,18] for global smooth solutions in one dimension with arbitrarily large initial data and in multi-dimensions with small perturbations of a given constant state, and [11,23] for local strong solutions with arbitrarily large initial data.

The question is how much regularity of the weak solutions is needed to ensure the energy equality (1.4)? In the context of incompressible Euler equations, this question is linked to a famous conjecture of Onsager [20]. It has been made great progress recently [1–3,8–10]. In the context of incompressible Navier-Stokes equations, Serrin [21] proved the energy conservation under the condition $\mathbf{u} \in L^p(0, T; L^q(\Omega))$, $\frac{2}{p} + \frac{N}{q} \leq 1$, where N is the dimension. Later, Shinbrot [22] removed the dimensional dependence, i.e., $\frac{2}{p} + \frac{2}{q} \leq 1$, where $q \geq 4$. When the magnetic field is ignored, i.e. $\mathbf{H} = 0$, system (1.1) becomes the compressible Navier-Stokes equations. Yu [24] proved the energy conservation (1.4) ($\mathbf{H} = 0$) of the Lions-Feireisl weak solutions (see [12,13,19]) for $\Omega = \mathbb{T}^3$ provided that

$$\begin{cases}
0 \leq \rho \leq \tilde{\rho} < \infty, \text{ and } \nabla\sqrt{\rho} \in L^\infty\big(0, T; L^2(\Omega)\big), \\
u \in L^p(0, T; L^q(\Omega)), \text{ for any } \frac{1}{p} + \frac{1}{q} \leq \frac{5}{12}, \text{ and } q \geq 6, \\
u_0 \in L^k(\Omega), \ \frac{1}{k} + \frac{1}{q} \leq \frac{1}{2},
\end{cases} \tag{1.5}$$

where $\tilde{\rho}$ is a positive constant. In [24], the case of density-dependence viscosity is also considered. Recently, Chen, Liang, Wang, Xu [5] nicely extended Yu's results to the Dirichlet problem.

The purpose of this paper is to provide a sufficient condition for the energy conservation of the weak solution of (1.1)-(1.2), which is motivated by Yu's work [24] (see also [5]).

Definition 1.1. (weak solution) $(\rho, \mathbf{u}, \mathbf{H})$ is called a weak solution to (1.1)-(1.2) over $\Omega \times (0,T)$, if $(\rho, \mathbf{u}, \mathbf{H})$ satisfies that

- (1.1) holds in $\mathcal{D}'\big(\Omega \times (0,T)\big)$ satisfying

$$\begin{cases} \rho \in L^\infty\big(0,T; L^\gamma(\Omega)\big), \quad \rho \geq 0, \\ \sqrt{\rho}\mathbf{u} \in L^\infty(0,T; L^2(\Omega)), \quad \nabla \mathbf{u} \in L^2(0,T; L^2(\Omega)), \\ \mathbf{H} \in L^\infty(0,T; L^2(\Omega)), \quad \nabla \mathbf{H} \in L^2(0,T; L^2(\Omega)), \end{cases} \tag{1.6}$$

  and

$$\rho \mathbf{u} \in C([0,T]; L^2_{weak}(\Omega)), \tag{1.7}$$

  and

$$\mathbf{H} \in C([0,T]; L^2_{weak}(\Omega)); \tag{1.8}$$

- the energy inequality (1.3) holds;
- (1.2) holds in $\mathcal{D}'(\Omega)$.

Our main result reads as follows.

Theorem 1.1. *Assume that*

$$\begin{cases} \rho(x,0) = \rho_0(x) \in L^1(\Omega) \cap L^\gamma(\Omega), \quad \rho_0(x) \geq 0 \ a.e. \ in \ \Omega, \\ \rho(x,0)\mathbf{u}(x,0) = \mathbf{m}_0(x) \in L^1(\Omega), \quad \mathbf{m}_0 = \mathbf{0} \ if \ \rho_0 = 0, \quad \frac{|\mathrm{m}_0|^2}{\rho_0} \in L^1(\Omega), \\ \mathbf{H}(x,0) = \mathbf{H}_0(x) \in L^2(\Omega), \quad \operatorname{div} \mathbf{H}_0 = 0 \ in \ \mathcal{D}'(\Omega). \end{cases} \tag{1.9}$$

In addition, we assume $\mathbf{u}_0 \in L^\kappa$, where $\kappa > 2$. Let $(\rho, \mathbf{u}, \mathbf{H})$ be a weak solution to (1.1)-(1.2) in the sense of Definition 1.1. Moreover, if

$$0 \leq \rho \leq \tilde{\rho} < \infty, \quad \nabla\sqrt{\rho} \in L^\infty(0,T; L^2(\Omega)), \tag{1.10}$$

and

$$\mathbf{u} \in L^p(0,T; L^q(\Omega)) \quad \text{for any } \frac{2}{p} + \frac{3}{q} \leq 1, \text{ with } q \geq 6, \tag{1.11}$$

then the weak solution $(\rho, \mathbf{u}, \mathbf{H})$ satisfies the energy equality (1.4) for $t \in [0,T]$.

2. Preliminaries

Define

$$\begin{aligned} \overline{f(x,t)} = \eta_\epsilon * f(x,t) &= \int_0^t \int_\Omega \eta_\epsilon(x-y, t-s) f(y,s) \,\mathrm{d}y \,\mathrm{d}s \\ &= \int_0^t \int_\Omega \frac{1}{\epsilon^4} \eta(x-y, t-s) f(y,s) \,\mathrm{d}y \,\mathrm{d}s \end{aligned}$$

where $\eta_\epsilon(x,t) = \frac{1}{\epsilon^4}\eta(\frac{x}{\epsilon}, \frac{t}{\epsilon})$, and $\eta(t,x) \geq 0$ is a smooth even function compactly supported in the space-time ball of radius 1, and with an integral equal to 1.

The following lemma will be useful in the proof of Theorem 1.1.

Lemma 2.1 *([19]). Let ∂ be a partial derivative in space or time. Let $f, \partial f \in L^p(\Omega \times \mathbb{R}^+), g \in L^q(\Omega \times \mathbb{R}^+)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have*

$$\|\overline{\partial(fg)} - \partial(f\overline{g})\|_{L^r(\Omega\times\mathbb{R}^+)} \leq C\|\partial f\|_{L^p(\Omega\times\mathbb{R}^+)}\|g\|_{L^q(\Omega\times\mathbb{R}^+)}$$

for some constant $C > 0$ independent of ϵ, f and g, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\overline{\partial(fg)} - \partial(f\overline{g}) \to 0 \ \textit{in}\ L^r(\Omega \times \mathbb{R}^+)$$

as $\epsilon \to 0$, if $r < \infty$.

3. Proof of main result

For a given test function $\psi(t) \in \mathcal{D}(0, +\infty)$, denote $\Phi = \overline{\psi(t)\overline{\mathbf{u}}}$. Since $\mathcal{D}(0, +\infty)$ is a class of all smooth compactly supported functions in $(0, +\infty)$, Φ is well defined on $(0, +\infty)$ for ϵ small enough. Finally, we will extend the result for $\psi(t) \in \mathcal{D}(-1, +\infty)$.

Step 1. Choosing Φ as the test function.

Using Φ as the test function of $(1.1)_2$, one obtains

$$\int_0^T\int_\Omega \Phi\big((\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P - (\nabla\times\mathbf{H})\times\mathbf{H} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u})\big)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.1}$$

which in turn yields

$$\int_0^T\int_\Omega \psi(t)\overline{\mathbf{u}}\cdot\left(\overline{(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P - (\nabla\times\mathbf{H})\times\mathbf{H} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u})}\right)\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.2}$$

where we used the fact $\eta(-t,-x) = \eta(t,x)$.

The first two terms in (3.2) yield that

$$\begin{aligned}
&\int_0^T\int_\Omega \psi(t)\overline{(\rho\mathbf{u})_t}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi(t)\overline{\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&\int_0^T\int_\Omega \psi(t)\Big(\overline{(\rho\mathbf{u})_t} - (\rho\overline{\mathbf{u}})_t\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})} - \mathrm{div}(\rho\mathbf{u}\otimes\overline{\mathbf{u}})\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega \psi(t)(\rho\overline{\mathbf{u}})_t\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi(t)\Big((\rho\mathbf{u}\cdot\nabla)\overline{\mathbf{u}} + \mathrm{div}(\rho\mathbf{u})\overline{\mathbf{u}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&A + B + \int_0^T\int_\Omega \psi(t)\left(\frac{1}{2}\rho\mid\overline{\mathbf{u}}\mid^2\right)_t\mathrm{d}x,
\end{aligned} \tag{3.3}$$

where

$$A = \int_0^T \int_\Omega \psi(t)\Big(\overline{(\rho\mathbf{u})_t} - (\rho\overline{\mathbf{u}})_t\Big) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t,$$
$$B = \int_0^T \int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})} - \mathrm{div}(\rho\mathbf{u}\otimes\overline{\mathbf{u}})\Big) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.4}$$

Next, we estimate the third term in (3.2) as follows

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(t)\overline{\nabla P} \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \\
=&\frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\overline{\rho\nabla(\rho^{\gamma-1})} \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \\
=&\frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\Big(\overline{\rho\nabla(\rho^{\gamma-1})} - \rho\nabla(\rho^{\gamma-1})\Big) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\rho\nabla(\rho^{\gamma-1}) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \\
=&D_1 + \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u})} - \mathrm{div}(\rho\overline{\mathbf{u}})\Big)\rho^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\overline{\rho_t}\rho^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t \\
=&D_1 + D_2 + \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)(\overline{\rho_t} - \rho_t)\rho^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\rho_t\rho^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t \\
=&D_1 + D_2 + D_3 + \frac{1}{\gamma-1}\int_0^T \int_\Omega \psi(t)(\rho^\gamma)_t\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.5}$$

where

$$D_1 = \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\Big(\overline{\rho\nabla(\rho^{\gamma-1})} - \rho\nabla(\rho^{\gamma-1})\Big) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t,$$
$$D_2 = \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u})} - \mathrm{div}(\rho\overline{\mathbf{u}})\Big)\overline{\rho^{\gamma-1}}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.6}$$
$$D_3 = \frac{\gamma}{\gamma-1}\int_0^T \int_\Omega \psi(t)(\overline{\rho_t} - \rho_t)\overline{\rho^{\gamma-1}}\,\mathrm{d}x\,\mathrm{d}t.$$

For the fifth item and the sixth item in (3.2), we have

$$\begin{aligned}
&-\int_0^T \int_\Omega \psi(t)\overline{\mu\Delta\mathbf{u}} \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \psi(t)\Big(\overline{(\lambda+\mu)\nabla(\mathrm{div}\mathbf{u})}\Big) \cdot \overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \\
=&\int_0^T \int_\Omega \mu\psi(t) \mid \overline{\nabla\mathbf{u}} \mid^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega (\lambda+\mu)\psi(t) \mid \overline{\mathrm{div}\mathbf{u}} \mid^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.7}$$

Finally, we handle the fourth item in (3.2).

$$
\begin{aligned}
&-\int_0^T\int_\Omega \psi(t)\Big(\overline{(\nabla\times\mathbf{H})\times\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&-\int_0^T\int_\Omega \psi(t)\overline{(\mathbf{H}\cdot\nabla)\mathbf{H}}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega \psi(t)\overline{\nabla\mid\mathbf{H}\mid^2}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&-\int_0^T\int_\Omega \psi(t)\overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{H})}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}\,\overline{\mathbf{H}\cdot\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.8}
$$

The first term in the last equality of (3.8) shows that

$$
\begin{aligned}
&-\int_0^T\int_\Omega \psi(t)\overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{H})}\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&-\int_0^T\int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{H})}-\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{H}})\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \psi(t)\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{H}})\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&I_1-\int_0^T\int_\Omega \psi(t)\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.9}
$$

And the second term in the last equality of (3.8) shows that

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}\,\overline{\mathbf{H}\cdot\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t\\
=&-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\overline{\mathbf{H}\cdot\mathbf{H}}-\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t\\
=&\frac{1}{2}\int_0^T\int_\Omega \psi(t)\overline{\mathbf{u}}\cdot(\overline{\nabla(\mathbf{H}\cdot\mathbf{H})}-\nabla(\mathbf{H}\cdot\overline{\mathbf{H}}))\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t\\
=&I_2-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.10}
$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$
\begin{aligned}
&-\int_0^T\int_\Omega \psi(t)\Big(\overline{(\nabla\times\mathbf{H})\times\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\
=&I_1+I_2-\int_0^T\int_\Omega \psi(t)\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{3.11}
$$

where

$$I_1 = \int_0^T \int_\Omega \psi(t)\Big(\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{H}}) - \overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{H})}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t,$$
$$I_2 = \frac{1}{2}\int_0^T \int_\Omega \psi(t)\overline{\mathbf{u}}\cdot\Big(\overline{\nabla(\mathbf{H}\cdot\mathbf{H})} - \nabla(\mathbf{H}\cdot\overline{\mathbf{H}})\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.12}$$

Combining (3.3), (3.5), (3.7) with (3.11), we can get the equality of (3.2) as follows

$$\begin{aligned}&\int_0^T \int_\Omega \psi(t)\Big(\frac{1}{2}\rho\mid\overline{\mathbf{u}}\mid^2 + \frac{\rho^\gamma}{\gamma-1}\Big)_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \psi(t)\Big(\mu\mid\overline{\nabla\mathbf{u}}\mid^2 + (\lambda+\mu)\mid\overline{\mathrm{div}\mathbf{u}}\mid^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega \psi(t)\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t + \mathbf{R}_\epsilon + I_1 + I_2 = 0,\end{aligned} \tag{3.13}$$

where

$$\mathbf{R}_\epsilon = A + B + D_1 + D_2 + D_3. \tag{3.14}$$

Now we are in a position to handle $(1.1)_3$. Here we introduce a new function $\Theta = \overline{\psi(t)\overline{\mathbf{H}}}$ as a test function of $(1.1)_3$. Then we get

$$\int_0^T\int_\Omega \psi(t)\Big(\overline{\mathbf{H}_t + \nabla\times(\nu\nabla\times\mathbf{H}) - \nabla\times(\mathbf{u}\times\mathbf{H})}\Big)\cdot\overline{\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.15}$$

The first term in (3.15) shows that

$$\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}_t}\cdot\overline{\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^T\int_\Omega \psi(t)(\mid\overline{\mathbf{H}}\mid^2)_t\,\mathrm{d}x\,\mathrm{d}t. \tag{3.16}$$

Similarly, the second term in (3.15) yields

$$\begin{aligned}\int_0^T\int_\Omega \psi(t)\overline{\nabla\times(\nu\nabla\times\mathbf{H})}\cdot\overline{\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t &= -\nu\int_0^T\int_\Omega \psi(t)\overline{\Delta\mathbf{H}}\cdot\overline{\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t\\ &= \nu\int_0^T\int_\Omega \psi(t)\mid\overline{\nabla\times\mathbf{H}}\mid^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.17}$$

Finally, the last term in (3.15) shows that

$$\begin{aligned}&-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\nabla\times(\mathbf{u}\times\mathbf{H})}\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\mathbf{u}(\mathrm{div}\mathbf{H}) - \mathbf{H}(\mathrm{div}\mathbf{u}) + (\mathbf{H}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{(\mathbf{H}\cdot\nabla)\mathbf{u}-\mathbf{H}(\operatorname{div}\mathbf{u})-(\mathbf{u}\cdot\nabla)\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{(\mathbf{H}\cdot\nabla)\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{(\mathbf{u}\cdot\nabla)\mathbf{H}+(\operatorname{div}\mathbf{u})\mathbf{H}}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\operatorname{div}(\mathbf{H}\otimes\mathbf{u})}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\operatorname{div}(\mathbf{u}\otimes\mathbf{H})}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.18}
$$

The first term in the last equality of (3.18) shows that

$$
\begin{aligned}
&-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\operatorname{div}(\mathbf{H}\otimes\mathbf{u})}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\overline{\operatorname{div}(\mathbf{H}\otimes\mathbf{u})}-\operatorname{div}(\mathbf{H}\otimes\overline{\mathbf{u}})\Big)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\operatorname{div}(\mathbf{H}\otimes\overline{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t \\
&= I_3-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.19}
$$

And the second term in the last equality of (3.18) shows that

$$
\begin{aligned}
&\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\operatorname{div}(\mathbf{u}\otimes\mathbf{H})}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\overline{\operatorname{div}(\mathbf{u}\otimes\mathbf{H})}-\operatorname{div}(\overline{\mathbf{u}}\otimes\mathbf{H})\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\operatorname{div}(\overline{\mathbf{u}}\otimes\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t \\
&= I_4+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\operatorname{div}(\overline{\mathbf{u}}\otimes\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t \\
&= I_4+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\operatorname{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.20}
$$

Substituting the above two equalities into (3.18), we have

$$
\begin{aligned}
&-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\overline{\nabla\times(\mathbf{u}\times\mathbf{H})}\,\mathrm{d}x\,\mathrm{d}t \\
&= I_3+I_4-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_0^T\int_\Omega \psi(t)\operatorname{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.21}
$$

Recalling (3.15), we have

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_\Omega \psi(t)(\mid \overline{\mathbf{H}}\mid^2)_t\,\mathrm{d}x\,\mathrm{d}t+\nu\int_0^T\int_\Omega \psi(t)\mid \overline{\nabla\times\mathbf{H}}\mid^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t+I_3+I_4=0,\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}I_3&=\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{u}})-\overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{u})}\Big)\,\mathrm{d}x\,\mathrm{d}t,\\ I_4&=\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\overline{\mathrm{div}(\mathbf{u}\otimes\mathbf{H})}-\mathrm{div}(\overline{\mathbf{u}}\otimes\mathbf{H})\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.23}$$

Combining (3.13) with (3.22), we have

$$\begin{aligned}&\int_0^T\int_\Omega \psi(t)\Big(\frac{1}{2}\rho\mid\overline{\mathbf{u}}\mid^2+\frac{\rho^\gamma}{\gamma-1}+\frac{1}{2}\mid\overline{\mathbf{H}}\mid^2\Big)_t\,\mathrm{d}x\,\mathrm{d}t+\nu\int_0^T\int_\Omega \psi(t)\mid\overline{\nabla\times\mathbf{H}}\mid^2\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^T\int_\Omega \mu\psi(t)\mid\overline{\nabla\mathbf{u}}\mid^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega(\lambda+\mu)\psi(t)\mid\overline{\mathrm{div}\mathbf{u}}\mid^2\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^T\int_\Omega \psi(t)\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t+R_\epsilon+I_\epsilon=0,\end{aligned}\tag{3.24}$$

where

$$I_\epsilon=I_1+I_2+I_3+I_4.\tag{3.25}$$

In equation (3.24), we continue to estimate the last four terms as follows.
On the one hand, we have

$$-\int_0^T\int_\Omega \psi(t)\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{H}}\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\mathbf{H}\cdot\nabla)\overline{\mathbf{u}}\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\tag{3.26}$$

On the other hand, we deduce

$$\int_0^T\int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega \psi(t)\mathrm{div}\overline{\mathbf{u}}(\mathbf{H}\cdot\overline{\mathbf{H}})\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \psi(t)\overline{\mathbf{H}} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_\Omega \psi(t)\overline{\mathbf{u}} \cdot \nabla(\mathbf{H} \cdot \overline{\mathbf{H}}) \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{2} \int_0^T \int_\Omega \psi(t)\overline{\mathbf{H}} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_\Omega \psi(t)\mathbf{H} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{H}} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{2} \int_0^T \int_\Omega \psi(t)\overline{\mathbf{H}} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_\Omega \psi(t)\mathbf{H} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^T \int_\Omega \psi(t)\mathbf{H} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_\Omega \psi(t)\mathbf{H} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{H}} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&= J_1 + J_2,
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
J_1 &= \frac{1}{2} \int_0^T \int_\Omega \psi(t)(\overline{\mathbf{H}} - \mathbf{H}) \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)\mathbf{H} \Big) \, \mathrm{d}x \, \mathrm{d}t, \\
J_2 &= \frac{1}{2} \int_0^T \int_\Omega \psi(t)\mathbf{H} \cdot \Big((\overline{\mathbf{u}} \cdot \nabla)(\mathbf{H} - \overline{\mathbf{H}}) \Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{3.28}
$$

By the above equalities and integration by parts, it yields

$$
\begin{aligned}
&- \int_0^T \int_\Omega \psi_t \Big(\frac{1}{2}\rho \mid \overline{\mathbf{u}} \mid^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2} \mid \overline{\mathbf{H}} \mid^2 \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi(t) \Big(\nu \mid \overline{\nabla \times \mathbf{H}} \mid^2 + \mu \mid \overline{\nabla \mathbf{u}} \mid^2 + (\lambda + \mu) \mid \overline{\mathrm{div} \mathbf{u}} \mid^2 \Big) \, \mathrm{d}x \, \mathrm{d}t + R_\epsilon + I_\epsilon + J_\epsilon = 0,
\end{aligned}
\tag{3.29}
$$

where $J_\epsilon = J_1 + J_2$.

Step 2. Passing to the limit in (3.29) as ϵ tends to zero.

Using Definition 1.1, (1.10) and (1.11), one obtains

$$
\begin{aligned}
&\int_0^T \int_\Omega \frac{1}{2}\rho \mid \overline{\mathbf{u}} \mid^2 \psi_t \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \frac{1}{2}\rho \mid \mathbf{u} \mid^2 \psi_t \, \mathrm{d}x \, \mathrm{d}t, \\
&\int_0^T \int_\Omega \frac{1}{2} \mid \overline{\mathbf{H}} \mid^2 \psi_t \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \frac{1}{2} \mid \mathbf{H} \mid^2 \psi_t \, \mathrm{d}x \, \mathrm{d}t, \\
&\int_0^T \int_\Omega \psi(t)\nu \mid \overline{\nabla \times \mathbf{H}} \mid^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \psi(t)\nu \mid \nabla \times \mathbf{H} \mid^2 \, \mathrm{d}x \, \mathrm{d}t, \\
&\int_0^T \int_\Omega \psi(t)\mu \mid \overline{\nabla \mathbf{u}} \mid^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \psi(t)\mu \mid \nabla \mathbf{u} \mid^2 \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
\tag{3.30}
$$

$$\int_0^T\int_\Omega \psi(t)(\lambda+\mu)\mid \overline{\mathrm{div}\mathbf{u}}\mid^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \psi(t)(\lambda+\mu)\mid \mathrm{div}\mathbf{u}\mid^2 \,\mathrm{d}x\,\mathrm{d}t,$$

as $\epsilon\to 0$.

The next goal is to make use of Lemma 2.1 to prove

$$R_\epsilon + I_\epsilon + J_\epsilon \to 0, \text{ as } \epsilon\to 0. \tag{3.31}$$

Firstly, we prove $R_\epsilon\to 0$, as $\epsilon\to 0$. We assume that $\mathbf{u}$ is bounded in $L^p(0,T;L^q(\Omega))$. On the one hand, due to (1.6), (1.10), we have

$$\rho_t = -(\rho\mathrm{div}\mathbf{u} + 2\sqrt{\rho}\mathbf{u}\cdot\nabla\sqrt{\rho}) \in L^2\left(0,T;L^2(\Omega)\right) + L^p\left(0,T;L^{\frac{2q}{q+2}}(\Omega)\right). \tag{3.32}$$

Thus, in view of Lemma 2.1, we have

$$\begin{aligned}
A+B \le \mid A\mid + \mid B\mid =& \Big|\int_0^T\int_\Omega \psi(t)\Big(\overline{(\rho\mathbf{u})_t} - (\rho\overline{\mathbf{u}})_t\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&+ \Big|\int_0^T\int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})} - \mathrm{div}(\rho\mathbf{u}\otimes\overline{\mathbf{u}})\Big)\cdot\overline{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \|\psi(t)\|_{L^\infty(0,T)}\int_0^T \|\mathbf{u}\|_{L^q(\Omega)}\bigg(\|\overline{(\rho\mathbf{u})_t} - (\rho\overline{\mathbf{u}})_t\|_{L^{\frac{q}{q-1}}(\Omega)} \\
&\quad + \|\overline{\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})} - \mathrm{div}(\rho\mathbf{u}\otimes\overline{\mathbf{u}})\|_{L^{\frac{q}{q-1}}(\Omega)}\bigg)\mathrm{d}t \\
&\le \|\psi(t)\|_{L^\infty(0,T)}\int_0^T \|\mathbf{u}\|_{L^q(\Omega)}\|\rho_t\|_{L^{\frac{2q}{q+2}}(\Omega)}\|\mathbf{u}\|_{L^{\frac{2q}{q-4}}(\Omega)}\,\mathrm{d}t \\
&\le C\int_0^T \|\mathbf{u}\|^4_{L^q(\Omega)} + \|\rho_t\|^2_{L^{\frac{2q}{q+2}}(\Omega)}\,\mathrm{d}t \\
&\le C\|\mathbf{u}\|^4_{L^p(0,T;L^q(\Omega))} + C\|\rho_t\|^2_{L^2(0,T;L^{\frac{2q}{q+2}}(\Omega))},
\end{aligned} \tag{3.33}$$

for any $p\ge 4$ and $q\ge 6$.

Thanks to Lemma 2.1, as $\epsilon\to 0$, we have

$$A+B\to 0. \tag{3.34}$$

For D_2, we get

$$\begin{aligned}
D_2 \le\mid D_2\mid =& \Big|\frac{\gamma}{\gamma-1}\int_0^T\int_\Omega \psi(t)\Big(\overline{\mathrm{div}(\rho\mathbf{u})} - \mathrm{div}(\rho\overline{\mathbf{u}})\Big)\overline{\rho^{\gamma-1}}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le C\|\psi(t)\|_{L^\infty(0,T)}\int_0^T \|\overline{\rho^{\gamma-1}}\|_{L^\infty(\Omega)}\|\overline{\mathrm{div}(\rho\mathbf{u})} - \mathrm{div}(\rho\overline{\mathbf{u}})\|_{L^1(\Omega)}\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq C \parallel \psi(t) \parallel_{L^\infty(0,T)} \int_0^T \parallel \sqrt{\rho}\nabla\sqrt{\rho} \parallel_{L^{\frac{q}{q-1}}(\Omega)} \parallel \mathbf{u} \parallel_{L^q(\Omega)} \mathrm{d}t \\
&\leq C \parallel \psi(t) \parallel_{L^\infty(0,T)} \int_0^T \parallel \nabla\sqrt{\rho} \parallel_{L^2(\Omega)} \parallel \mathbf{u} \parallel_{L^q(\Omega)} \mathrm{d}t \\
&\leq C \parallel \nabla\sqrt{\rho} \parallel_{L^\infty(0,T;L^2(\Omega))} \parallel \mathbf{u} \parallel_{L^p(0,T;L^q(\Omega))},
\end{aligned} \tag{3.35}$$

for any $q \geq 6$. Similarly, by Lemma 2.1, we have that D_2 converges to zero, as $\epsilon \to 0$.

For D_1, by (1.10), and (1.11), we have

$$\begin{aligned}
D_1 \leq | D_1 | &= \left| \frac{\gamma}{\gamma-1} \int_0^T \int_\Omega \psi(t) \Big(\overline{\rho\nabla(\rho^{\gamma-1})} - \rho\nabla(\rho^{\gamma-1}) \Big) \cdot \overline{\mathbf{u}} \,\mathrm{d}x\,\mathrm{d}t \right| \\
&= \left| \gamma \int_0^T \int_\Omega \psi(t) \big[(\overline{\rho^{\gamma-1}\nabla\rho} - \rho^{\gamma-1}\nabla\rho) \cdot \overline{\mathbf{u}} \big] \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq C \parallel \psi(t) \parallel_{L^\infty(0,T)} \int_0^T \parallel \overline{\mathbf{u}} \parallel_{L^q(\Omega)} \parallel \overline{\rho^{\gamma-1}\nabla\rho} - \rho^{\gamma-1}\nabla\rho \parallel_{L^{\frac{q}{q-1}}(\Omega)} \mathrm{d}t \\
&\leq C \int_0^T \parallel \overline{\mathbf{u}} \parallel_{L^q(\Omega)} \parallel \overline{\rho^{\gamma-1}\nabla\rho} - \rho^{\gamma-1}\nabla\rho \parallel_{L^2(\Omega)} \mathrm{d}t \\
&\leq C \parallel \overline{\mathbf{u}} \parallel_{L^p(0,T;L^q(\Omega))} \parallel \overline{\rho^{\gamma-1}\nabla\rho} - \rho^{\gamma-1}\nabla\rho \parallel_{L^{\frac{p}{p-1}}(0,T;L^2(\Omega))} \to 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned} \tag{3.36}$$

Using $\rho_t \in L^2(0,T;L^{\frac{2q}{q+2}}(\Omega))$ and $\rho \leq \tilde{\rho}$, we have D_3 goes to zero as ϵ tends to zero. Thus $R_\epsilon \to 0$, as $\epsilon \to 0$.

Secondly, we prove $I_\epsilon + J_\epsilon \to 0$, as $\epsilon \to 0$. By the Gagliardo-Nirenberg inequality and (1.6), for any $\alpha_2 \in (0,1)$, we obtain

$$\mathbf{H} \in L^{r_2}(0,T;L^{r_1}(\Omega)), \tag{3.37}$$

where $\frac{1}{r_1} = \frac{1}{6} + \frac{\alpha_2}{3}$, $\frac{1}{r_2} = \frac{1-\alpha_2}{2}$. In fact, for any $0 < \alpha_2 < 1$, we have

$$\begin{aligned}
\parallel \mathbf{H} \parallel^{r_2}_{L^{r_2}(0,T;L^{r_1}(\Omega))} &= \int_0^T \parallel \mathbf{H} \parallel^{r_2}_{L^{r_1}(\Omega)} \mathrm{d}t \\
&\leq \int_0^T \big(\parallel \mathbf{H} \parallel^{\alpha_2}_{L^2(\Omega)} \parallel \mathbf{H} \parallel^{1-\alpha_2}_{H^1(\Omega)} \big)^{r_2} \mathrm{d}t \\
&\leq C + \parallel \mathbf{H} \parallel^{\alpha_2 r_2}_{L^\infty(0,T;L^2(\Omega))} \int_0^T \parallel \nabla\mathbf{H} \parallel^{r_2(1-\alpha_2)}_{L^2(\Omega)} \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^T \| \nabla \mathbf{H} \|_{L^2(\Omega)}^2 \,\mathrm{d}t + C \\
&\le C,
\end{aligned}
$$

where $\frac{\alpha_2}{2} + (\frac{1}{2} - \frac{1}{3})(1-\alpha_2) = \frac{1}{r_1}$, and $r_2(1-\alpha_2) = 2$.

Firstly, we prove $I_\epsilon = I_1 + I_2 + I_3 + I_4 \to 0$, as $\epsilon \to 0$.

By virtue of the assumption that $\mathbf{u} \in L^p(0,T;L^q(\Omega))$ where p, q will be determined later, and Hölder inequality, Lemma 2.1 and (3.37), we get

$$
\begin{aligned}
I_\epsilon =& I_1 + I_2 + I_3 + I_4 \le | I_1 | + | I_2 | + | I_3 | + | I_4 | \\
=& \Big| \int_0^T \int_\Omega \psi(t)\Big(\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{H}}) - \overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{H})}\Big)\cdot \overline{\mathbf{u}} \,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \frac{1}{2}\int_0^T \int_\Omega \psi(t)\overline{\mathbf{u}}\cdot\Big(\nabla\overline{(\mathbf{H}\cdot\mathbf{H})} - \nabla(\mathbf{H}\cdot\overline{\mathbf{H}})\Big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&+ \Big| \int_0^T \int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\mathrm{div}(\mathbf{H}\otimes\overline{\mathbf{u}}) - \overline{\mathrm{div}(\mathbf{H}\otimes\mathbf{u})}\Big)\,\mathrm{d}x\,\mathrm{d}t \Big| + \Big| \int_0^T \int_\Omega \psi(t)\overline{\mathbf{H}}\cdot\Big(\overline{\mathrm{div}(\mathbf{u}\otimes\mathbf{H})} - \mathrm{div}(\overline{\mathbf{u}}\otimes\mathbf{H})\Big)\,\mathrm{d}x\,\mathrm{d}t \Big| \\
\le& C \| \psi(t) \|_{L^\infty(0,T)} \int_0^T \| \mathbf{u} \|_{L^q(\Omega)} \| \nabla\mathbf{H} \|_{L^2(\Omega)} \| \mathbf{H} \|_{L^{\frac{2q}{q-2}}(\Omega)} \,\mathrm{d}t \\
\le& C \| \mathbf{u} \|_{L^p(0,T;L^q(\Omega))} \| \nabla\mathbf{H} \|_{L^2(0,T;L^2(\Omega))} \| \mathbf{H} \|_{L^{\frac{2p}{p-2}}(0,T;L^{r_1}(\Omega))} \\
\le& C \| \mathbf{u} \|_{L^p(0,T;L^q(\Omega))} \| \nabla\mathbf{H} \|_{L^2(0,T;L^2(\Omega))} \| \mathbf{H} \|_{L^{r_2}(0,T;L^{r_1}(\Omega))},
\end{aligned}
$$

where $\frac{2q}{q-2} \le r_1$, and $\frac{2p}{p-2} \le r_2$.

Thanks to Lemma 2.1, as ϵ tend to zero, we have

$$
I_1 + I_2 + I_3 + I_4 \to 0. \tag{3.38}
$$

Next we prove $J_\epsilon = J_1 + J_2 \to 0$, as $\epsilon \to 0$.

For J_1, we obtain

$$
\begin{aligned}
| J_1 | =& \Big| \frac{1}{2}\int_0^T \int_\Omega \psi(t)(\overline{\mathbf{H}} - \mathbf{H})\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)\mathbf{H}\Big)\mathrm{d}x\mathrm{d}t \Big| \\
&\le C \| \psi(t) \|_{L^\infty(0,T)} \int_0^T \| \mathbf{u} \|_{L^q(\Omega)} \| \nabla\mathbf{H} \|_{L^2(\Omega)} \| \overline{\mathbf{H}} - \mathbf{H} \|_{L^{\frac{2q}{q-2}}(\Omega)} \,\mathrm{d}t \\
&\le C \| \mathbf{u} \|_{L^p(0,T;L^q(\Omega))} \| \nabla\mathbf{H} \|_{L^2(0,T;L^2(\Omega))} \| \overline{\mathbf{H}} - \mathbf{H} \|_{L^{\frac{2p}{p-2}}(0,T;L^{r_1}(\Omega))} \\
&\le C \| \mathbf{u} \|_{L^p(0,T;L^q(\Omega))} \| \nabla\mathbf{H} \|_{L^2(0,T;L^2(\Omega))} \| \overline{\mathbf{H}} - \mathbf{H} \|_{L^{r_2}(0,T;L^{r_1}(\Omega))} \to 0, \quad \text{as} \quad \epsilon \to 0,
\end{aligned}
$$

where $\frac{2q}{q-2} \le r_1$, and $\frac{2p}{p-2} \le r_2$.

For J_2, we have

$$
| J_2 | = \Big| \frac{1}{2}\int_0^T \int_\Omega \psi(t)\mathbf{H}\cdot\Big((\overline{\mathbf{u}}\cdot\nabla)(\mathbf{H} - \overline{\mathbf{H}})\Big)\, dxdt \Big|
$$

$$
\begin{aligned}
&\leq C \parallel \psi(t) \parallel_{L^\infty(0,T)} \int_0^T \parallel \mathbf{u} \parallel_{L^q(\Omega)} \parallel \mathbf{H} \parallel_{L^{\frac{2q}{q-2}}(\Omega)} \parallel \nabla(\overline{\mathbf{H}} - \mathbf{H}) \parallel_{L^2(\Omega)} \ dt \\
&\leq C \parallel \mathbf{u} \parallel_{L^p(0,T;L^q(\Omega))} \parallel \mathbf{H} \parallel_{L^{\frac{2p}{p-2}}(0,T;L^{r_1}(\Omega))} \parallel \nabla(\overline{\mathbf{H}} - \mathbf{H}) \parallel_{L^2(0,T;L^2(\Omega))} \\
&\leq C \parallel \mathbf{u} \parallel_{L^p(0,T;L^q(\Omega))} \parallel \mathbf{H} \parallel_{L^{r_2}(0,T;L^{r_1}(\Omega))} \parallel \nabla(\overline{\mathbf{H}} - \mathbf{H}) \parallel_{L^2(0,T;L^2(\Omega))} \to 0, \quad \text{as} \ \ \epsilon \to 0,
\end{aligned}
$$

where $\frac{2q}{q-2} \leq r_1$, and $\frac{2p}{p-2} \leq r_2$.

Thus, we have

$$
J_\epsilon \to 0, \ \text{as} \ \epsilon \to 0. \tag{3.39}
$$

In fact, because $\frac{1}{r_1} = \frac{1}{6} + \frac{\alpha_1}{3}, \frac{1}{r_2} = \frac{1-\alpha_1}{2}$ is equivalent to $\frac{3}{r_1} + \frac{2}{r_2} = \frac{3}{2}, r_1 < 6$, and $\frac{1}{r_1} \leq \frac{1}{2} - \frac{1}{q}, \frac{1}{r_2} \leq \frac{1}{2} - \frac{1}{p}$, we obtain

$$
\frac{3}{2} = \frac{3}{r_1} + \frac{2}{r_2} \leq \frac{5}{2} - \left(\frac{3}{q} + \frac{2}{p}\right).
$$

Thus we need $\mathbf{u} \in L^p(0,T;L^q(\Omega))$, where $\frac{3}{q} + \frac{2}{p} \leq 1$ for any $p \geq 4$, $q \geq 6$.

We are ready to pass to the limits in (3.29). Let ϵ go to zero, and using (3.38)-(3.39), what we have proved is that in the limit

$$
\begin{aligned}
&-\int_0^T \int_\Omega \psi_t \left(\frac{1}{2}\rho \mid \mathbf{u} \mid^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2} \mid \mathbf{H} \mid^2 \right) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi(t) \Big(\nu \mid \nabla \times \mathbf{H} \mid^2 + \mu \mid \nabla \mathbf{u} \mid^2 + (\lambda + \mu) \mid \mathrm{div}\mathbf{u} \mid^2 \Big) \mathrm{d}x \, \mathrm{d}t = 0
\end{aligned} \tag{3.40}
$$

for any test function $\psi \in \mathcal{D}(0, +\infty)$.

Step 3. Extending the result (3.40) for $\psi \in \mathcal{D}(-1, +\infty)$.

The final goal is to extend our result (3.40) for the test function $\psi \in \mathcal{D}(-1, +\infty)$. To this end, it is necessary for us to have the continuity of $\rho(t), (\sqrt{\rho}\mathbf{u})(t)$ and $\mathbf{H}$ in the strong topology at $t = 0$. Adopting a similar argument to that of [24], what we expected can be done.

Using $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega))$, and (1.10), we have

$$
\rho_t \in L^2(0,T;H^{-1}(\Omega)), \ \text{and} \ \nabla\rho \in L^2(0,T;L^2(\Omega)).
$$

Hence

$$
\rho \in C([0,T];L^2(\Omega)). \tag{3.41}
$$

By energy inequality (1.3), we have $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \mathrm{div}\mathbf{u} \in L^2(0,T;L^2(\Omega))$, and $\rho \leq \tilde{\rho}$. Recall $(1.1)_1$, we obtain

$$
(\sqrt{\rho})_t = -\mathrm{div}(\sqrt{\rho}\mathbf{u}) + \frac{1}{2}\sqrt{\rho}\mathrm{div}\mathbf{u}.
$$

Hence, we deduce

$$(\sqrt{\rho})_t \in L^2(0,T;H^{-1}(\Omega)).$$

Meanwhile, due to $\nabla\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega))$, we get $\sqrt{\rho} \in C([0,T];L^2(\Omega))$. More generally, in view of $\rho \in L^\infty(0,T;L^\infty(\Omega))$, we deduce

$$\rho \in C([0,T];L^s(\Omega)), \tag{3.42}$$

where $1 \le s < \infty$.

On the other hand,

$$\begin{aligned}
&\frac{1}{2} ess \limsup_{t\to 0} \int_\Omega |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx + ess\limsup_{t\to 0}\int_\Omega |\mathbf{H}-\mathbf{H}_0|^2\,dx \\
=&\underbrace{ess\limsup_{t\to 0}\left[\int_\Omega \left(\frac{1}{2}\rho\mathbf{u}^2 + \frac{\rho^\gamma}{\gamma-1} + |\mathbf{H}|^2\right)dx - \int_\Omega\left(\frac{1}{2}\rho_0\mathbf{u}_0^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\mathbf{H}_0|^2\right)dx\right]}_{II_1} \\
&\underbrace{+\, ess\limsup_{t\to 0}\int_\Omega \mathbf{u}_0(\rho_0\mathbf{u}_0 - \sqrt{\rho_0\rho}\mathbf{u})dx + 2ess\limsup_{t\to 0}\int_\Omega \mathbf{H}_0(\mathbf{H}_0-\mathbf{H})dx}_{II_2} \\
&\underbrace{+\, ess\limsup_{t\to 0}\int_\Omega\left(\frac{\rho_0^\gamma}{\gamma-1} - \frac{\rho^\gamma}{\gamma-1}\right)dx}_{II_3}.
\end{aligned}$$

Using energy inequality (1.3), we get $II_1 \le 0$.

By (3.42), we obtain $II_3 = 0$.

Hence, we have

$$\begin{aligned}
&\frac{1}{2} ess \limsup_{t\to 0} \int_\Omega |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx + ess\limsup_{t\to 0}\int_\Omega |\mathbf{H}-\mathbf{H}_0|^2\,dx \\
\le& ess\limsup_{t\to 0}\int_\Omega \mathbf{u}_0(\rho_0\mathbf{u}_0 - \sqrt{\rho_0\rho}\mathbf{u})dx + 2ess\limsup_{t\to 0}\int_\Omega \mathbf{H}_0(\mathbf{H}_0-\mathbf{H})dx \\
=& II_2^1 + II_2^2.
\end{aligned}$$

For II_2^1, taking $\mathbf{u}_0 \in L^\kappa,\ \kappa > 2$, it follows

$$\begin{aligned}
II_2^1 =& ess\limsup_{t\to 0}\int_\Omega \mathbf{u}_0(\rho_0\mathbf{u}_0 - \rho\mathbf{u}) + ess\limsup_{t\to 0}\int_\Omega \mathbf{u}_0\sqrt{\rho}\mathbf{u}(\sqrt{\rho}-\sqrt{\rho_0}) \\
=&0,
\end{aligned}$$

where we used the fact $\rho\mathbf{u} \in C([0,T];L^2_{weak}(\Omega))$, $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega))$, and (3.42).

For II_2^2, using $\mathbf{H} \in C([0,T];L^2_{weak}(\Omega))$, we get $II_2^2 = 0$.

So we get

$$\frac{1}{2} ess \limsup_{t\to 0} \int_\Omega |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx + ess\limsup_{t\to 0}\int_\Omega |\mathbf{H}-\mathbf{H}_0|^2\,dx \le 0,$$

which gives us

$$\sqrt{\rho}\mathbf{u} \in C([0,T];L^2(\Omega)), \text{ and } \mathbf{H} \in C([0,T];L^2(\Omega)). \tag{3.43}$$

As in [5], for any given $t_0 > 0$, we choose the text function $\psi_\tau(t)$ as below.

$$\psi_\tau(t) = \begin{cases} 0, & 0 \le t \le \tau \\ \frac{t-\tau}{\epsilon}, & \tau \le t \le \tau+\epsilon \\ 1, & \tau+\epsilon \le t \le t_0 \\ \frac{t_0+\epsilon-t}{\epsilon}, & t_0 \le t \le t_0+\epsilon \\ 0, & t_0+\epsilon \le t, \end{cases} \tag{3.44}$$

where positive τ and ϵ satisfying $\tau + \epsilon < t_0$.

Substituting (3.44) into (3.40), we have

$$\begin{aligned} &\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\int_\Omega \Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + |\,\mathbf{H}\,|^2\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_\tau^{t_0+\epsilon}\int_\Omega \psi_\tau(t)\Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ =&\frac{1}{\epsilon}\int_\tau^{\tau+\epsilon}\int_\Omega \Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + |\,\mathbf{H}\,|^2\Big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.45}$$

We deal with the second item on the left hand side

$$\begin{aligned} &\int_\tau^{t_0+\epsilon}\int_\Omega \psi_\tau(t)\Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ =&\int_\tau^{t_0+\epsilon}\int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_\tau^{\tau+\epsilon}\int_\Omega \Big(\frac{t-\tau}{\epsilon} - 1\Big)\Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\int_\Omega (t_0 - t)\Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Passing to the limit as $\epsilon \to 0$ in (3.45), and using (3.41), (3.43) and Lebesgue theorem, one deduces

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + |\,\mathbf{H}\,|^2\Big)(t_0)\,\mathrm{d}x + \int_\tau^{t_0}\int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ =&\int_\Omega \Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + |\,\mathbf{H}\,|^2\Big)(\tau)\,\mathrm{d}x, \end{aligned} \tag{3.46}$$

where

$$\begin{aligned}&\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon}\int_{\Omega}(t-\tau)\Big(\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathrm{div}\mathbf{u}|^2+\nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\le\int_{\tau}^{\tau+\epsilon}\int_{\Omega}\Big(\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathrm{div}\mathbf{u}|^2+\nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\to0,\end{aligned}$$

and similarly

$$\begin{aligned}&\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\int_{\Omega}(t_0-t)\Big(\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathrm{div}\mathbf{u}|^2+\nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\ge\int_{t_0}^{t_0+\epsilon}\int_{\Omega}\Big(\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathrm{div}\mathbf{u}|^2+\nu|\nabla\times\mathbf{H}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\to0,\end{aligned}$$

as $\epsilon\to0$. Finally setting $\tau\to0$ in (3.46), from (3.43) we get (1.4).

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