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Normalized solutions for Schrödinger-Poisson equations with general nonlinearities ☆

Sitong Chen, Xianhua Tang *, Shuai Yuan

School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China

A R T I C L E I N F O

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A B S T R A C T

In this paper, we prove the existence of normalized solutions to the following Schrödinger-Poisson equation

$$-\Delta u+\left(|x|^{-1}*|u|^2\right)u-f(u)=\lambda u, \quad x\in\mathbb{R}^3,\ \lambda\in\mathbb{R},$$

where $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$ satisfies more general conditions which cover the case $f(u)\sim|u|^{q-2}u$ with $q\in(3,\frac{10}{3})\cup(\frac{10}{3},6)$. Especially, some new analytical techniques are presented to overcome the difficulties due to the presence of three terms in the corresponding energy functional which scale differently.

1. Introduction

This paper deals with the existence of normalized solutions to the following Schrödinger-Poisson equation:

$$-\Delta u+\left(|x|^{-1}*|u|^2\right)u-f(u)=\lambda u, \quad x\in\mathbb{R}^3 \tag{1.1}$$

where $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$. This class of Schrödinger-type equations with a repulsive nonlocal Coulomb potential is obtained by approximation of the Hartree-Fock equation describing a quantum mechanical system of many particles; see [10–12,19,24,25].

In (1.1), when $\lambda\in\mathbb{R}$ is a fixed and assigned a parameter or even with an additional external and fixed potential $V(x)$, the existence of solutions of (1.1) has been intensively studied during the last decade; see, for example, [2–4,27] for radial or coercive potentials; [1,13,17,30] for periodic or asymptotically periodic potentials; we also refer to [26,31,33–35,39] for more similar variational problems. In this case, solutions can be obtained as critical points of the corresponding energy functional, however, nothing can be given a priori on the L^2-norm of the solutions.

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* Corresponding author.
E-mail addresses: mathsitongchen@163.com (S. Chen), tangxh@mail.csu.edu.cn (X. Tang), ys950526@163.com (S. Yuan).

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Nowadays, since physicists are interested in normalized solutions, mathematical researchers began to focus on solutions having a prescribed L^2-norm, that is, solutions which satisfy $\|u\|_2^2 = c > 0$ for a priori given c. Such solutions of (1.1) can be obtained by looking for critical points of the following functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \tag{1.2}$$

on the constraint

$$\mathcal{S}_c = \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c\right\}, \tag{1.3}$$

where $F(u) = \int_0^u f(t)\mathrm{d}t$. In this case, the parameter $\lambda \in \mathbb{R}$ cannot longer be fixed but instead appears as a Lagrange multiplier, and each critical point $u_c \in \mathcal{S}_c$ of $I|_{\mathcal{S}_c}$, corresponds a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that (u_c, λ_c) solves (weakly) (1.1). In particular, if $u_c \in \mathcal{S}_c$ is a minimizer of problem

$$\sigma(c) := \inf_{u\in\mathcal{S}_c} I(u), \tag{1.4}$$

then there exists $\lambda_c \in \mathbb{R}$ such that $I'(u_c) = \lambda_c u_c$, namely, (u_c, λ_c) is a solution of (1.1).

For the following Schrödinger equation

$$-\Delta u - f(u) = \lambda u \text{ in } \mathbb{R}^N, \tag{1.5}$$

Stuart [29] and Jeanjean [20] obtained the existence of normalized solutions by solving the minimization problem

$$\inf_{u\in H^1(\mathbb{R}^N), \|u\|_2 = c} \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla u|^2 - F(u)\right] \mathrm{d}x$$

and by using a mountain pass argument on the constraint $\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\}$, respectively. Later, these results on normalized solutions of (1.5) were extended in [7–9,21,40] to the following special form of (1.1):

$$-\Delta u + \left(|x|^{-1} * |u|^2\right) u - |u|^{q-2}u = \lambda u, \qquad x \in \mathbb{R}^3, \tag{1.6}$$

where $q \in (2, 6)$. We also refer to [22] for quasi-linear Schrödinger equations; [23] for Choquard equations; [37,38] for Kirchhoff-type equations; [5,6] for Schrödinger systems. In particular, owing to the presence of three terms in the corresponding energy functional which scale differently, it is more complicated to deal with the existence of normalized solutions for the Schrödinger-Poisson equation. Let us introduce and review the known results in this respect.

For the case $2 < q < \frac{10}{3}$, normalized solutions can be found by considering the minimization problem:

$$\sigma(c) = \inf_{u\in\mathcal{S}_c} I(u), \tag{1.7}$$

since the functional I is bounded from below and coercive on $\mathcal{S}_c$. Bellazzini and Siciliano in [8] and [9] proved that $\sigma(c)$ is achieved when $c > 0$ is small and $2 < q < 3$ and when $c > 0$ is large and $3 < q < \frac{10}{3}$, respectively. Subsequently, for the range $2 \le q \le \frac{10}{3}$, Jeanjean and Luo in [21] explicated a threshold value of $c > 0$ separating existence and nonexistence of minimizers of $\sigma(c)$. Using techniques introduced by Catto and Lions in [12], Sánchez and Soler in [28] showed that minimizers of $\sigma(c)$ exist for $q = \frac{8}{3}$ provided that

$c \in (0, \hat{c})$ for a suitable $\hat{c} > 0$ small enough. If an additional potential $V(x)$ with $\inf_{x\in\mathbb{R}^3} V(x) = 0$ and $\lim_{|x|\to\infty} V(x) = \infty$ is added to the left side of (1.6), with the compactness of the Sobolev embeddings in the working space, Zeng and Zhang in [40] obtained the existence of normalized solutions of (1.6) for the case $2 < q < \frac{10}{3}$. However, for the case $\frac{10}{3} < q < 6$, the functional I is no more bounded from below on $\mathcal{S}_c$. As far as we know, there seems to be only one paper [7] dealt with this case. Precisely, Bellazzini, Jeanjean and Luo in [7] found critical points of I on $\mathcal{S}_c$ by looking at the mountain-pass level for $c > 0$ sufficiently small. To prove this result, they first established the mountain-pass geometry of I on $\mathcal{S}_c$, and then constructed the special bounded Palais-Smale sequence $\{u_n\}$ at the level $\gamma(c)$ which concentrates around the set:

$$\mathcal{M}_c = \left\{ u \in \mathcal{S}_c : J(u) := \frac{\mathrm{d}}{\mathrm{d}t} I(u^t)\Big|_{t=1} = 0 \right\}, \tag{1.8}$$

that is $J(u_n) = o(1)$, where $u^t(x) = t^{3/2} u(tx)$. In addition, they proved that $\mathcal{M}_c$ acts as a natural constrain and $\gamma(c)$ equals to

$$m(c) = \inf_{u\in\mathcal{M}_c} I(u). \tag{1.9}$$

In spite of this fact, it does not seem possible to rule out the dichotomy of any minimizing sequence of $m(c)$, that is to rule out

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3) \text{ and } 0 < \|u\|_2^2 < c, \tag{1.10}$$

which is the main difficulty. For this, information on the derivative of I along the sequence seems necessary and that is why the authors introduced Palais-Smale sequences to solve the minimization problem. To overcome this difficulty, with the Implicit Function Theorem, the authors in [7] proved that $\gamma(c)$ is nonincreasing on $(0, \infty)$, and the associated Lagrange multiplier $\lambda_c \in \mathbb{R}$ is a nonzero, here to do the latter, it is necessary that $c > 0$ is sufficiently small. However, the approach relies heavily on the q-homogeneity and differentiability of the nonlinearity f (see [7, Lemmas 5.2 and 5.3]), it does not work for (1.1) with more general f.

A natural question is whether the above result obtained in [7] on the existence of normalized solutions to (1.6) with $\frac{10}{3} < q < 6$ can be generalized to (1.1) with more general f. One purpose of the present paper is to address this question. To this end, we introduce the following assumptions:

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist $\mathcal{C} > 0$ and $q \in (2, 6)$ such that $|f(t)| \le \mathcal{C}(1 + |t|^{q-1})$ for all $t \in \mathbb{R}$;
(F2) $\lim_{|t|\to 0} \frac{F(t)}{|t|^2} = 0$ and $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{\frac{10}{3}}} = +\infty$;
(F3) there exists a constant $p \in (\frac{10}{3}, 6)$ such that $[f(t)t - 2F(t)]/|t|^{p-1}t$ is nondecreasing on $(-\infty, 0)$ and $(0, +\infty)$.

Our first result is as follows.

Theorem 1.1. *Assume that* (F1)-(F3) *hold. Then there exists $c_0 > 0$ such that for any $c \in (0, c_0]$,* (1.1) *has a couple of solutions $(\bar{v}_c, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}^-$ such that*

$$I(\bar{v}_c) = \inf_{v\in\mathcal{M}_c} I(v) = \inf_{v\in\mathcal{S}_c} \max_{t>0} I(v^t) > 0.$$

Another purpose of this paper is to improve and generalize the previous results on the existence of a global minimizer of I on $\mathcal{S}_c$ in the case $f(u) = |u|^{q-2}u$ with $3 < q < \frac{10}{3}$ to a general nonlinearity satisfying the following conditions:

(F4) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $\lim_{|t|\to 0} \frac{F(t)}{|t|^3} = 0$ and there exist constants $\mathcal{C}_0 > 0$ and $q_0 \in (3, \frac{10}{3})$ such that $|f(t)| \leq \mathcal{C}_0(1 + |t|^{q_0-1})$ for all $t \in \mathbb{R}$;
(F5) there exists a constant $p_0 \in (3, \frac{10}{3})$ such that $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{p_0}} > 0$;
(F6) $f(t)t \geq 3F(t) \geq 0$ for all $t \in \mathbb{R}$.

Let

$$c^* := \inf \{c \in (0, +\infty), \sigma(c) < 0\}. \tag{1.11}$$

In this direction, we have the following theorem.

Theorem 1.2. *Assume that* (F4)-(F6) *hold. Then $c^* > 0$, and I admits a critical point u_c on $\mathcal{S}_c$ which is a global minimum of I when $c \in [c_*, +\infty)$. In particular, $\sigma(c^*) = 0$. Moreover, for the above critical point u_c, there exists Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that (u_c, λ_c) is a solution of* (1.1).

Remark 1.3. Unlike previous work on other elliptic PDEs, it does not seem possible to reduce the problem to the classical vanishing-dichotomy-compactness scenario and to the check of the associated strict subadditivity inequalities due to the different scaling rates of each term in $I(u)$. Theorem 1.1 and Theorem 1.2 improve and extend to the main existence results in [7] and [8,9,21], respectively.

Now, we give our main idea for the proof of Theorem 1.1. Since (F1)-(F3) imply that I is no more bounded from below on $\mathcal{S}_c$, we shall look for a critical point satisfying a minimax characterization, i.e., we try to prove that I possesses a mountain pass geometry on $\mathcal{S}_c$.

Definition 1.4. For given $c > 0$, we say that $I(u)$ possesses a mountain pass geometry on $\mathcal{S}_c$ if there exists $\rho_c > 0$ such that

$$\gamma(c) = \inf_{g\in\Gamma_c} \max_{\tau\in[0,1]} I(g(\tau)) > \max_{g\in\Gamma_c} \max\{I(g(0)), I(g(1))\}, \tag{1.12}$$

where $\Gamma_c = \{g \in \mathcal{C}([0, 1], \mathcal{S}_c) : \|\nabla g(0)\|_2^2 \leq \rho_c, I(g(1)) < 0\}$.

Let us mention that, to do that, the authors in [7] constructed the nice 'shape' of some sequence of paths $\{g_n\} \subset \Gamma_c$, and obtained a localization lemma for a specific (PS) sequence, in which Taylor's formula was used that relies on $I \in \mathcal{C}^2(H^1(\mathbb{R}^3), \mathbb{R})$. In the present paper, different from [7], we consider the following auxiliary functional:

$$\tilde{I}(v, t) = I(\beta(v, t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 + \frac{e^t}{4} \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y - \frac{1}{e^{3t}} \int\limits_{\mathbb{R}^3} F(e^{\frac{3t}{2}} v)\mathrm{d}x,$$

and prove that $\tilde{I}$ possesses the same mountain pass geometry on $\mathcal{S}_c \times \mathbb{R}$ as the functional $I\big|_{\mathcal{S}_c}$, on this basis, obtain the $(\mathrm{PS})_{\gamma(c)}$ sequence $\{u_n\}$ satisfying the extra information $J(u_n) \to 0$, and then prove the convergence of $\{u_n\}$, this idea comes back to [20] in which the classical Schrödinger equation (1.5) was studied. Let us point out that the adaptation of the idea to our problem is not trivial at all because of the presence of three terms in I which scale differently. In fact, to derive the convergence of the above $\{u_n\}$, a key step is to show that $\gamma(c)$ is nonincreasing. But now, the scaling technique introduced in [20] does not work for I. Instead, we first show that $\gamma(c) = m(c)$ and then prove that the add in a suitable way L^2-norm does not increase the mountain-pass level. This information permits to reduce the problem of convergence to that of showing that the associated Lagrange multiplier $\lambda \in \mathbb{R}$ is a nonzero. This approach is reminiscent of the one developed in [7] and but here the fact I may be not $\mathcal{C}^2$ prevents us from using the Implicit Function Theorem, thus new techniques and more subtle analyses are required for more general $f \notin \mathcal{C}^1$. Theorem 1.1 will be proved in Section 2.

Theorem 1.2 is a generalization of the result from [8,9,21], in which the case $f(u) = |u|^{q-2}u$ with $3 < q < \frac{10}{3}$ was considered. To obtain the achievability of $\sigma(c) = \inf_{\mathcal{S}_c} I$, it is necessary to rule out the dichotomy of the minimizing sequence, that is the case (1.10) does not occur. For this purpose, as in [8,9,21], a key step is to prove

$$\sigma(tc) \leq t^3\sigma(c), \quad \forall\, t > 1. \tag{1.13}$$

But, the fact that f has no homogeneity property makes the proof more delicate. In addition, the case $c = c^*$ requires a special treatment since $\sigma(c^*) = 0$. Theorem 1.2 will be proved in Section 3.

Throughout the paper we make use of the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\, u, v \in H^1(\mathbb{R}^3);$$

- $L^s(\mathbb{R}^3)(1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s\mathrm{d}x\right)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^3)$, $u^t(x) := t^{3/2}u(tx)$ and $u_t(x) := t^2u(tx)$;
- For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;
- $S = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \|\nabla u\|_2^2/\|u\|_6^2$;
- $C_1, C_2, \cdots$ denote positive constants possibly different in different places.

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

First, we prove that I has a mountain pass geometry on the constraint $\mathcal{S}_c$.

Lemma 2.1. *Assume that* (F1)-(F3) *hold. Then for any $c > 0$, there exist $0 < k_1 < k_2$ and $u_1, u_2 \in \mathcal{S}_c$ such that $u_1 \in \mathcal{A}_{k_1}$ and $u_2 \in \mathcal{A}^{k_2}$, where*

$$\mathcal{A}_{k_1} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq k_1, I(u) > 0\right\} \tag{2.1}$$

and

$$\mathcal{A}^{k_2} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 > k_2, I(u) < 0\right\}. \tag{2.2}$$

Moreover, I has a mountain pass geometry on the constraint $\mathcal{S}_c$.

Proof. For any $k > 0$, set

$$\mathcal{B}_k = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq k\right\}. \tag{2.3}$$

We first claim that there exist $0 < k_1 < k_2$ such that

$$I(u) > 0, \quad \forall\, u \in \mathcal{B}_{k_2} \text{ and } \sup_{u\in\mathcal{B}_{k_1}} I(u) < \inf_{u\in\partial\mathcal{B}_{k_2}} I(u). \tag{2.4}$$

On the one hand, by the Gagliardo-Nirenberg inequality and the Sobolev embedding inequality, we have

$$
\begin{aligned}
I(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \varepsilon\|u\|_2^2 - C_\varepsilon\|u\|_q^q \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \varepsilon\|u\|_2^2 - C_\varepsilon C(q)\|\nabla u\|_2^{\frac{3(q-2)}{2}}\|u\|_2^{\frac{6-q}{2}}.
\end{aligned}
\tag{2.5}
$$

Since ε is arbitrary and $\frac{3(q-2)}{2} > 2$, it follows from (2.5) that there exist $k_2 > 0$ small and $\rho > 0$ such that

$$
\inf_{u\in\partial\mathcal{B}_{k_2}} I(u) \geq \rho > 0 \text{ and } I(u) > 0 \text{ for } u \in \mathcal{B}_{k_2}. \tag{2.6}
$$

On the other hand, the Hardy-Littlewood-Sobolev inequality and the Gagliardo-Nirenberg inequality give

$$
\begin{aligned}
I(u) &\leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \frac{1}{2}\|\nabla u\|_2^2 + C_1\|u\|_{12/5}^4 \\
&\leq \frac{1}{2}\|\nabla u\|_2^2 + C_2\|\nabla u\|_2\|u\|_2^3,
\end{aligned}
\tag{2.7}
$$

which implies

$$
\sup_{u\in\mathcal{B}_k}|I(u)| \to 0 \text{ as } k \to 0. \tag{2.8}
$$

Combining (2.6) with (2.8), there exists $k_1 \in (0, k_2)$ small such that

$$
\sup_{u\in\mathcal{B}_{k_1}} I(u) < \rho \leq \inf_{u\in\partial\mathcal{B}_{k_2}} I(u).
$$

Hence, we have proved the above claim, that is (2.4) holds. Let

$$
u^t(x) = t^{3/2}u(tx), \qquad \forall\, t > 0, u \in H^1(\mathbb{R}^3). \tag{2.9}
$$

Then $\|u^t\|_2 = \|u\|_2$, and so $u^t \in \mathcal{S}_c$ for any $u \in \mathcal{S}_c$ and $t > 0$. Note that

$$
I(u^t) = \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{t^3}\int_{\mathbb{R}^3}F(t^{3/2}u)\mathrm{d}x. \tag{2.10}
$$

Using (F3) and (2.10), it is easy to see that $I(u^t) \to +\infty$ as $t \to -\infty$. For any $u \in \mathcal{S}_c$, there exist $t_1 > 0$ small and $t_2 > 1$ large such that

$$
\|\nabla u^{t_1}\|_2^2 = t_1^2\|\nabla u\|_2^2 \leq k_1, \quad \|\nabla u^{t_2}\|_2^2 = t_2^2\|\nabla u\|_2^2 > k_2 \text{ and } I(u^{t_2}) < 0. \tag{2.11}
$$

Set $u_1 = u^{t_1}$ and $u_2 = u^{t_2}$. Then (2.11) yields

$$
\|\nabla u_1\|_2^2 \leq k_1, \qquad \|\nabla u_2\|_2^2 > k_2.
$$

This shows that $u_1 \in \mathcal{A}_{k_1}$ and $u_2 \in \mathcal{A}^{k_2}$.

We next prove that I has a mountain pass geometry on $\mathcal{S}_c$ (see [26,36]). For

$$
\Gamma_c := \left\{g \in \mathcal{C}([0,1], \mathcal{S}_c) : \|\nabla g(0)\|_2^2 \leq k_1, I(g(1)) < 0\right\},
$$

if $\Gamma_c \neq \emptyset$, then for any $g \in \Gamma_c$, (2.4) implies $\|\nabla g(0)\|_2^2 \leq k_1 < k_2 < \|\nabla g(1)\|_2^2$. Thus, by the intermediate value theorem, there exists $\tau_0 \in (0,1)$ such that $\|\nabla g(\tau_0)\|_2^2 = k_2$, i.e., $g(\tau_0) \in \partial\mathcal{B}_{k_2}$. We conclude from (2.4) that

$$\max_{t\in[0,1]} I(g(t)) \geq I(g(\tau_0)) \geq \inf_{u\in\partial\mathcal{B}_{k_2}} I(u) > \sup_{u\in\mathcal{B}_{k_1}} I(u), \qquad \forall\, g \in \Gamma_c,$$

which, together with the arbitrariness of $g \in \Gamma_c$, implies

$$\gamma(c) = \inf_{g\in\Gamma_c} \max_{t\in[0,1]} I(g(t)) > \max_{g\in\Gamma_c} \max\{I(g(0)), I(g(1))\}. \tag{2.12}$$

Thus, to obtain the desired conclusion, it suffices to show that $\Gamma_c \neq \emptyset$. For any $u \in \mathcal{S}_c$, set

$$g_0(\tau) = u^{(1-\tau)t_1+\tau t_2}, \qquad \forall\, \tau \in [0,1].$$

It follows from (2.11) that $g_0 \in \gamma(c)$. Hence, $\Gamma_c \neq \emptyset$ and the proof is completed. $\square$

Second, inspired by [14,16,20], we will find a (PS) sequence for the functional I on $\mathcal{S}_c$ with the extra information $J(u_n) \to 0$, where

$$J(u) = \|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{3}{2}\int_{\mathbb{R}^3} [f(u)u - 2F(u)]\,\mathrm{d}x, \qquad \forall\, u \in H^1(\mathbb{R}^3). \tag{2.13}$$

To this end, we define a continuous map $\beta : H := H^1(\mathbb{R}^3) \times \mathbb{R} \to H^1(\mathbb{R}^3)$ by

$$\beta(v,t)(x) = e^{\frac{3t}{2}} v(e^t x) \ \text{ for } v \in H^1(\mathbb{R}^3),\ t \in \mathbb{R},\ \text{and } x \in \mathbb{R}^3, \tag{2.14}$$

where H is a Banach space equipped with the product norm $\|(v,t)\|_H := \left(\|v\|^2 + |t|^2\right)^{1/2}$. We consider the following auxiliary functional:

$$\tilde{I}(v,t) = I(\beta(v,t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 + \frac{e^t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{e^{3t}}\int_{\mathbb{R}^3} F(e^{\frac{3t}{2}}v)\mathrm{d}x. \tag{2.15}$$

It is easy to check that $\tilde{I} \in \mathcal{C}^1(H,\mathbb{R})$, and for any $(w,s) \in H$,

$$\begin{aligned}\left\langle \tilde{I}'(v,t),(w,s)\right\rangle = {} & e^{2t}\int_{\mathbb{R}^3} \nabla v\cdot\nabla w\mathrm{d}x + e^{2t}s\|\nabla v\|_2^2 + \frac{e^t s}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\ & + e^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v(y)w(y)}{|x-y|}\mathrm{d}x\mathrm{d}y + \frac{3s}{e^{3t}}\int_{\mathbb{R}^3} F(e^{\frac{3t}{2}}v)\mathrm{d}x \\ & - \frac{1}{e^{3t}}\int_{\mathbb{R}^3} f(e^{\frac{3t}{2}}v)e^{\frac{3t}{2}}w\mathrm{d}x.\end{aligned} \tag{2.16}$$

For the sets $\mathcal{A}_{k_1}$ and $\mathcal{A}^{k_2}$ defined in Lemma 2.1, set

$$\tilde{\gamma}(c) := \inf_{\tilde{g}\in\tilde{\Gamma}_c} \max_{\tau\in[0,1]} \tilde{I}(\tilde{g}(\tau)), \tag{2.17}$$

where

$$\tilde{\Gamma}_c = \{\tilde{g} \in \mathcal{C}([0,1], \mathcal{S}_c \times \mathbb{R}) : \tilde{g}(0) \in \mathcal{A}_{k_1} \times \{0\}, \tilde{g}(1) \in \mathcal{A}^{k_2} \times \{0\}\}.$$

For any $g \in \Gamma_c$, let $\tilde{g}_0(\tau) = (g(\tau), 0)$ for $\tau \in [0,1]$. Then $\tilde{g}_0 \in \tilde{\Gamma}_c$, and so $\tilde{\Gamma}_c \neq \emptyset$. Since $\Gamma_c = \left\{\beta \circ \tilde{g} : \tilde{g} \in \tilde{\Gamma}_c\right\}$, the minimax values of I and $\tilde{I}$ coincide, i.e., $\gamma(c) = \tilde{\gamma}(c)$, moreover, (2.12) leads to

$$\tilde{\gamma}(c) = \gamma(c) > \max_{g \in \Gamma_c} \max\{I(g(0)), I(g(1))\} = \max_{\tilde{g} \in \tilde{\Gamma}_c} \max\{\tilde{I}(\tilde{g}(0)), \tilde{I}(\tilde{g}(1))\}. \tag{2.18}$$

Following [36], we recall that for any $c > 0$, $\mathcal{S}_c$ is a submanifold of $H^1(\mathbb{R}^3)$ with codimension 1 and the tangent space at $\mathcal{S}_c$ is defined as

$$T_u = \left\{ v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} uv \mathrm{d}x = 0 \right\}. \tag{2.19}$$

The norm of the derivative of the $\mathcal{C}^1$ restriction functional $I|_{\mathcal{S}_c}$ is defined by

$$\|I|'_{\mathcal{S}_c}(u)\| = \sup_{v \in T_u, \|v\|=1} \langle I'(u), v \rangle . \tag{2.20}$$

Similarly, the tangent space at $(u,t) \in \mathcal{S}_c \times \mathbb{R}$ is given as

$$\tilde{T}_{u,t} = \left\{ (v,s) \in H : \int_{\mathbb{R}^3} uv \mathrm{d}x = 0 \right\}. \tag{2.21}$$

The norm of the derivative of the $\mathcal{C}^1$ restriction functional $\tilde{I}|_{\mathcal{S}_c \times \mathbb{R}}$ is defined by

$$\|\tilde{I}|'_{\mathcal{S}_c \times \mathbb{R}}(u,t)\| = \sup_{(v,s) \in \tilde{T}_{u,t}, \|(v,s)\|_H = 1} \left\langle \tilde{I}|'_{\mathcal{S}_c \times \mathbb{R}}(u,t), (v,s) \right\rangle . \tag{2.22}$$

As in [20, Proposition 2.2], we have the following proposition.

Proposition 2.2. *Assume that $\tilde{I}$ has a mountain pass geometry on the constraint $\mathcal{S}_c \times \mathbb{R}$. Let $\tilde{g}_n \in \tilde{\Gamma}_c$ be such that*

$$\max_{\tau \in [0,1]} \tilde{I}(\tilde{g}_n(\tau)) \le \tilde{\gamma}(c) + \frac{1}{n}. \tag{2.23}$$

Then there exists a sequence $\{(u_n, t_n)\} \subset \mathcal{S}_c \times \mathbb{R}$ such that

(i) $\tilde{I}(u_n, t_n) \in \left[\tilde{\gamma}(c) - \frac{1}{n}, \tilde{\gamma}(c) - \frac{1}{n}\right]$;
(ii) $\min_{\tau \in [0,1]} \|(u_n, t_n) - \tilde{g}_n(\tau)\|_H \le \frac{1}{\sqrt{n}}$;
(iii) $\|\tilde{I}|'_{\mathcal{S}_c \times \mathbb{R}}(u_n, t_n)\| \le \frac{2}{\sqrt{n}}$, *i.e.,*

$$|\langle \tilde{I}'(u_n, t_n), (v,s) \rangle| \le \frac{2}{\sqrt{n}} \|(v,s)\|_H, \qquad \forall\ (v,s) \in \tilde{T}_{u_n, t_n}.$$

Applying Proposition 2.2 to $\tilde{I}$, we have the following key lemma.

Lemma 2.3. *Assume that* (F1)-(F3) *hold. Then for any $c > 0$, there exists a sequence $\{v_n\} \subset \mathcal{S}_c$ such that*

$$I(v_n) \to \gamma(c) > 0, \quad I|'_{\mathcal{S}_c}(v_n) \to 0 \quad \textit{and} \quad J(v_n) \to 0. \tag{2.24}$$

Proof. Let $\{g_n\} \subset \Gamma_c$ satisfy

$$\max_{\tau\in[0,1]} I(g_n(\tau)) \le \gamma(c) + \frac{1}{n}. \tag{2.25}$$

To obtain the desired sequence, we first apply Proposition 2.2 to $\tilde{I}$. For this purpose, we define

$$\tilde{g}_n(\tau) = (g_n(\tau), 0), \qquad \forall\, \tau \in [0,1].$$

It is easy to see that $\tilde{g}_n \in \tilde{\Gamma}_c$ and $\tilde{I}(\tilde{g}_n(\tau)) = I(g_n(\tau))$. Since $\tilde{\gamma}(c) = \gamma(c)$, it follows from (2.25) that

$$\max_{\tau\in[0,1]} \tilde{I}(\tilde{g}_n(\tau)) \le \tilde{\gamma}(c) + \frac{1}{n}. \tag{2.26}$$

From Proposition 2.2, there exists a sequence $\{(u_n, t_n)\} \subset \mathcal{S}_c \times \mathbb{R}$ such that

(i) $\tilde{I}(u_n, t_n) \to \tilde{\gamma}(c)$;
(ii) $\min_{\tau\in[0,1]} \|(u_n, t_n) - (g_n(\tau), 0)\|_H \to 0$;
(iii) $\|\tilde{I}|'_{\mathcal{S}_c\times\mathbb{R}}(u_n, t_n)\| \le \frac{2}{\sqrt{n}}$.

Set $v_n := \beta(u_n, t_n)$, where the definition of β is given in (2.14). Since $v_n \in \mathcal{S}_c$ and $\tilde{\gamma}(c) = \gamma(c)$, it follows from (i) that

$$I(v_n) \to \gamma(c). \tag{2.27}$$

From (2.16) and (ii), we derive

$$\langle I'(v_n), w\rangle = \left\langle \tilde{I}'(u_n, t_n), (\beta(w, -t_n), 0)\right\rangle \le \frac{2}{\sqrt{n}}\|(\beta(w, -t_n), 0)\|_H, \qquad \forall\, w \in T_{v_n}. \tag{2.28}$$

To prove $I|'_{\mathcal{S}_c}(v_n) \to 0$, by (2.28), it suffices to show that $\{(\beta(w, -t_n), 0)\} \subset \tilde{\Gamma}_{u_n,t_n}$ and $\{(\beta(w, -t_n), 0)\}$ is uniformly bounded in H. We next prove the conclusion holds. For any $w \in T_{v_n}$, i.e.,

$$\int_{\mathbb{R}^3} v_n w \mathrm{d}x = \int_{\mathbb{R}^3} e^{\frac{3t_n}{2}} u_n(e^{t_n}x) w(x) \mathrm{d}x = 0,$$

we have

$$\int_{\mathbb{R}^3} u_n(x)\beta(w, -t_n)\mathrm{d}x = \int_{\mathbb{R}^3} u_n(x) e^{\frac{-3t_n}{2}} w(e^{-t_n}x)\mathrm{d}x = \int_{\mathbb{R}^3} e^{\frac{3t_n}{2}} u_n(e^{t_n}x) w(x)\mathrm{d}x = 0,$$

which implies

$$(\beta(w, -t_n), 0) \in \tilde{\Gamma}_{u_n,t_n}. \tag{2.29}$$

Moreover, by (ii), we have

$$|t_n| \le \min_{\tau\in[0,1]} \|(u_n, t_n) - \tilde{g}_n(\tau)\|_H \le 1 \text{ for large } n \in \mathbb{N},$$

which leads to

$$\|(\beta(w,-t_n),0)\|_H^2 = \|\beta(w,-t_n)\|^2 = e^{-2t_n}\|\nabla w\|_2^2 + \|w\|_2^2 \le e^2\|w\|^2 \text{ for large } n \in \mathbb{N}. \tag{2.30}$$

This shows that $\{(\beta(w,-t_n),0)\} \subset \tilde{\Gamma}_{u_n,t_n}$ is uniformly bounded in H, and so $I|'_{\mathcal{S}_c}(v_n) \to 0$. Finally, by (iii), we have

$$\left|\langle \tilde{I}'(u_n,t_n),(0,1)\rangle\right| = J(\beta(u_n,t_n)) = J(v_n) = o(1). \tag{2.31}$$

Hence, $\{v_n\}$ satisfies (2.24). $\square$

Next, we will give an additional minimax characterization of $\gamma(c)$. Before this, we establish some new inequalities.

Lemma 2.4. *Assume that* (F1)-(F3) *hold. Then*

$$\begin{aligned} h(t,\tau) &:= t^{-3}F(t^{3/2}\tau) - F(\tau) + \frac{1-t^{\frac{3(p-2)}{2}}}{p-2}[f(\tau)\tau - 2F(\tau)] \\ &\ge 0, \quad \forall\, t>0,\ \tau\in\mathbb{R} \end{aligned} \tag{2.32}$$

and

$$\frac{F(t)}{|t|^{p-1}t} \text{ is nondecreasing on both } (-\infty,0) \text{ and } (0,+\infty). \tag{2.33}$$

Proof. For any $\tau \in \mathbb{R}$, by (F1) and (F3), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}h(t,\tau) &= \frac{3}{2}t^{-4}\left[f(t^{3/2}\tau)t^{3/2}\tau - 2F(t^{3/2}\tau)\right] - \frac{3}{2}t^{\frac{3(p-2)}{2}-1}[f(\tau)\tau - 2F(\tau)] \\ &= \frac{3}{2}t^{\frac{3(p-2)}{2}-1}|\tau|^p\left[\frac{f(t^{3/2}\tau)t^{3/2}\tau - 2F(t^{3/2}\tau)}{|t^{\frac{3}{2}}\tau|^p} - \frac{f(\tau)\tau-2F(\tau)}{|\tau|^p}\right] \\ &\begin{cases} \ge 0, & t\ge 1, \\ \le 0, & 0<t<1, \end{cases} \end{aligned}$$

which implies that $h(t,\tau) \ge h(1,\tau) = 0$ for all $t>0$ and $\tau\in\mathbb{R}$. This shows that (2.32) holds. Moreover, (F2) and (2.32) give

$$h(0,\tau) := \lim_{|t|\to 0} h(t,\tau) = \frac{1}{p-2}[f(\tau)\tau - pF(\tau)] \ge 0, \quad \forall\, \tau\in\mathbb{R}, \tag{2.34}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{F(t)}{|t|^{p-1}t} = \frac{1}{|t|^{p+1}}[f(t)t - pF(t)] \ge 0.$$

This shows that (2.33) holds. $\square$

By the scaling (2.9), one has

$$I(u^t) = \frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - t^{-3}\int_{\mathbb{R}^3}F(t^{3/2}u)\mathrm{d}x. \tag{2.35}$$

It is checked easily that $J(u) = \frac{\mathrm{d}}{\mathrm{d}t} I(u^t)\Big|_{t=1}$, where the definition of J is given in (2.13). Let

$$h_1(t) := 4t^{\frac{3(p-2)}{2}} - 3(p-2)t^2 + 3p - 10, \qquad h_2(t) := 2t^{\frac{3(p-2)}{2}} - 3(p-2)t + 3p - 8, \qquad \forall\ t \geq 0. \tag{2.36}$$

By simple calculations, one has

$$h_1(1) = h_2(1) = 0, \qquad h_1(t) > 0, \qquad h_2(t) > 0, \qquad \forall\ t \in [0,1) \cup (1,+\infty). \tag{2.37}$$

Inspired by [15,18,32], we prove the following lemma.

Lemma 2.5. *Assume that* (F1)-(F3) *hold. Then*

$$\begin{aligned} I(u) \geq I\left(u^t\right) &+ \frac{2\left[1 - t^{\frac{3(p-2)}{2}}\right]}{3(p-2)} J(u) + \frac{h_1(t)}{6(p-2)} \|\nabla u\|_2^2 \\ &+ \frac{h_2(t)}{12(p-2)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y, \qquad \forall\ u \in H^1(\mathbb{R}^3),\ t > 0 \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} I(u) \geq \frac{2}{3(p-2)} J(u) + \frac{3p-10}{6(p-2)} \|\nabla u\|_2^2 + \frac{3p-8}{12(p-2)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y, \\ \forall\ u \in H^1(\mathbb{R}^3). \end{aligned} \tag{2.39}$$

Proof. By (1.2), (2.13), (2.32), (2.33), (2.35) and (2.36), we have

$$\begin{aligned} I(u) - I\left(u^t\right) &= \frac{1-t^2}{2} \|\nabla u\|_2^2 + \frac{1-t}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3} \left[t^{-3} F\left(t^{3/2} u\right) - F(u)\right] \mathrm{d}x \\ &= \frac{2\left[1 - t^{\frac{3(p-2)}{2}}\right]}{3(p-2)} J(u) + \left\{ \frac{1-t^2}{2} - \frac{2\left[1 - t^{\frac{3(p-2)}{2}}\right]}{3(p-2)} \right\} \|\nabla u\|_2^2 \\ &\quad + \left\{ 1 - t - \frac{2\left[1 - t^{\frac{3(p-2)}{2}}\right]}{3(p-2)} \right\} \frac{1}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3} \left\{ t^{-3} F(t^{3/2} u) - F(u) + \frac{1 - t^{\frac{3(p-2)}{2}}}{p-2} [f(u)u - 2F(u)] \right\} \mathrm{d}x \\ &\geq \frac{2\left[1 - t^{\frac{3(p-2)}{2}}\right]}{3(p-2)} J(u) + \frac{h_1(t)}{6(p-2)} \|\nabla u\|_2^2 \\ &\quad + \frac{h_2(t)}{12(p-2)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y, \quad \forall\ u \in H^1(\mathbb{R}^3), \quad t > 0. \end{aligned} \tag{2.40}$$

This shows that (2.38) holds. Letting $t \to 0$ in (2.38), we derive that (2.39) holds. $\square$

From Lemma 2.5, we have the following corollary.

Corollary 2.6. *Assume that* (F1)-(F3) *hold. Then*

$$I(u) = \max_{t>0} I\left(u^t\right), \qquad \forall\, u \in \mathcal{M}_c. \tag{2.41}$$

Lemma 2.7. *Assume that* (F1)-(F3) *hold. Then for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $u^{t_u} \in \mathcal{M}_c$.*

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta(t) := I\left(u^t\right)$ on $(0, \infty)$. Clearly, by (2.35) and (2.13), we have

$$\begin{aligned} \zeta'(t) = 0 \;\Leftrightarrow\;\; & t\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\ & -\frac{3}{2t^4}\int_{\mathbb{R}^3}\left[f(t^{3/2}u)t^{3/2}u - 2F(t^{3/2}u)\right]\mathrm{d}x = 0 \\ \Leftrightarrow\;\; & \frac{1}{t}J\left(u^t\right) = 0 \;\;\Leftrightarrow\;\; u^t \in \mathcal{M}_c. \end{aligned} \tag{2.42}$$

Note that (2.33) leads to

$$F(t^{3/2}\tau) \le t^{\frac{3p}{2}}F(\tau), \qquad \forall\, t \in (0,1),\ \tau \in \mathbb{R}. \tag{2.43}$$

From (2.35) and (2.43), we derive that

$$I(u^t) \ge \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - t^{\frac{3}{2}(p-2)}\int_{\mathbb{R}^3}F(u)\mathrm{d}x, \qquad \forall\, t \in (0,1), \tag{2.44}$$

which, together with $2 < \frac{3}{2}(p-2) < 6$ implies that $\zeta(t) > 0$ for $t > 0$ small. Moreover, by (F1), (F2) and (2.35), it is easy to verify that $\lim_{t\to 0}\zeta(t) = 0$ and $\zeta(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at $t_u > 0$ so that $\zeta'(t_u) = 0$ and $u^{t_u} \in \mathcal{M}_c$.

Next we claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Otherwise, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exist positive constants $t_1 \ne t_2$ such that $u^{t_1}, u^{t_2} \in \mathcal{M}_c$, i.e. $J\left(u^{t_1}\right) = J\left(u^{t_2}\right) = 0$, then (2.37) and (2.38) lead to

$$\begin{aligned} I\left(u^{t_1}\right) &> I\left(u^{t_2}\right) + \frac{2\left[t_1^{\frac{3(p-2)}{2}} - t_2^{\frac{3(p-2)}{2}}\right]}{3(p-2)t_1^{\frac{3(p-2)}{2}}}J\left(u^{t_1}\right) = I\left(u^{t_2}\right) \\ &> I\left(u^{t_1}\right) + \frac{2\left[t_2^{\frac{3(p-2)}{2}} - t_1^{\frac{3(p-2)}{2}}\right]}{3(p-2)t_2^{\frac{3(p-2)}{2}}}J\left(u^{t_2}\right) = I\left(u^{t_1}\right). \end{aligned} \tag{2.45}$$

This contradiction shows that $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. □

Combining Corollary 2.6 with Lemma 2.7, we have the following lemma.

Lemma 2.8. *Assume that* (F1)-(F3) *hold. Then*

$$\inf_{u\in\mathcal{M}_c} I(u) = m(c) = \inf_{u\in\mathcal{S}_c}\max_{t>0} I\left(u^t\right).$$

Lemma 2.9. *Assume that* (F1)-(F3) *hold. The function* $c \mapsto m(c)$ *is nonincreasing on* $(0,\infty)$.

Proof. To prove this lemma, it is enough to verify that for any $c_1 < c_2$ and $\varepsilon > 0$ arbitrary,

$$m(c_2) \le m(c_1) + \varepsilon \tag{2.46}$$

By the definition of $m(c_1)$, there exists $u \in \mathcal{M}_{c_1}$ such that $I(u) \le m(c_1) + \varepsilon/4$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be such that

$$\eta(x) = \begin{cases} 1, & |x| \le 1, \\ \in [0,1], & 1 \le |x| < 2, \\ 0, & |x| \ge 2. \end{cases}$$

For any small $\delta \in (0,1]$, let

$$u_\delta(x) = \eta(\delta x)\cdot u(x). \tag{2.47}$$

It is easy to check that $u_\delta \to u$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$. Then we have

$$I(u_\delta) \to I(u) \le m(c_1) + \frac{\varepsilon}{4}, \qquad J(u_\delta) \to J(u) = 0. \tag{2.48}$$

From Lemma 2.7, for any $\delta > 0$, there exists $t_\delta > 0$ such that ${u_\delta}^{t_\delta} \in \mathcal{M}_c$. We claim that $\{t_\delta\}$ is bounded. In fact, if $t_\delta \to \infty$ as $\delta \to 0$, since $u_\delta \to u \ne 0$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$, by (F2), we have

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\frac{I(u_\delta^{t_\delta})}{t_\delta^2} = \frac{1}{2}\|\nabla u\|_2^2 + \lim_{\delta\to 0}\left[\frac{1}{4t_\delta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_\delta^2(x)u_\delta^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\frac{F(t_\delta^{\frac{3}{2}}u_\delta)}{t_\delta^5}\mathrm{d}x\right] \\ &= -\infty, \end{aligned}$$

which is impossible. Then we may assume that up to a subsequence, $t_\delta \to \bar{t}$ as $\delta \to 0$, and so $J({u_\delta}^{t_\delta}) \to J(u^{\bar{t}})$. This, jointly with $J(u) = 0$, shows that $\bar{t} = 1$. By (2.37) and (2.38), we have

$$\begin{aligned} I(u_\delta^{t_\delta}) \le I(u_\delta) &- \frac{2(1 - t_\delta^{\frac{3(p-2)}{2}})}{3(p-2)}J(u_\delta) + \frac{h_1(t_\delta)}{6(p-2)}\|\nabla u_\delta\|_2^2 \\ &+ \frac{h_2(t_\delta)}{12(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\delta(x)|^2|u_\delta(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

which, together with (2.48), implies that there exists $\delta_0 \in (0,1)$ small enough such that

$$I\left(u_{\delta_0}^{t_{\delta_0}}\right) \le I(u_{\delta_0}) + \frac{\varepsilon}{8} \le I(u) + \frac{\varepsilon}{4} \le m(c_1) + \frac{\varepsilon}{2}. \tag{2.49}$$

Let $v \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be such that $\mathrm{supp} v \subset B_{2R_{\delta_0}} \setminus B_{R_{\delta_0}}$ with $R_{\delta_0} = 2/\delta_0$. Define

$$v_0 = \frac{c_2 - \|u_{\delta_0}\|_2^2}{\|v\|_2^2}v,$$

for which we have $\|v_0\|_2^2 = c_2 - \|u_{\delta_0}\|_2^2$. For $\lambda \in (0,1)$, we define $w_\lambda = u_{\delta_0} + v_0^\lambda$ with $\|v_0^\lambda\|_2 = \|v_0\|_2$. Noting that

$$\operatorname{dist}\left\{\operatorname{supp} u_{\delta_0}, \operatorname{supp} v_0^\lambda\right\} \geq \frac{2R_{\delta_0}}{\lambda} - R_{\delta_0} = \frac{2}{\delta_0}\left(\frac{2}{\lambda} - 1\right) > 0, \tag{2.50}$$

we have

$$|w_\lambda(x)|^2 = |u_{\delta_0}(x) + v_0^\lambda(x)|^2 = |u_{\delta_0}(x)|^2 + |v_0^\lambda(x)|^2, \tag{2.51}$$

$$\|w_\lambda\|_2^2 = \|u_{\delta_0} + v_0^\lambda\|_2^2 = \|u_{\delta_0}\|_2^2 + \|v_0^\lambda\|_2^2 = \|u_{\delta_0}\|_2^2 + \|v_0\|_2^2, \tag{2.52}$$

$$\|\nabla w_\lambda\|_2^2 = \|\nabla u_{\delta_0} + \nabla v_0^\lambda\|_2^2 = \|\nabla u_{\delta_0}\|_2^2 + \|\nabla v_0^\lambda\|_2^2 = \|\nabla u_{\delta_0}\|_2^2 + \lambda^2\|\nabla v_0\|_2^2, \tag{2.53}$$

$$\begin{aligned}\int_{\mathbb{R}^3} F(w_\lambda)\mathrm{d}x &= \int_{\mathbb{R}^3} F(u_{\delta_0} + v_0^\lambda)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x + \int_{\mathbb{R}^3} F(v_0^\lambda)\mathrm{d}x \\ &= \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x + \lambda^{-3}\int_{\mathbb{R}^3} F(\lambda^{\frac{3}{2}} v_0)\mathrm{d}x\end{aligned} \tag{2.54}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left[|w_\lambda(x)|^2|w_\lambda(y)|^2 - |u_{\delta_0}(x)|^2|u_{\delta_0}(y)|^2 - |v_0^\lambda(x)|^2|v_0^\lambda(y)|^2\right]}{|x-y|}\mathrm{d}x\mathrm{d}y \\ &= 2\int_{\operatorname{supp} u_{\delta_0}}\int_{\operatorname{supp} v_0^\lambda} \frac{|u_{\delta_0}(x)|^2|v_0^\lambda(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y \\ &\leq \frac{\delta_0\lambda}{2-\lambda}\int_{\operatorname{supp} u_{\delta_0}}\int_{\operatorname{supp} v_0^\lambda} |u_{\delta_0}(x)|^2|v_0^\lambda(y)|^2\mathrm{d}x\mathrm{d}y \\ &\leq \lambda\|u_{\delta_0}\|_2^2\|v_0^\lambda\|_2^2 = \lambda\|u_{\delta_0}\|_2^2\|v_0\|_2^2.\end{aligned} \tag{2.55}$$

Then (2.53), (2.54) and (2.55) imply that as $\lambda \to 0$,

$$\|\nabla w_\lambda\|_2^2 \to \|\nabla u_{\delta_0}\|^2, \qquad \int_{\mathbb{R}^3} F(w_\lambda)\mathrm{d}x \to \int_{\mathbb{R}^3} F(u_{\delta_0})\mathrm{d}x \tag{2.56}$$

and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|w_\lambda(x)|^2|w_\lambda(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_{\delta_0}(x)|^2|u_{\delta_0}(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{2.57}$$

which lead to

$$I(w_\lambda) \to I(u_{\delta_0}) \quad \text{and} \quad J(w_\lambda) \to J(u_{\delta_0}). \tag{2.58}$$

By (2.52), we have $w_\lambda \in \mathcal{S}_{c_2}$. From Lemma 2.7, there exists $t_\lambda > 0$ such that ${w_\lambda}^{t_\lambda} \in \mathcal{M}_{c_2}$. Similarly to the previous proof, we deduce that $\{t_\lambda\}$ is bounded. Then we may assume that up to a subsequence, $t_\lambda \to \hat{t}$ as $\lambda \to 0$. Note that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \mathcal{C}_1\|u\|_{12/5}^4, \qquad \forall\, u \in H^1(\mathbb{R}^3). \tag{2.59}$$

Using (2.56) and (2.59), a standard argument shows that as $\lambda \to 0$,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|{w_\lambda}^{t_\lambda}(x)|^2|{w_\lambda}^{t_\lambda}(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|{u_{\delta_0}}^{\hat{t}}(x)|^2|{u_{\delta_0}}^{\hat{t}}(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y \tag{2.60}$$

and

$$\int_{\mathbb{R}^3} F\left({w_\lambda}^{t_\lambda}\right)\mathrm{d}x \to \int_{\mathbb{R}^3} F\left({u_{\delta_0}}^{\hat{t}}\right)\mathrm{d}x. \tag{2.61}$$

From (2.60) and (2.61), there exists $\lambda_0 \in (0,1)$ small enough such $I\left({w_\lambda}^{t_\lambda}\right) \le I\left({u_{\delta_0}}^{\hat{t}}\right) + \varepsilon/2$. Thus, it follows from (2.41) and (2.49) that

$$\begin{aligned} m(c_2) &\le I\left({w_\lambda}^{t_\lambda}\right) \le I\left({u_{\delta_0}}^{\hat{t}}\right) + \frac{\varepsilon}{2} \\ &\le \max_{t>0} I({u_{\delta_0}}^{t}) + \frac{\varepsilon}{2} = I({u_{\delta_0}}^{t_{\delta_0}}) + \frac{\varepsilon}{2} \\ &\le m(c_1) + \varepsilon. \end{aligned} \tag{2.62}$$

The proof is completed. $\square$

Lemma 2.10. *Assume that* (F1)-(F3) *hold. Then* $\gamma(c) = m(c)$ *for any* $c > 0$.

Proof. From (2.11), for any $u \in \mathcal{M}_c$, there exist $t_1 < 0$ small and $t_2 > 1$ large such that $u^{t_1} \in \mathcal{A}_{k_1}$ and $u^{t_2} \in \mathcal{A}^{k_2}$. Letting

$$\bar{g}(\tau) = u^{(1-\tau)t_1+\tau t_2}, \qquad \forall\, \tau \in [0,1],$$

we have $\bar{g} \in \Gamma_c$. By (2.41), we have

$$\gamma(c) \le \max_{\tau\in[0,1]} I(\bar{g}(\tau)) = I(u),$$

and so $\gamma(c) \le \inf_{u\in\mathcal{M}_c} I(u) = m(c)$ for any $c > 0$.

On the other hand, by (2.39), one has

$$J(u) \le \frac{3(p-2)}{2} I(u) - \frac{3p-10}{4}\|\nabla u\|_2^2 - \frac{3p-8}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \qquad \forall\, u \in \mathcal{S}_c,$$

which implies

$$J(g(1)) \le \frac{3(p-2)}{2} I(g(1)) < 0, \qquad \forall\, g \in \Gamma_c.$$

Moreover, it is checked easily that there exists $u_0 \in \mathcal{B}_{k_1}$ such that $J(u_0) > 0$. Hence, any path in Γ_c has to cross $\mathcal{M}_c$. This shows that

$$\max_{\tau\in[0,1]} I(g(\tau)) \ge \inf_{u\in\mathcal{M}_c} I(u) = m(c), \qquad \forall\, g \in \Gamma_c,$$

and so $\gamma(c) \ge m(c)$ for any $c > 0$. Therefore, $\gamma(c) = m(c)$ for any $c > 0$. $\square$

Proposition 2.11 *([7, Proposition 4.1]). Let $\{v_n\} \subset \mathcal{S}_c$ be a bounded (PS) sequence for I restricted to $\mathcal{S}_c$ such that $I(v_n) \to \gamma(c) > 0$. Then there is a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that, up to a subsequence,*

(1) $v_n \rightharpoonup \bar{v}_c$ *in* $H^1(\mathbb{R}^3)$ *and* $\lambda_n \to \bar{\lambda}_c$ *in* $\mathbb{R}$;
(2) $-\Delta v_n - \lambda_n v_n + \left(|x|^{-1} * |v_n|^2\right) v_n - f(v_n) \to 0$ *in* $H^{-1}(\mathbb{R}^3)$;
(3) $-\Delta v_n - \bar{\lambda}_c v_n + \left(|x|^{-1} * |v_n|^2\right) v_n - f(v_n) \to 0$ *in* $H^{-1}(\mathbb{R}^3)$;
(4) $-\Delta \bar{v}_c - \bar{\lambda}_c \bar{v}_c + \left(|x|^{-1} * |\bar{v}_c|^2\right) \bar{v}_c - f(\bar{v}_c) = 0$ *in* $H^{-1}(\mathbb{R}^3)$.

Lemma 2.12. *Assume that* (F1)-(F3) *hold and v is a weak solution of* (1.1). *Then $J(v) = 0$. Furthermore, there exists a constant $c_0 > 0$ independent on $\lambda \in \mathbb{R}$ such that if $\|v\|_2^2 \le c_0$, then $\lambda < 0$.*

Proof. Let v be a weak solution of (1.1), the following Pohozaev-type identity holds

$$\frac{1}{2}\|\nabla v\|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - 3F(v) = \frac{3\lambda}{2}\|v\|_2^2. \tag{2.63}$$

By multiplying (1.1) by v and integrating, we derive the following identity

$$\|\nabla v\|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3} f(v)v\mathrm{d}x = \lambda\|v\|_2^2. \tag{2.64}$$

By multiplying (2.64) by $\frac{3}{2}$ and minus (2.63), we obtain $J(v) = 0$. Using $J(v) = 0$ and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\nabla v\|_2^2 - C(p)\|\nabla v\|_2^{\frac{3(p-2)}{2}}\|v\|_2^{\frac{6-p}{2}} &\le \|\nabla v\|_2^2 - \frac{3}{2}\int_{\mathbb{R}^3}[f(v)v - 2F(v)]\mathrm{d}x \\ &= -\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le 0,\end{aligned}$$

which implies

$$\|\nabla v\|_2^{\frac{10-3p}{2}} \le C(p)\|v\|_2^{\frac{6-p}{2}}. \tag{2.65}$$

By multiplying (2.63) by $\frac{p}{3}$ and minus (2.64), we obtain

$$\begin{aligned}&\frac{p-6}{6}\|\nabla v\|_2^2 + \frac{5p-12}{12}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}[f(v)v - pF(v)]\mathrm{d}x \\ &= \frac{(p-2)\lambda}{2}\|v\|_2^2.\end{aligned} \tag{2.66}$$

The Hardy-Littlewood-Sobolev inequality and the Gagliardo-Nirenberg inequality give

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le C_3\|\nabla v\|_2\|v\|_2^3. \tag{2.67}$$

From (2.34), (2.66) and (2.67), we deduce

$$\lambda\|v\|_2^2 \le \frac{p-6}{3(p-2)}\|\nabla v\|_2^2 + C_4\|\nabla v\|_2\|v\|_2^3,$$

which, together with Young's inequality, leads to

$$\lambda\|v\|_2^2 \le \frac{3(p-6)}{2(p-2)}\|\nabla v\|_2^2 + C_5\|v\|_2^6. \tag{2.68}$$

Noting that (2.65) implies that for any solution v of (1.1) with small L^2-norm, $\|\nabla v\|_2$ must be large, it follows that the left-hand side of (2.68) is negative when $\|\nabla v\|_2$ is sufficiently small. Hence, there exists a constant $c_0 > 0$ independent on $\lambda \in \mathbb{R}$ such that if a solution v of (1.1) satisfies $\|v\|_2^2 \le c_0$, then $\lambda < 0$. This completes the proof. □

Proof of Theorem 1.1. In view of Lemmas 2.3 and 2.10, for each $c \in (0, c_0]$, there exists a sequence $\{v_n\} \subset \mathcal{S}_c$ such that

$$I(v_n) \to m(c) > 0, \quad I|'_{\mathcal{S}_c}(v_n) \to 0 \ \text{ and } \ J(v_n) \to 0. \tag{2.69}$$

By (2.39) and (2.69), we have

$$m(c) + o(1) = I(v_n) - \frac{2}{3(p-2)}J(v_n) \ge \frac{3p-10}{6(p-2)}\|\nabla v_n\|_2^2, \tag{2.70}$$

which, together with $\|v_n\|_2^2 = c$, implies $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Then there exists $v \in H^1(\mathbb{R}^3)$ such that, passing to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 \le s < 6$ and $v_n \to v$ a.e. in $\mathbb{R}^3$. Since $m(c) = \gamma(c) > 0$, by Lions' concentration compactness principle [36, Lemma 1.21] and a standard argument, we deduce that $\{v_n\}$ is non-vanishing, and so there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |v_n|^2 \mathrm{d}x > \delta$. Let $\bar{v}_n(x) = v_n(x + y_n)$. Then we have $\|\bar{v}_n\| = \|v_n\|$ and

$$I(\bar{v}_n) \to m(c), \qquad J(\bar{v}_n) = o(1), \qquad \int_{B_1(0)} |\bar{v}_n|^2 \mathrm{d}x > \delta. \tag{2.71}$$

Therefore, there exists $\bar{v} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \bar{v}_n \rightharpoonup \bar{v}, & \text{in } H^1(\mathbb{R}^3); \\ \bar{v}_n \to \bar{v}, & \text{in } L^s_{\text{loc}}(\mathbb{R}^3), \ \forall\, s \in [1,6); \\ \bar{v}_n \to \bar{v}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{2.72}$$

Let $w_n = \bar{v}_n - \bar{v}$. Then (2.72) and the Brezis-Lieb type Lemma yield

$$\|\bar{v}\|_2^2 := \bar{c} \le c, \qquad \|w_n\|_2^2 := \bar{c}_n \le c \text{ for large } n \in \mathbb{N} \tag{2.73}$$

and

$$I(\bar{v}_n) = I(\bar{v}) + I(w_n) + o(1) \ \text{ and } \ J(\bar{v}_n) = J(\bar{v}) + J(w_n) + o(1). \tag{2.74}$$

Let

$$\begin{aligned}\Psi(u) &:= I(u) - \frac{2}{3(p-2)}J(u)\\ &= \frac{3p-10}{6(p-2)}\|\nabla u\|_2^2 + \frac{3p-8}{12(p-2)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\\ &\quad + \frac{1}{p-2}\int_{\mathbb{R}^3}[f(u)u - pF(u)]\,\mathrm{d}x, \qquad \forall\, u \in H^1(\mathbb{R}^3).\end{aligned} \tag{2.75}$$

Then $\Psi(u) > 0$ for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Moreover, it follows from (2.71), (2.74) and (2.75) that

$$\Psi(w_n) = m(c) - \Psi(\bar{v}) + o(1), \qquad J(w_n) = -J(\bar{v}) + o(1). \tag{2.76}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, from (2.34), (2.75), (2.76), the weak semicontinuity of norm and Fatou's lemma, we then deduce that $\|\nabla\bar{v}_n - \nabla\bar{v}\|_2 \to 0$. Next, we prove that this still holds for $w_n \neq 0$. Let us assume that $w_n \neq 0$. We claim that $J(\bar{v}) \leq 0$. Otherwise, if $J(\bar{v}) > 0$, then (2.76) implies $J(w_n) < 0$ for large n. In view of Lemma 2.7, there exists $t_n > 0$ such that $(w_n)^{t_n} \in \mathcal{M}_{\bar{c}_n}$. Then it follows from (1.2), (2.13), (2.38), (2.75), (2.76), Lemmas 2.9 and 2.10 that

$$\begin{aligned}m(c) - \Psi(\bar{v}) + o(1) \geq \Psi(w_n) &= I(w_n) - \frac{2}{3(p-2)}J(w_n)\\ &\geq I\left((w_n)^{t_n}\right) - \frac{t_n^{\frac{3(p-2)}{2}}}{3(p-2)}J(w_n)\\ &\geq m(\bar{c}_n) - \frac{t_n^{\frac{3(p-2)}{2}}}{3(p-2)}J(w_n)\\ &\geq m(c) + o(1),\end{aligned}$$

which is impossible due to $\Psi(\bar{v}) > 0$. This shows that $J(\bar{v}) \leq 0$. In view of Lemma 2.7, there exists $\bar{t} > 0$ such that $\bar{v}^{\bar{t}} \in \mathcal{M}_{\bar{c}}$. Then it follows from (2.38), (2.75), the weak semicontinuity of norm, Fatou's lemma and Lemma 2.9 that

$$\begin{aligned}m(c) &= \lim_{n\to\infty}\left[I(\bar{v}_n) - \frac{2}{3(p-2)}J(\bar{v}_n)\right] = \lim_{n\to\infty}\Psi(\bar{v}_n)\\ &\geq \Psi(\bar{v}) = I(\bar{v}) - \frac{2}{3(p-2)}J(\bar{v})\\ &\geq I\left(\bar{v}^{\bar{t}}\right) - \frac{\bar{t}^{\frac{3(p-2)}{2}}}{3(p-2)}J(\bar{v}) \geq m(\bar{c}) \geq m(c),\end{aligned}$$

which implies $\|\nabla\bar{v}_n - \nabla\bar{v}\|_2 \to 0$ for $w_n \neq 0$. Finally, we prove that $\|\bar{v}_n - \bar{v}\|_2 \to 0$. Applying Proposition 2.11, there exists $\bar{\lambda}_c \in \mathbb{R}$ such that

$$\langle I'(\bar{v}_n), \bar{v}_n\rangle = \bar{\lambda}_c\|\bar{v}_n\|_2^2 + o(1) \text{ and } \langle I'(\bar{v}), \bar{v}\rangle = \bar{\lambda}_c\|\bar{v}\|_2^2. \tag{2.77}$$

Since $\|\nabla\bar{v}_n - \nabla\bar{v}\|_2 \to 0$, a standard argument shows that

$$\langle I'(\bar{v}_n), \bar{v}_n\rangle = \langle I'(\bar{v}), \bar{v}\rangle + o(1). \tag{2.78}$$

Moreover, Lemma 2.12 leads to $\bar{\lambda}_c < 0$ for $\|\bar{v}\|_2^2 \leq c_0$. Jointly with (2.77) and (2.78), we have $\|\bar{v}_n - \bar{v}\|_2 \to 0$. Hence, for any $c \in (0, c_0]$, (1.1) has a couple of solutions $(\bar{v}_c, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}^-$ such that

$$I(\bar{v}_c) = \inf_{v \in \mathcal{M}_c} I(v) = \inf_{v \in \mathcal{S}_c} \max_{t>0} I(v^t) > 0.$$

This completes the proof. □

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

Lemma 3.1. *Assume that* (F4) *holds. Then*

(i) *for any $c > 0$, $\sigma(c) = \inf_{u \in \mathcal{S}_c} I(u)$ is well defined and $\sigma(c) \le 0$;*
(ii) *$\sigma(c)$ is continuous on $(0, \infty)$;*
(iii) *there exists $\mathcal{C}_1 > 0$ such that $\sigma(c) < 0$ for any $c > \mathcal{C}_1$ if* (F5) *holds.*

Proof. (i) By (F4), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(t)| \le \varepsilon |t|^3 + C_\varepsilon |t|^{q_0}, \qquad \forall\, t \in \mathbb{R}. \tag{3.1}$$

By the Gagliardo-Nirenberg inequality, one has

$$\|u\|_s^s \le C(s) \|\nabla u\|_2^{\frac{3(s-2)}{2}} \|u\|_2^{\frac{6-s}{2}}, \qquad \forall\, s \in (2, 6). \tag{3.2}$$

In view of [21, (2.11)], we have

$$\int_{\mathbb{R}^3} |u|^3 \mathrm{d}x \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y + \frac{1}{16\pi} \|\nabla u\|_2^2, \qquad \forall\, u \in H^1(\mathbb{R}^3). \tag{3.3}$$

Letting $\varepsilon = \frac{1}{4}$ in (3.1), it follows from (1.2), (3.1), (3.2) and (3.3) that

$$\begin{aligned} I(u) \;&\ge\; \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \left(\frac{1}{4}\|u\|_3^3 + C_1 \|u\|_{p_0}^{p_0}\right) \\ &\ge\; \left(\frac{1}{2} - \frac{1}{64\pi}\right)\|\nabla u\|_2^2 - C_2 c^{\frac{6-q_0}{4}} \|\nabla u\|_2^{\frac{3(q_0-2)}{2}}, \qquad \forall\, u \in \mathcal{S}_c,\ c > 0, \end{aligned} \tag{3.4}$$

which, together with $0 < \frac{3(q_0-2)}{2} < 2$, shows that I is bounded from below on $\mathcal{S}_c$ for any $c > 0$, that is $\sigma(c)$ is well defined. Since $u^t \in \mathcal{S}_c$ for all $u \in \mathcal{S}_c$, from (2.35) and (3.1), we deduce that $I(u^t) \to 0$ as $t \to 0$, and so $\sigma(c) \le 0$ for any $c > 0$.

(ii) For any $c > 0$, let $c_n > 0$ and $c_n \to c$. For every $n \in \mathbb{N}$, let $u_n \in \mathcal{S}_{c_n}$ such that $I(u_n) < \sigma(c_n) + \frac{1}{n} \le \frac{1}{n}$. Then (3.4) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, moreover, we have

$$\sigma(c) \le I\left(\sqrt{\frac{c}{c_n}} u_n\right) = I(u_n) + o(1) \le \sigma(c_n) + o(1). \tag{3.5}$$

On the other hand, given a minimizing sequence $\{v_n\} \subset \mathcal{S}_c$ for I, we have

$$\sigma(c_n) \le I\left(\sqrt{\frac{c_n}{c}} v_n\right) \le I(v_n) + o(1) = \sigma(c) + o(1),$$

which, jointly with (3.5), gives $\lim_{n\to\infty} \sigma(c_n) = \sigma(c)$.

(iii) By (F4) and (F5), there exist $\delta_0 > 0$ and $\varrho_0 > 0$ such that

$$|F(t)| \geq \delta_0 |t|^{p_0}, \quad \forall\ |t| \geq \varrho_0. \tag{3.6}$$

Set

$$u_t(x) := t^2 u(tx), \quad \forall\ u \in H^1(\mathbb{R}^3),\ t > 0. \tag{3.7}$$

For any $c > 0$, we choose a function $w \in \mathcal{C}_1^\infty(\mathbb{R}^3, [\varrho_0, +\infty))$ satisfying $\|w\|_2^2 = c$. Then it follows from (3.6) and (3.7) that

$$\begin{aligned} I(w_t) &= \frac{t^3}{2}\|\nabla w\|_2^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w^2(x)w^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{t^3}\int_{\mathbb{R}^3} F(t^2 w)\mathrm{d}x \\ &\leq \frac{t^3}{2}\|\nabla w\|_2^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w^2(x)w^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \delta_0 t^{2p_0-3}\|w\|_{p_0}^{p_0}, \quad \forall\ t > 1, \end{aligned} \tag{3.8}$$

which, together with $2p_0 - 3 > 3$, implies that $I(w_t) \to -\infty$ as $t \to +\infty$. Noting that $\|w_t\|_2^2 = t\|w\|_2^2$ for $t > 0$, there exists $\mathcal{C}_1 > 0$ such that $\sigma(c) < 0$ for any $c > \mathcal{C}_1$. $\square$

Noting that Lemma 3.1 implies

$$\{c \in (0, +\infty), \sigma(c) < 0\} \neq \emptyset, \tag{3.9}$$

we have

$$c^* = \inf\{c \in (0, +\infty), \sigma(c) < 0\}$$

is well-defined.

Lemma 3.2. *Assume that* (F4)-(F6) *hold. Then for any* $c > 0$*,*

$$\sigma(tc) \leq t^3\sigma(c), \quad \forall\ t > 1. \tag{3.10}$$

Proof. Letting $\{u_n\} \subset \mathcal{S}_c$ be such that $I(u_n) \to \sigma(c)$ for $c > c^*$, it follows from (3.4) and Lemma 3.1 that $\sigma(c) < 0$, and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. By (F6), one has

$$\frac{F(t)}{t^3} \text{ is nondecreasing on } (-\infty, 0) \text{ and } (0, \infty). \tag{3.11}$$

By (1.2) and (3.11), one has

$$\begin{aligned} I(u_t) &= \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - t^{-3}\int_{\mathbb{R}^3} F(t^2 u)\mathrm{d}x \\ &\leq \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t^3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - t^3\int_{\mathbb{R}^3} F(u)\mathrm{d}x \\ &= t^3 I(u), \quad \forall\ u \in \mathcal{S}_c,\ c > 0,\ t > 1, \end{aligned} \tag{3.12}$$

where the definition of u_t is given in (3.7). Since $\|(u_n)_t\|_2^2 = t\|u_n\|_2^2 = tc$ for all $t > 0$, it follows from (3.12) and Lemma 3.1 (i) that

$$\sigma(tc) \leq I((u_n)_t) \leq t^3 I(u_n) = t^3\sigma(c) + o(1), \qquad \forall\, t > 1,\ c > 0$$

which implies that (3.10) holds for $c > 0$. $\square$

Lemma 3.3. *Assume that* (F4)-(F6) *hold. Then $\sigma(c)$ has a minimizer for any $c \geq c^*$ and $\sigma(c^*) = 0$.*

Proof. We first prove that $\sigma(c) < 0$ has a minimizer for any $c > c^*$ by the definition of c^*. Let $\{u_n\} \subset \mathcal{S}_c$ be such that $I(u_n) \to \sigma(c)$ for any $c > c^*$. Then (3.4) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. We claim that

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \mathrm{d}x > 0. \tag{3.13}$$

In fact, if $\delta = 0$, then by Lions' concentration compactness principle [36, Lemma 1.21], one has $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$, and so (3.1) implies that $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to 0$. Then by (1.2) and (2.59), we have

$$0 > \sigma(c) = \lim_{n\to\infty} I(u_n) = \frac{1}{2}\lim_{n\to\infty} \|\nabla u_n\|_2^2 \geq 0.$$

This contradiction shows that $\delta > 0$, and there exists $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2}. \tag{3.14}$$

Letting $\bar{u}_n(x) = u_n(x + y_n)$, we have

$$\bar{u}_n \in \mathcal{S}_c, \qquad I(\bar{u}_n) \to \sigma(c). \tag{3.15}$$

In view of (3.14), we may assume that there exists $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \bar{u}_n \rightharpoonup \bar{u}, & \text{in } H^1(\mathbb{R}^3); \\ \bar{u}_n \to \bar{u}, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ \forall\, s \in [1, 6); \\ \bar{u}_n \to \bar{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.16}$$

By Lemma 3.2, we have

$$\begin{aligned} \sigma(c) &= \lim_{n\to\infty} I(\bar{u}_n) = I(\bar{u}) + \lim_{n\to\infty} I(\bar{u}_n - \bar{u}) \\ &\geq \sigma(\|\bar{u}\|_2^2) + \lim_{n\to\infty} \sigma(\|\bar{u}_n - \bar{u}\|_2^2) \\ &= \sigma(\|\bar{u}\|_2^2) + \sigma(c - \|\bar{u}\|_2^2). \end{aligned} \tag{3.17}$$

If $\|\bar{u}\|_2^2 < c$, then (3.17) and Lemma 3.2 imply

$$\begin{aligned} \sigma(c) &\geq \sigma\left(\frac{c}{\|\bar{u}\|_2^2}\|\bar{u}\|_2^2\right)\left(\frac{\|\bar{u}\|_2^2}{c}\right)^3 + \sigma\left(\frac{c}{c - \|\bar{u}\|_2^2}(c - \|\bar{u}\|_2^2)\right)\left(\frac{c - \|\bar{u}\|_2^2}{c}\right)^3 \\ &= \sigma(c)\frac{\|\bar{u}\|_2^6 + \left(c - \|\bar{u}\|_2^2\right)^3}{c^3} > \sigma(c), \end{aligned} \tag{3.18}$$

which is impossible. This shows $\|\bar{u}\|_2^2 = c = \|u_n\|_2^2$, i.e., $\bar{u} \in \mathcal{S}_c$, and so $u_n \to \bar{u}$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$. Thus, it follows from (2.59) and the weak semicontinuity of norm that

$$\sigma(c) = \lim_{n\to\infty} I(\bar{u}_n) \ge I(\bar{u}) \ge \sigma(c),$$

which leads to $\bar{u} \in \mathcal{S}_c$ and $I(\bar{u}) = \sigma(c)$ for any $c > c^*$. Hence, $\sigma(c)$ has a minimizer for any $c > c^*$.

We next prove that $\sigma(c^*)$ is also attained. Let $c_n = c^* + \frac{1}{n}$. By Lemma 3.1 (iii), one has $\sigma(c_n) < 0$ for every $n \in \mathbb{N}$. In view of the previous proof, there exists $\{u_n\} \subset \mathcal{S}_{c_n}$ such that

$$I(u_n) = \sigma(c_n) < 0 \quad \text{for every } n \in \mathbb{N}. \tag{3.19}$$

By the definition of c^* and Lemma 3.1 (ii), we have $I(u_n) = \sigma(c_n) \to \sigma(c^*) = 0$. Then (3.4) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. We claim that (3.13) holds. Otherwise, if δ=0, then by Lions' concentration compactness principle [36, Lemma 1.21], one has $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Then we derive easily $\|\nabla u_n\|_2 \to 0$ due to $I(u_n) \to 0$. Similarly to (3.4), we have

$$I(u_n) \ge \left(\frac{1}{2} - \frac{1}{64\pi}\right)\|\nabla u_n\|_2^2 - C_4(c^*)^{\frac{6-q_0}{4}}\|\nabla u_n\|_2^{\frac{3(q_0-2)}{2}},$$

which, together with $0 < \frac{3(q_0-2)}{2} < 2$, implies that $I(u_n) \ge 0$ for $n \in \mathbb{N}$ sufficiently large. This contradicts (3.19), and so (3.13) holds. Then there exists $\{y_n\} \subset \mathbb{R}^3$ such that (3.14) holds. Letting $\bar{u}_n(x) = u_n(x+y_n)$, we have

$$\bar{u}_n \in \mathcal{S}_{c^*}, \qquad I(\bar{u}_n) \to \sigma(c^*) = 0, \tag{3.20}$$

and there exists $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence, (3.16) holds. Since $0 < \|\bar{u}\|_2^2 \le c^*$, we deduce from Lemma 3.1 (ii) that

$$\begin{aligned} 0 &= \sigma(c^*) = \lim_{n\to\infty} I(\bar{u}_n) = I(\bar{u}) + \lim_{n\to\infty} I(\bar{u}_n - \bar{u}) \\ &\ge \sigma(\|\bar{u}\|_2^2) + \lim_{n\to\infty} \sigma(\|\bar{u}_n - \bar{u}\|_2^2) \\ &= \sigma(\|\bar{u}\|_2^2) + \sigma(c^* - \|\bar{u}\|_2^2) = 0, \end{aligned} \tag{3.21}$$

which leads to $I(\bar{u}) = \sigma(\|\bar{u}\|_2^2) = 0$. Let $\bar{t} = \frac{c^*}{\|\bar{u}\|_2^2}$. Then $\bar{t} \ge 1$ and $\|\bar{u}_{\bar{t}}\|_2^2 = \bar{t}\|\bar{u}\|_2^2 = c^*$ by the scaling (3.7). Jointly with (3.4), we have

$$0 = \sigma(c^*) \le I(\bar{u}_{\bar{t}}) \le \bar{t}^3 I(\bar{u}) = 0,$$

which implies $\bar{u}_{\bar{t}} \in \mathcal{S}_{c^*}$ and $I(\bar{u}_{\bar{t}}) = \sigma(c^*) = 0$. Hence, $\sigma(c^*)$ has a minimizer. The proof is now complete. □

Proof of Theorem 1.2. Note that if u_c is a critical point of $I|_{\mathcal{S}_c}$, then there exists $\lambda_c \in \mathbb{R}$ such that $I'(u_c) - \lambda_c u_c = 0$. Hence, Theorem 1.2 follows directly from Lemma 3.3. □

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