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Functional inequalities for the ratio of complete p-elliptic integrals

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In this paper we show monotonicity and convexity properties of a function involving the ratio of complete p-elliptic integrals which reduces to the modulus of the Grötzsch ring. As applications we obtain several sharp functional inequalities.

1. Introduction

Let $1 < p < \infty$. Define

$$\operatorname{arcsin}_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}} dt, \quad 0 \le x \le 1,$$

and

$$\frac{\pi_p}{2} = \operatorname{arcsin}_p(1) \equiv \int_0^1 \frac{1}{(1-t^p)^{1/p}} dt = \frac{\pi/p}{\sin(\pi/p)} = \frac{1}{p} B(1/p, 1-1/p),$$

where B is the beta function. We define the function $\sin_p$ on $[0, \pi_p/2]$ as the inverse of arcsin_p and extend it on $(-\infty, \infty)$ as the classical sine function. The function $\sin_p$ is called the *generalized sine function* (see [12,13]).

The complete p-elliptic integrals of the first and second kind are respectively defined as follows: for $p \in (1, \infty)$ and $r \in [0, 1)$,

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$$\mathcal{K}_p(r) = \int_0^{\pi_p/2} \frac{d\theta}{(1 - r^p \sin_p^p \theta)^{1-1/p}} = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-r^p t^p)^{1-1/p}} \tag{1.1}$$

and

$$\mathcal{E}_p(r) = \int_0^{\pi_p/2} (1 - r^p \sin_p^p \theta)^{1/p} d\theta = \int_0^1 \left(\frac{1-r^p t^p}{1-t^p}\right)^{1/p} dt. \tag{1.2}$$

For real numbers $a, b,$ and c with $c \neq 0, -1, -2 \cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad |x| < 1.$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the *shifted factorial function*

$$(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{N} \equiv \{k : k \text{ is a positive integer}\}$.

The complete p-elliptic integrals can be represented by the Gaussian hypergeometric function [17, Proposition 2.8]: for $r \in [0,1)$,

$$\mathcal{K}_p(r) = \frac{\pi_p}{2} F(1/p, 1-1/p; 1; r^p) \tag{1.3}$$

and

$$\mathcal{E}_p(r) = \frac{\pi_p}{2} F(1/p, -1/p; 1; r^p). \tag{1.4}$$

As is traditional, we always use the notation $r' = (1-r^p)^{1/p}$ for $r \in [0,1]$. The complementary integrals ${\mathcal{K}_p}'(r)$ and ${\mathcal{E}_p}'(r)$ are defined by ${\mathcal{K}_p}'(r) = \mathcal{K}_p(r')$ and ${\mathcal{E}_p}'(r) = \mathcal{E}_p(r')$. Then we have the following beautiful Legendre relation [17, Theorem 1.1]:

$$\mathcal{K}_p(r){\mathcal{E}_p}'(r) + {\mathcal{K}_p}'(r)\mathcal{E}_p(r) - \mathcal{K}_p(r){\mathcal{K}_p}'(r) = \frac{\pi_p}{2}.$$

We define two related functions m_p and μ_p as follows: for $0 < r < 1$,

$$m_p(r) = \frac{2}{\pi_p} {r'}^p \mathcal{K}_p(r){\mathcal{K}_p}'(r), \tag{1.5}$$

$$\mu_p(r) = \frac{\pi_p}{2} \frac{{\mathcal{K}_p}'(r)}{\mathcal{K}_p(r)}. \tag{1.6}$$

For $p = 2$, these functions reduce to well-known special cases. The function $\mu(r) = \mu_2(r)$ is the modulus of the Grötzsch ring domain in the plane, which has numerous applications in the conformal invariants and the theory of quasiconformal mappings [3,14]. The function $\mu(r)$ also appears in the classical modular equations [6,7]. Many noteworthy monotonicity and convexity properties of functions defined in terms of the modulus of the Grötzsch ring are presented in the monograph [3]. Applications of these results lead to various sharp functional inequalities for the function μ. These sharp inequalities of the functions $m_2(r)$ and $\mu_2(r)$ can be used to deduce very good estimates of quasiconformal distortion functions.

Note that there are several different forms of the generalized complete elliptic integrals and generalized Grötzsch function. Many well-known properties and functional inequalities have been extended to these generalized functions, see [2,5,7,10,11,16–18,24–26].

Recently, several authors investigated the properties of convexity and concavity of special functions from which many interesting and elegant functional inequalities have been derived, see [4,8,9,19–22].

In this paper, we show the monotonicity and convexity properties of a function involving the ratio of complete p-elliptic integrals. As applications we obtain several sharp functional inequalities which extend the results proved by Alzer and Richards [1].

2. Monotonicity and convexity

The functions $\mathcal{K}_p$ and $\mathcal{E}_p$ satisfy a system of differential equations [17, Proposition 2.1]:

$$\frac{d\mathcal{K}_p}{dr} = \frac{\mathcal{E}_p - r'^p\mathcal{K}_p}{rr'^p}, \quad \frac{d\mathcal{E}_p}{dr} = \frac{\mathcal{E}_p - \mathcal{K}_p}{r}. \tag{2.1}$$

From (2.1), it is easy to get the following derivative formula:

$$\frac{d}{dr}(\mathcal{E}_p - r'^p\mathcal{K}_p) = (p-1)r^{p-1}\mathcal{K}_p, \quad \frac{d}{dr}(\mathcal{K}_p - \mathcal{E}_p) = \frac{r^{p-1}\mathcal{E}_p}{r'^p}. \tag{2.2}$$

We also have the derivative formula [26, Lemma 3.12]

$$\frac{d\,\mu_p(r)}{d\,r} = -\frac{\pi_p^2}{4rr'^p\mathcal{K}_p(r)^2}. \tag{2.3}$$

Lemma 2.4. *[26, Theorem 3.16(1)(2)] Let $p > 1$. Then the function*

(1) $f_1(r) = m_p(r) + \log r$ is decreasing and concave from $(0, 1)$ onto $(0, R(1/p)/p)$.
(2) $f_2(r) = m_p(r)/\log(1/r)$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.

Theorem 2.5. *For given $r \in (0, 1)$, the function $G_r(\alpha) = \mu_p(r^\alpha)/\alpha$ is strictly decreasing and log-convex from $(0, \infty)$ onto $(-\log r, +\infty)$.*

Proof. Let $t = r^\alpha$. By differentiation, we get

$$\begin{aligned}\alpha^2\frac{d}{d\alpha}G_r(\alpha) &= \frac{d\mu_p(t)}{dt}\frac{dt}{d\alpha}\alpha - \mu_p(t)\\ &= -\frac{\pi_p^2}{4tt'^p\mathcal{K}_p(t)^2}t\log t - \frac{\pi_p}{2}\frac{\mathcal{K}'_p(t)}{\mathcal{K}_p(t)}\\ &= -\frac{\pi_p^2}{4t'^p\mathcal{K}_p(t)^2}(\log t + m_p(t))\end{aligned}$$

which is negative by Lemma 2.4(1). It follows that the function $G_r(\alpha)$ is strictly decreasing on $(0, \infty)$.

Using l'Hopital rule and (2.1), we have

$$\lim_{t\to 1-}\frac{\mathcal{K}_p(t)}{-\log(1-t)} = \lim_{t\to 1-}\frac{\mathcal{E}_p - t'^p\mathcal{K}_p}{t}\frac{1-t}{1-t^p} = \frac{1}{p}$$

and

$$\lim_{t\to 1-}\frac{\mathcal{K}_p(t)}{\log(1/t')}=\lim_{t\to 1-}\frac{\mathcal{K}_p(t)}{-\log(1-t)}\lim_{t\to 1-}\frac{p\log(1-t)}{\log(1-t^p)}=1.$$

We write the function $G_r(\alpha)$ in two representations as follows:

$$G_r(\alpha)=\frac{\pi_p}{2\alpha}\frac{\mathcal{K}_p'(t)}{-\log(1-t)}\frac{-\log(1-t)}{\mathcal{K}_p(t)}$$

and

$$G_r(\alpha)=\frac{\pi_p}{2}(-\log r)\frac{\mathcal{K}_p'(t)}{-\log t}\frac{1}{\mathcal{K}_p(t)}.$$

Then we get the limiting values

$$\lim_{\alpha\to 0^+}G_r(\alpha)=\lim_{\alpha\to 0+}\frac{\pi_p}{2\alpha}\lim_{t\to 1-}\frac{\mathcal{K}_p(t)}{-\log(1-t')}\lim_{t\to 1-}\frac{-\log(1-t)}{\mathcal{K}_p(t)}=+\infty$$

and

$$\lim_{\alpha\to +\infty}G_r(\alpha)=\frac{\pi_p}{2}(-\log r)\lim_{t\to 1-}\frac{\mathcal{K}_p(t)}{-\log t'}\lim_{t\to 0+}\frac{1}{\mathcal{K}_p(t)}=-\log r.$$

For the log-convexity,

$$\begin{aligned}
\frac{d}{d\alpha}\log G_r(\alpha)&=\frac{1}{G_r(\alpha)}\frac{dG_r(\alpha)}{d\alpha}\\
&=\frac{\alpha}{\mu_p(t)}\frac{1}{\alpha}\frac{-\pi_p^2}{4t'^p\mathcal{K}_p(t)^2}(\log t+m_p(t))\\
&=\frac{1}{\alpha}\frac{-\pi_p^2}{4t'^p\mu_p(t)\mathcal{K}_p(t)^2}(\log t+m_p(t))\\
&=\frac{\log r}{\log t}\frac{-1}{m_p(t)}(\log t+m_p(t))\\
&=\frac{1}{m_p(t)}\left(\frac{m_p(t)}{\log(1/t)}-1\right)\log r
\end{aligned}$$

which is strictly decreasing in t by Lemma 2.4(2) and the monotonicity of $m_p(t)$, and hence strictly increasing in α. Therefore, we get the log-convexity of the function G_r. □

Theorem 2.6. *Given $r\in(0,1)$, the function $f(\alpha)=\frac{1}{G_r(\alpha)}$ is strictly concave on $(0,+\infty)$.*

Proof. Let $t=r^\alpha$. By differentiation,

$$\begin{aligned}
\frac{d}{d\alpha}\frac{1}{G_r(\alpha)}&=\frac{d}{d\alpha}\left(\frac{\alpha}{\mu_p(t)}\right)\\
&=\frac{\mu_p(t)-\frac{d\mu_p(t)}{dt}\frac{t}{\alpha}\alpha}{\mu_p(t)^2}\\
&=\frac{1}{\mu_p(t)}\left(1-\frac{(t\log t)\mu_p'(t)}{\mu_p(t)}\right)
\end{aligned}$$

$$= \frac{1}{\mu_p(t)} \left(1 - \frac{-\pi_p^2 (t \log t) / \left(4tt'^p \mathcal{K}_p(t)^2 \right)}{\pi_p \mathcal{K}_p(t') / (2\mathcal{K}_p(t))} \right)$$
$$= \frac{1}{\mu_p(t)} \left(1 - \frac{\log(1/t)}{m_p(t)} \right)$$

which is positive and strictly increasing in t, and hence decreasing in α. Therefore, the function $f(\alpha)$ is strictly concave on $(0, +\infty)$. □

3. Inequalities

The following lemmas provide functional inequalities for convex functions, which are from [15, p. 22] and [23], respectively.

Lemma 3.1.

(1) If $f : \mathbb{R} \to \mathbb{R}$ is strictly convex, then for $xy > 0$,

$$f(x) + f(y) < f(0) + f(x + y).$$

(2) Let f be a positive function, monotone or convex, in some interval $I \subset \mathbb{R}$. Then, for $x, y, z \in I$,

$$(x - y)(x - z)f(x) + (y - x)(y - z)f(y) + (z - x)(z - y)f(z) \geq 0,$$

with equality if and only if $x = y = z$.

Theorem 3.2. *For $0 < \alpha < \beta$ and $r \in (0, 1)$,*

$$\frac{\alpha}{\beta} < \frac{\mu_p(r^\alpha)}{\mu_p(r^\beta)} < 1.$$

Proof. Since $\mu_p(r)$ is strictly decreasing in r, we have $\mu_p(r^\alpha) < \mu_p(r^\beta)$ for $0 < \alpha < \beta$ and hence $\mu_p(r^\alpha)/\mu_p(r^\beta) < 1$. By Theorem 2.5, it follows from the monotonicity of the function $G_r(\alpha)$ that $\mu_p(r^\alpha)/\alpha > \mu_p(r^\beta)/\beta$ and then

$$\frac{\mu_p(r^\alpha)}{\mu_p(r^\beta)} > \frac{\alpha}{\beta}.$$

For the sharpness of the inequalities,

$$\lim_{r \to 0^+} \frac{\mu_p(r^\alpha)}{\mu_p(r^\beta)} = \lim_{r \to 0^+} \frac{\mathcal{K}_p(r^\beta)}{\mathcal{K}_p(r^\alpha)} \frac{\mathcal{K}_p((1 - r^{\alpha p})^{\frac{1}{p}})}{\mathcal{K}_p((1 - r^{\beta p})^{\frac{1}{p}})}$$
$$= \lim_{r \to 0^+} \frac{\mathcal{K}_p((1 - r^{\alpha p})^{\frac{1}{p}})}{\mathcal{K}_p((1 - r^{\beta p})^{\frac{1}{p}})}$$
$$= \lim_{r \to 0^+} \frac{\mathcal{K}_p((1 - r^{\alpha p})^{\frac{1}{p}})}{-\log(r^\alpha)} \frac{-\log(r^\beta)}{\mathcal{K}_p((1 - r^{\beta p})^{\frac{1}{p}})} \frac{\alpha}{\beta}$$
$$= \frac{\alpha}{\beta}.$$

Similarly, we have

$$\begin{aligned}
\lim_{r\to 1^-}\frac{\mu_p(r^\alpha)}{\mu_p(r^\beta)} &= \lim_{r\to 1^-}\frac{k_p(r^\beta)}{k_p(r^\alpha)}\frac{k_p((1-r^{\alpha p})^{\frac{1}{p}})}{k_p((1-r^{\beta p})^{\frac{1}{p}})}\\
&= \lim_{r\to 1^-}\frac{k_p(r^\beta)}{k_p(r^\alpha)}\\
&= \lim_{r\to 1^-}\frac{k_p(r^\beta)}{-\log(1-r^\beta)}\frac{-\log(1-r^\alpha)}{k_p(r^\alpha)}\frac{\log(1-r^\beta)}{\log(1-r^\alpha)}\\
&= \lim_{r\to 1^-}\frac{1}{p}p\frac{\log(1-r^\beta)}{\log(1-r^\alpha)} = 1. \quad \square
\end{aligned}$$

Theorem 3.3. *Let $r\in\left(0,\frac{1}{e}\right]$. For $\alpha,\beta\in(0,1)$ or $\alpha,\beta>1$,*

$$\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{\frac{1}{\sqrt{\alpha}}}\left(\frac{\mu_p(r^\beta)}{\beta}\right)^{\frac{1}{\sqrt{\beta}}} < \mu_p(r)\left(\frac{\mu_p(r^{\alpha\beta})}{\alpha\beta}\right)^{\frac{1}{\sqrt{\alpha\beta}}}.$$

Proof. Let $w_r(\alpha)=\log G_r(\alpha)$ and $\phi_r(t)=e^{\frac{t}{2}}w_r(e^{-t})$. Then

$$\phi_r'(t)=\frac{1}{2}e^{\frac{t}{2}}w_r(e^{-t})-e^{-\frac{t}{2}}w_r'(e^{-t})$$

and

$$\phi_r''(t)=\frac{1}{4}e^{\frac{t}{2}}w_r(e^{-t})+e^{-\frac{3t}{2}}w_r''(e^{-t}).$$

By Theorem 2.5, $G_r(e^{-t})>\log\frac{1}{r}$ and $w_r(e^{-t})>\log(\log\frac{1}{r})$. Since $r\in(0,\frac{1}{e}]$ and $\frac{1}{r}\in(e,+\infty)$, $w_r(e^{-t})>\log(\log\frac{1}{r})>\log\log e=0$. The log-convexity of $G_r(\alpha)$ implies that $w_r''(e^{-t})\geq 0$ and $\phi_r''(t)>0$. Hence $\phi_r(t)=e^{\frac{t}{2}}w_r(e^{-t})$ is strictly convex in $t\in R$. By Lemma 3.1(1), for $s,t>0$ or $s,t<0$, $\phi_r(s)+\phi_r(t)<\phi_r(0)+\phi_r(s+t)$. Let $s=-\log\alpha$ and $t=-\log\beta$. Then $e^{-s}=\alpha$, $\quad e^{-\frac{\log\alpha}{2}}=\alpha^{-\frac{1}{2}}$,

$$\phi_r(s)=\log\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{\frac{1}{\sqrt{\alpha}}}\quad\text{and}\quad\phi_r(0)=\log\mu_p(r),$$

and hence

$$\log\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{\frac{1}{\sqrt{\alpha}}}+\log\left(\frac{\mu_p(r^\beta)}{\beta}\right)^{\frac{1}{\sqrt{\beta}}}<\log\mu_p(r)+\log\left(\frac{\mu_p(r^{\alpha\beta})}{\alpha\beta}\right)^{\frac{1}{\sqrt{\alpha\beta}}}$$

and

$$\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{\frac{1}{\sqrt{\alpha}}}\left(\frac{\mu_p(r^\beta)}{\beta}\right)^{\frac{1}{\sqrt{\beta}}} < \mu_p(r)\left(\frac{\mu_p(r^{\alpha\beta})}{\alpha\beta}\right)^{\frac{1}{\sqrt{\alpha\beta}}}.$$

Let $\beta=\alpha$. Then

$$\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{\frac{2}{\sqrt{\alpha}}}<\mu_p(r)\left(\frac{\mu_p(r^{\alpha^2})}{\alpha^2}\right)^{\frac{1}{\alpha}}$$

Let $\alpha\to 1$, it is easy to see that $\mu_p(r)$ can not be replaced by a constant. $\square$

Theorem 3.4. *Let $r \in \left(0, \frac{1}{e}\right]$. For $\alpha, \beta, \gamma > 0$,*

$$1 \leq \left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{(\alpha-\beta)(\alpha-\gamma)} \left(\frac{\mu_p(r^\beta)}{\beta}\right)^{(\beta-\alpha)(\beta-\gamma)} \left(\frac{\mu_p(r^\gamma)}{\gamma}\right)^{(\gamma-\alpha)(\gamma-\beta)}$$

with equality if and only if $\alpha = \beta = \gamma$.

Proof. Let $r \in \left(0, \frac{1}{e}\right]$. The function $\alpha \to \log G_r(\alpha)$ is positive and strictly decreasing on $(0, +\infty)$. By Lemma 3.1(2),

$$\begin{aligned}
0 &\leq (\alpha-\beta)(\alpha-\gamma)\log\frac{\mu_p(r^\alpha)}{\alpha} + (\beta-\alpha)(\beta-\gamma)\log\frac{\mu_p(r^\beta)}{\beta} \\
&\quad + (\gamma-\alpha)(\gamma-\beta)\log\frac{\mu_p(r^\gamma)}{\gamma} \\
&= \log\left(\left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{(\alpha-\beta)(\alpha-\gamma)} \left(\frac{\mu_p(r^\beta)}{\beta}\right)^{(\beta-\alpha)(\beta-\gamma)} \left(\frac{\mu_p(r^\gamma)}{\gamma}\right)^{(\gamma-\alpha)(\gamma-\beta)}\right).
\end{aligned}$$

Hence

$$1 \leq \left(\frac{\mu_p(r^\alpha)}{\alpha}\right)^{(\alpha-\beta)(\alpha-\gamma)} \left(\frac{\mu_p(r^\beta)}{\beta}\right)^{(\beta-\alpha)(\beta-\gamma)} \left(\frac{\mu_p(r^\gamma)}{\gamma}\right)^{(\gamma-\alpha)(\gamma-\beta)}. \quad \square$$

Theorem 3.5. *For $x, y \in (0, 1)$,*

$$\mu_p(\sqrt{xy}) \leq \frac{-\log\sqrt{xy}}{\sqrt{\log x \log y}}\sqrt{\mu_p(x)\mu_p(y)}$$

with equality if and only if $x = y$.

Proof. By the log-convexity of $G_r(\alpha)$, for $\alpha, \beta > 0$,

$$\log G_r\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{2}(\log G_r(\alpha) + \log G_r(\beta)) \tag{3.6}$$

with equality if and only if $\alpha = \beta$. Let $\alpha = \frac{\log x}{\log r}$ and $\beta = \frac{\log y}{\log r}$. Then $x = r^\alpha$, $y = r^\beta$ and $\sqrt{xy} = r^{\frac{\alpha+\beta}{2}}$.

$$G_r(\alpha) = \frac{\log r}{\log x}\mu_p(x), \quad G_r(\beta) = \frac{\log r}{\log y}\mu_p(y), \quad G_r\left(\frac{\alpha+\beta}{2}\right) = \frac{2\log r}{\log(xy)}\mu_p(\sqrt{xy}).$$

Hence (3.6) yields that

$$\log\left(\frac{2\log r}{\log(xy)}\mu_p(\sqrt{xy})\right) \leq \frac{1}{2}\left(\log\left(\frac{\log r}{\log x}\mu_p(x)\right) + \log\left(\frac{\log r}{\log y}\mu_p(y)\right)\right).$$

Therefore,

$$\mu_p(\sqrt{xy}) \leq \frac{-\log\sqrt{xy}}{\sqrt{\log x \log y}}\sqrt{\mu_p(x)\mu_p(y)}. \quad \square$$

Theorem 3.7. *For $x, y \in (0,1)$,*

$$\frac{\log(xy)}{\mu_p(\sqrt{xy})} \le \frac{\log x}{\mu_p(x)} + \frac{\log y}{\mu_p(y)}$$

with the equality if and only if $x = y$.

Proof. By Theorem $r \in (0,1), \alpha \to \frac{1}{G_r(\alpha)}$ is strictly concave on $(0, +\infty)$ which implies that

$$\frac{1}{2}\left(\frac{1}{G_r(\alpha)} + \frac{1}{G_r(\beta)}\right) \le \frac{1}{G_r(\frac{\alpha+\beta}{2})} \tag{3.8}$$

with equality if and only if $\alpha = \beta$. Setting $\alpha = \frac{\log x}{\log r}$, $\beta = \frac{\log y}{\log r}$, we have $x = r^\alpha$, $y = r^\beta$, $\frac{\alpha+\beta}{2} = \frac{\log\sqrt{xy}}{\log r}$, and $\sqrt{xy} = r^{\frac{\alpha+\beta}{2}}$. Then

$$\frac{1}{G_r(\alpha)} = \frac{\log x}{\mu_p(x)\log r}, \quad \frac{1}{G_r(\beta)} = \frac{\log y}{\mu_p(y)\log r}, \quad \frac{1}{G_r(\frac{\alpha+\beta}{2})} = \frac{\log(\sqrt{xy})}{\mu_p(\sqrt{xy})\log r}.$$

Hence (3.8) yields

$$\frac{1}{2}\left(\frac{\log x}{\mu_p(x)\log r} + \frac{\log y}{\mu_p(y)\log r}\right) \le \frac{\log(\sqrt{xy})}{\mu_p(\sqrt{xy})\log r}.$$

Since $\log r < 0$, we have

$$\frac{\log x}{\mu_p(x)} + \frac{\log y}{\mu_p(y)} \ge \frac{2\log\sqrt{xy}}{\mu_p(\sqrt{xy})} = \frac{\log(xy)}{\mu_p(\sqrt{xy})}. \quad \square$$

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