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Circle embeddings with restrictions on Fourier coefficients

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ABSTRACT

This paper continues the investigation of the relation between the geometry of a circle embedding and the values of its Fourier coefficients. First, we answer a question of Kovalev and Yang concerning the support of the Fourier transform of a starlike embedding. An important special case of circle embeddings are homeomorphisms of the circle onto itself. Under a one-sided bound on the Fourier support, such homeomorphisms are rational functions related to Blaschke products. We study the structure of rational circle homeomorphisms and show that they form a connected set in the uniform topology.

1. Introduction

Every continuous map f of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ to the complex plane extends to a harmonic map F of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the Taylor coefficients of F are given by the Fourier coefficients of f, namely $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta$. This simple but important relation has consequences both for geometric function theory and for the theory of minimal surfaces [3,7,9], especially when f is an embedding, i.e., an injective continuous map. For example, when f is a sense-preserving embedding with a convex image, the Radó-Kneser-Choquet theorem [3, p. 29] states that F is a sense-preserving diffeomorphism and in particular $\hat{f}(1) \neq 1$. On the other hand, for every integer N there exists a sense-preserving embedding $f : \mathbb{T} \to \mathbb{C}$ such that $\hat{f}(n) = 0$ whenever $|n| \leq N$ [10, Theorem 5.1]. Thus, some restrictions on the shape of $f(\mathbb{T})$ are necessary to obtain a non-vanishing result for $\hat{f}$.

By a circle embedding we mean an injective continuous map $f : \mathbb{T} \to \mathbb{C}$. In the special case $f(\mathbb{T}) = \mathbb{T}$ the map f is called a circle homeomorphism. The curve $f(\mathbb{T})$ is called shar-shaped about $w_0 \in \mathbb{C}$ if the argument of $f(e^{i\theta}) - w_0$ is a monotone function of θ. In this case f is called a starlike embedding. In this paper we answer Question 5.1 in [10] by proving that for a starlike embedding f, at least one of the coefficients $\hat{f}(1)$ and $\hat{f}(-1)$ is nonzero.

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If the Fourier series of a sense-preserving circle homeomorphism $f\colon \mathbb{T} \to \mathbb{T}$ terminates in either direction, then f is the restriction of the quotient of two Blaschke products [10, Proposition 3.2]. It is difficult to describe exactly which ratios of Blaschke products restrict to circle homeomorphisms. We give some sufficient conditions in Section 5. In Section 4, specifically Theorems 4.1 and 4.3 we show that the set of rational circle homeomorphisms is connected in the uniform topology.

2. Main results and preliminaries

Hall [7, Theorem 2] proved that $|\hat{f}(1)| + |\hat{f}(0)| > 0$ when f is a sense-preserving embedding and $f(\mathbb{T})$ is star-shaped about 0. The first of our main results, Theorem 2.1, allows $f(\mathbb{T})$ to be star-shaped about any point; it also applies to sense-reversing embeddings.

Theorem 2.1. *Let $f\colon \mathbb{T} \to \mathbb{C}$ be a starlike embedding. Then*

$$|\hat{f}(1)| + |\hat{f}(-1)| > 0.$$

Under the assumptions of this theorem, each of the individual coefficients $\hat{f}(1)$ and $\hat{f}(-1)$ may vanish [10, Proposition 5.1].

A complex-valued function is called harmonic if its real and imaginary parts are harmonic. For any continuous function on $\mathbb{T}$, convolution with the Poisson kernel

$$P(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}, \quad z \in \mathbb{D},\ \zeta \in \mathbb{T} \tag{2.1}$$

provides a harmonic extension in $\mathbb{D}$ [1, Theorem 4.22]. This fact is restated as a proposition for future references.

Proposition 2.2. *[2, Section 19.1] If $f:\ \mathbb{T} \to \mathbb{C}$ is continuous, then the series*

$$F(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n + \sum_{n=1}^{\infty} \hat{f}(-n) \overline{z}^n$$

defines a harmonic function in $\mathbb{D}$, for which f provides a continuous boundary extension.

Theorem 2.1 has an implication for the harmonic extension of starlike embeddings.

Corollary 2.3. *Suppose $f\colon \mathbb{T} \to \mathbb{C}$ is a starlike embedding. Let F be the harmonic extension of f, that is the harmonic map $F\colon \mathbb{D} \to \mathbb{C}$ that agrees with f on the boundary of $\mathbb{D}$. Then F has nonvanishing total derivative, meaning that $|F_z| + |F_{\bar{z}}| > 0$ in $\mathbb{D}$.*

In contrast to the Radó-Kneser-Choquet theorem, the harmonic map in Corollary 2.3 need not be a diffeomorphism, see Example 3.1.

The circle homeomorphisms whose Fourier series terminates in one direction can be completely described in terms of finite Blaschke products. A Blaschke product of degree n is a rational function of the form

$$B(z) = \sigma \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z},$$

where $z_1, \dots, z_n \in \mathbb{D}$ and $\sigma \in \mathbb{T}$. The book [4] is a convenient reference on the properties of these products.

Lemma 2.4. *[10, Proposition 3.2] Suppose that $f : \mathbb{T} \to \mathbb{T}$ is a sense-preserving circle homeomorphism. The set $\{n \in \mathbb{Z} \colon \hat{f}(n) \neq 0\}$ is:*

(a) bounded below if and only if $f(\zeta) = B(\zeta)/\zeta^{n-1}$ for some integer $n \geq 0$, where B is a Blaschke product of degree n;

(b) bounded above if and only if $f(\zeta) = \zeta^{n+1}/B(\zeta)$ for some integer $n \geq 0$, where B is a Blaschke product of degree n.

Not every Blaschke product B induces a circle homeomorphism in the way described in Lemma 2.4. To treat the two cases of Lemma 2.4 in a unified way, Kovalev and Yang gave the necessary and sufficient condition for the quotient of two finite Blaschke products to be a circle homeomorphism, which is recorded in the following.

Lemma 2.5. *[10, Lemma 3.1] Suppose that B_1, B_2 are finite Blaschke products:*

$$B_1(z) = \sigma_1 \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \quad B_2(z) = \sigma_2 \prod_{k=1}^{m} \frac{z - w_k}{1 - \overline{w_k} z},$$

where $z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_m \in \mathbb{D}$ and $\sigma_1, \sigma_2 \in \mathbb{T}$. Then the quotient $B_1(\zeta)/B_2(\zeta)$ is a sense-preserving circle homeomorphism if and only if $n - m = 1$ and

$$\sum_{k=1}^{n} P(z_k, \zeta) \geq \sum_{k=1}^{m} P(w_k, \zeta) \quad \text{for all} \quad \zeta \in \mathbb{T}, \tag{2.2}$$

where P is the Poisson kernel for the unit disk $\mathbb{D}$, see (2.1).

Because condition (2.2) is difficult to check in practice, in Section 5 we continue to consider the problem of describing the Blaschke products that induce circle homeomorphisms, and obtain some explicit sufficient conditions in terms of the Blaschke zeros. Moreover, in Section 4 we prove that both sets of homeomorphisms described in Lemma 2.4 are connected, as is the larger set in Lemma 2.5. The connectedness is understood in the topology of $C(\mathbb{T})$ induced by the uniform norm.

Since replacing the function $f(e^{i\theta})$ with $f(e^{-i\theta})$, which reverses the orientation of f, only changes the indexing of its Fourier coefficients, throughout the rest of this paper we will assume that f is sense-preserving.

3. Fourier coefficients of starlike embeddings

Proof of Theorem 2.1. Suppose that the curve $f(\mathbb{T})$ is star-shaped about some point w_0. Without loss of generality, we may assume that $w_0 = 0$. Write $f(e^{i\theta}) = R(\theta)e^{i\phi(\theta)}$, where ϕ is non-decreasing and

$$\phi(2\pi-) - \phi(0) = 2\pi.$$

Since f is an embedding, the function ϕ is continuous on $[0, 2\pi)$. Therefore the function $\psi(\theta) := \phi(\theta + \pi) - \phi(\theta) - \pi$ is continuous on $[0, \pi)$. Since $\psi(0) + \psi(\pi-) = 0$, the intermediate value theorem implies that there exists $\theta_0 \in [0, \pi)$ such that $\psi(\theta_0) = 0$, that is

$$\phi(\theta_0 + \pi) = \phi(\theta_0) + \pi.$$

Let $z_0 = e^{i\theta_0}$ and $g(e^{i\theta}) = e^{i\phi(\theta)}$. Then we have $f(e^{i\theta}) = R(\theta)g(e^{i\theta})$ and

$$g(-z_0) = g(e^{i(\theta_0+\pi)}) = e^{i\phi(\theta_0+\pi)} = e^{i(\phi(\theta_0)+\pi)} = -e^{i\phi(\theta_0)} = -g(e^{i\theta_0}) = -g(z_0).$$

Let $\eta = g(z_0)$ and γ be the open half-circle traced counterclockwise from the point z_0 to $-z_0$. Note that $\mathbb{T} \setminus \overline{\gamma}$ is a complementary open half-circle traced counterclockwise from $-z_0$ to z_0. Then $\overline{\eta} g(e^{i\theta})$ maps γ onto the upper half-circle and $\mathbb{T} \setminus \overline{\gamma}$ onto the lower half-circle, for g is sense-preserving.

Let

$$h(e^{i\theta}) = \frac{\overline{z_0} e^{i\theta}}{1 - \overline{z_0}^2 e^{2i\theta}}.$$

Then h maps γ onto the half-line starting from the point $\frac{1}{2}i$ to infinity along the upper imaginary axis, and $\mathbb{T} \setminus \overline{\gamma}$ onto the half-line starting from the point $-\frac{1}{2}i$ to infinity along the lower imaginary axis. Then $\frac{1}{h(e^{i\theta})} = (1 - \overline{z_0}^2 e^{2i\theta}) e^{-i\theta} z_0$ maps γ onto the segment $(0, -2i]$ and $\mathbb{T} \setminus \gamma$ onto the segment $(0, 2i]$. In addition, $1/h(\pm z_0) = 0$.

The above statements yield that for all $e^{i\theta} \in \mathbb{T}$, we have

$$\operatorname{Re}\left(\overline{\eta} g(e^{i\theta}) \frac{1}{h(e^{i\theta})} \right) = \operatorname{Re}\left(z_0 \overline{\eta} g(e^{i\theta})(1 - \overline{z_0}^2 e^{2i\theta}) e^{-i\theta} \right) \geq 0,$$

where the equality holds if and only if $e^{i\theta} \in \{z_0, -z_0\}$.

We now consider

$$\overline{\eta} z_0 \left(\hat{f}(1) - \overline{z_0}^2 \hat{f}(-1) \right) = \frac{1}{2\pi} \int_0^{2\pi} R(\theta) \overline{\eta} g(e^{i\theta}) \frac{1}{h(e^{i\theta})} d\theta,$$

which implies that

$$|\hat{f}(1)| + |\hat{f}(-1)| \geq \operatorname{Re}\left(\overline{\eta} z_0 \left(\hat{f}(1) - \overline{z_0}^2 \hat{f}(-1) \right) \right) = \frac{1}{2\pi} \int_0^{2\pi} R(\theta) \operatorname{Re}\left(\overline{\eta} g(e^{i\theta}) \frac{1}{h(e^{i\theta})} \right) d\theta > 0.$$

Therefore, the proof is completed. $\square$

Proof of Corollary 2.3. Since F is the harmonic extension of f, by Proposition 2.2 we have $F_z(0) = \hat{f}(1)$ and $F_{\bar{z}}(0) = \hat{f}(-1)$. By Theorem 2.1, $|F_z(0)| + |F_{\bar{z}}(0)| > 0$. Given any point $a \in \mathbb{D}$, consider the Möbius transformation $g(z) = \frac{z+a}{1+\bar{a}z}$. The composition $F \circ g$ is a harmonic map with boundary values $f \circ g$. By the above, the total derivative of $F \circ g$ at 0 does not vanish. Since $g(0) = a$, the chain rule shows that $|F_z(a)| + |F_{\bar{z}}(a)| > 0$. $\square$

Example 3.1. The embedding $f \colon \mathbb{T} \to \mathbb{C}$, defined by

$$f(e^{i\theta}) = e^{i\theta} + e^{2i\theta} + \frac{1}{2} e^{-2i\theta}$$

is starlike with respect to the point $w = 1$. Its harmonic extension $F(z) = z + z^2 + \bar{z}^2/2$ has Jacobian of variable sign in $\mathbb{D}$: in particular, F_z vanishes at $-1/2$ while $F_{\bar{z}}$ vanishes at 0. Therefore, F is not a diffeomorphism.

For the sake of completeness we show that f is indeed starlike with respect to 1. Its values on $\mathbb{T}$ agree with the rational function $h(z) = z^2 + z + z^{-2}/2$. Nevanlinna's criterion for starlikeness [6, Theorem 8.3.1] requires $zh'(z)/(h(z) - 1)$ to have nonnegative real part on $\mathbb{T}$. We have

$$\operatorname{Re} \frac{zh'(z)}{h(z) - 1} = \operatorname{Re} \frac{2z^2 + z - z^{-2}}{z^2 + z + z^{-2}/2 - 1}.$$

Multiplying the numerator of this fraction by the conjugate of its denominator and writing $z = e^{i\theta}$, we arrive at

$$\mathrm{Re}\left\{(2z^2 + z - z^{-2})(z^{-2} + z^{-1} + z^2/2 - 1)\right\} = \frac{5}{2} + 2\cos\theta - \cos 2\theta - \frac{1}{2}\cos 3\theta.$$

The latter expression factors as $(5 - 2\cos 2\theta)\cos^2(\theta/2)$ and is therefore nonnegative.

4. The set of rational circle homeomorphisms is connected

The space $C(\mathbb{T})$ of continuous mappings from $\mathbb{T}$ into $\mathbb{C}$ is equipped with the topology induced by the norm $\|f\| = \sup_{\mathbb{T}} |f|$. Let $H_+(\mathbb{T}) \subset C(\mathbb{T})$ denote the group of all sense-preserving circle homeomorphisms $f\colon \mathbb{T} \to \mathbb{T}$. The group $H_+(\mathbb{T})$ contains the rotation group $SO(2,\mathbb{R})$ which consists of the maps $z \mapsto \sigma z$, $\sigma \in \mathbb{T}$. Proposition 4.2 in [5] shows that $SO(2,\mathbb{R})$ is a deformation retract of $H_+(\mathbb{T})$, meaning there exists a continuous map $F\colon H_+(\mathbb{T}) \times [0,1] \to H_+(\mathbb{T})$ such that $F(\cdot,1)$ is the identity and $F(\cdot,0)$ is a retraction from $H_+(\mathbb{T})$ onto $SO(2,\mathbb{R})$. In particular, $H_+(\mathbb{T})$ is a connected set. We will show that similar results hold for certain subsets of $H_+(\mathbb{T})$ that consist of rational functions.

Theorem 4.1. *For each $n \geq 1$, the sets*

$$H_{n1}(\mathbb{T}) = H_+(\mathbb{T}) \cap \left\{B(\zeta)/\zeta^{n-1}\colon B \text{ is a Blaschke product of degree } n\right\}$$

and

$$H_{n2}(\mathbb{T}) = H_+(\mathbb{T}) \cap \left\{\zeta^{n+1}/B(\zeta)\colon B \text{ is a Blaschke product of degree } n\right\}$$

contain $SO(2,\mathbb{R})$ as a deformation retract. In particular, both sets are connected. The unions $\bigcup_{n=1}^{\infty} H_{n1}(\mathbb{T})$ and $\bigcup_{n=1}^{\infty} H_{n2}(\mathbb{T})$ are connected as well.

The proof is based on the semigroup property of the Poisson kernel P, namely $P_{r\rho} = P_r * P_\rho$ for $r, \rho \in [0,1)$.

Lemma 4.2. *Let*

$$B(t,\zeta) = \sigma \prod_{k=1}^{n} \frac{\zeta - tz_k}{1 - \overline{tz_k}\zeta},$$

where $z_1, z_2, \dots, z_n \in \mathbb{D}$, $0 \leq t \leq 1$ and $\sigma \in \mathbb{T}$.

(a) If $B(1,\zeta)/\zeta^{n-1}$ is a circle homeomorphism, then for any $t \in [0,1)$, $B(t,\zeta)/\zeta^{n-1}$ is still a circle homeomorphism.

(b) If $\zeta^{n+1}/B(1,\zeta)$ is a circle homeomorphism, then for any $t \in [0,1)$, $\zeta^{n+1}/B(t,\zeta)$ is still a circle homeomorphism.

Proof. In part (a), by Lemma 2.5 we have

$$\sum_{k=1}^{n} P(z_k,\zeta) \geq n-1, \quad \zeta \in \mathbb{T} \tag{4.1}$$

and the goal is to show that for any $t \in [0,1)$,

$$\sum_{k=1}^{n} P(tz_k,\zeta) \geq n-1, \quad \zeta \in \mathbb{T}. \tag{4.2}$$

By the semigroup property of P,

$$\begin{aligned}\sum_{k=1}^{n} P(tz_k,\zeta) &= \int_{\mathbb{T}} \sum_{k=1}^{n} P(z_k,\xi)P(t,\zeta/\xi)\frac{|d\xi|}{2\pi} \\ &\geq \int_{\mathbb{T}} (n-1)P(t,\zeta/\xi)\frac{|d\xi|}{2\pi} = n-1.\end{aligned} \tag{4.3}$$

The proof of part (b) follows the same process except the inequalities (4.1) and (4.2) are changed from $\cdots \geq n-1$ to $\cdots \leq n+1$. □

Proof of Theorem 4.1. We introduce a homotopy $F\colon H_{n1}(\mathbb{T}) \times [0,1] \to H_{n1}(\mathbb{T})$ using the notation of Lemma 4.2: if $\psi(\zeta) = B(1,\zeta)/\zeta^{n-1}$, then $F(\psi,t)$ is the map $\zeta \mapsto B(t,\zeta)/\zeta^{n-1}$. In view of Lemma 4.2 and the fact that $B(0,\zeta)/\zeta^{n-1}$ is a rotation, the map F is indeed a deformation retraction onto $SO(2,\mathbb{R})$. In particular, each set $H_{n1}(\mathbb{T})$ is connected. Since all these sets contain the identity map, their union is connected as well. The same reasoning applies to $H_{n2}(\mathbb{T})$. □

We have a similar result for the larger set of all circle homeomorphisms that are restrictions of rational functions. Recall that if a rational function maps $\mathbb{T}$ into $\mathbb{T}$, it must be a quotient of Blaschke products [4, Corollary 3.5.4].

Theorem 4.3. *The set of all circle homeomorphisms of the form B_1/B_2, where B_1, B_2 are finite Blaschke products, is connected.*

Proof. We will prove that each circle homeomorphism of the form B_1/B_2, where B_1, B_2 are finite Blaschke products, can be connected to a linear map $z \mapsto \sigma z$. By Lemma 2.5, we may assume that

$$B_1(z) = \sigma_1 \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}, \quad B_2(z) = \sigma_2 \prod_{k=1}^{n-1} \frac{z-w_k}{1-\overline{w_k}z},$$

where $z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_{n-1} \in \mathbb{D}$ and $\sigma_1, \sigma_2 \in \mathbb{T}$, and we have

$$\sum_{k=1}^{n} P(z_k,\zeta) \geq \sum_{k=1}^{n-1} P(w_k,\zeta) \quad \text{for all} \quad \zeta \in \mathbb{T}. \tag{4.4}$$

It suffices to prove that for all $t \in [0,1)$,

$$\sum_{k=1}^{n} P(tz_k,\zeta) \geq \sum_{k=1}^{n-1} P(tw_k,\zeta) \quad \text{for all} \quad \zeta \in \mathbb{T}.$$

By similar reasoning as in the proof of Lemma 4.2, we have

$$\begin{aligned}\sum_{k=1}^{n} P(tz_k,\zeta) - \sum_{k=1}^{n-1} P(tw_k,\zeta) &= \int_{\mathbb{T}} \sum_{k=1}^{n} P(z_k,\xi)P(t,\zeta/\xi)\frac{|d\xi|}{2\pi} - \int_{\mathbb{T}} \sum_{k=1}^{n-1} P(w_k,\xi)P(t,\zeta/\xi)\frac{|d\xi|}{2\pi} \\ &= \int_{\mathbb{T}} \left(\sum_{k=1}^{n} P(z_k,\xi) - \sum_{k=1}^{n-1} P(w_k,\xi) \right) P(t,\zeta/\xi)\frac{|d\xi|}{2\pi} \geq 0\end{aligned}$$

for all $\zeta \in \mathbb{T}$. Therefore, the proof is completed. □

5. Sufficient conditions for rational circle homeomorphisms

In this section we give some sufficient conditions for the ratio of Blaschke products to be a circle homeomorphism. In certain cases these conditions are also necessary.

Theorem 5.1. *Suppose that*

$$B(\zeta) = \sigma \prod_{k=1}^{n} \frac{\zeta - z_k}{1 - \overline{z_k}\zeta},$$

where $z_1, z_2, \ldots, z_n \in \mathbb{D}$ *and* $\sigma \in \mathbb{T}$. *If*

$$\sum_{k=1}^{n} \frac{1 - |z_k|}{1 + |z_k|} \geq n - 1, \tag{5.1}$$

then $B(\zeta)/\zeta^{n-1}$ *is a circle homeomorphism. In particular, this holds if* $|z_k| \leq \frac{1}{2n-1}$ *for all* k.

If all numbers z_k *have the same argument, the condition* (5.1) *is also necessary for* $B(\zeta)/\zeta^{n-1}$ *to be a circle homeomorphism.*

Proof. For $k = 1, \ldots, n$, we have

$$P(z_k, \zeta) = \frac{1 - |z_k|^2}{|\zeta - z_k|^2} \geq \frac{1 - |z_k|}{1 + |z_k|}. \tag{5.2}$$

If (5.1) holds, then (5.2) implies $\sum_{k=1}^{n} P(z_k, \zeta) \geq n - 1$. It follows from Lemma 2.5 that $B(\zeta)/\zeta^{n-1}$ is a circle homeomorphism.

In the case when all numbers z_k have the same argument, we can choose $\zeta \in \mathbb{T}$ such that $\overline{\zeta} z_k \leq 0$ for every k. Then equality holds in (5.2), which shows that (5.1) is necessary in order to have $\sum_{k=1}^{n} P(z_k, \zeta) \geq n - 1$. $\square$

In general it seems difficult to describe precisely when $B(\zeta)/\zeta^{n-1}$ is a circle homeomorphism, but for Blaschke products of degree 2 the picture is more complete.

Theorem 5.2. *Suppose that*

$$B(\zeta) = \sigma \frac{\zeta - a}{1 - \overline{a}\zeta} \frac{\zeta - b}{1 - \overline{b}\zeta},$$

where $a, b \in \mathbb{D} \cap \mathbb{R}$ *and* $\sigma \in \mathbb{T}$. *Then* $B(\zeta)/\zeta$ *is a circle homeomorphism if and only if one of the following conditions holds.*

(a) $ab \geq 0$ *and* $1 - |a| - |b| - 3ab \geq 0$;
(b) $ab \leq 0$ *and* $1 + ab - a^2 - b^2 \geq 0$.

Proof. Part (a) follows from Theorem 5.1 after observing that

$$\frac{1 - |a|}{1 + |a|} + \frac{1 - |b|}{1 + |b|} - 1 = \frac{1 - |a| - |b| - 3|ab|}{(1 + |a|)(1 + |b|)}.$$

In part (b), our goal is to determine when $P(a,\zeta)+P(b,\zeta)\ge 1$ for all $\zeta\in\mathbb{T}$. The quantity

$$P(a,\zeta)+P(b,\zeta)-1=\operatorname{Re}\left(\frac{\zeta+a}{\zeta-a}+\frac{2b}{\zeta-b}\right)=\operatorname{Re}\frac{\zeta^2+(a+b)\zeta-3ab}{\zeta^2-(a+b)\zeta+ab}$$

is nonnegative if and only if the product

$$\left(\zeta^2+(a+b)\zeta-3ab\right)\left(\bar\zeta^2-(a+b)\bar\zeta+ab\right) \tag{5.3}$$

has nonnegative real part. Expand (5.3) and rewrite it in terms of $x=\operatorname{Re}\zeta$, using the identity $\operatorname{Re}(\zeta^2)=2x^2-1$. The result is the quadratic function

$$\begin{aligned} g(x)&=-2ab(2x^2-1)+4ab(a+b)x+1-(a+b)^2-3a^2b^2\\ &=-4abx^2+4ab(a+b)x+1-a^2-b^2-3a^2b^2 \end{aligned}$$

Since $ab\le 0$, the minimum of g is attained at $x=(a+b)/2\in(-1,1)$. Thus, the inequality $\min_{[-1,1]} g\ge 0$ is equivalent to $g((a+b)/2)\ge 0$. Since

$$g\left(\frac{a+b}{2}\right)=(1-ab)(1+ab-a^2-b^2)$$

and $1-ab>0$, part (b) is proved. $\square$

The second natural way to form a circle homeomorphism from a Blaschke product of degree n is $\zeta\mapsto\zeta^{n+1}/B(\zeta)$. For this construction we have a statement that parallels Theorem 5.1.

Theorem 5.3. *Suppose that*

$$B(\zeta)=\sigma\prod_{k=1}^{n}\frac{\zeta-z_k}{1-\overline{z_k}\zeta},$$

where $z_1,z_2,\dots,z_n\in\mathbb{D}$ *and* $\sigma\in\mathbb{T}$. *If*

$$\sum_{k=1}^{n}\frac{1+|z_k|}{1-|z_k|}\le n+1, \tag{5.4}$$

then $\zeta^{n+1}/B(\zeta)$ *is a circle homeomorphism. In particular, this holds if* $|z_k|\le\frac{1}{2n+1}$ *for all* k.

If all numbers z_k *have the same argument, the condition* (5.4) *is also necessary for* $\zeta^{n+1}/B(\zeta)$ *to be a circle homeomorphism.*

Proof. For $k=1,\dots,n$, we have

$$P(z_k,\zeta)=\frac{1-|z_k|^2}{|\zeta-z_k|^2}\le\frac{1+|z_k|}{1-|z_k|}. \tag{5.5}$$

If (5.4) holds, then (5.5) implies $\sum_{k=1}^{n}P(z_k,\zeta)\le n+1$, and the conclusion follows from Lemma 2.5.

When all numbers z_k have the same argument, we can choose $\zeta\in\mathbb{T}$ such that $\bar\zeta z_k\ge 0$ for every k. Then equality holds in (5.5), which shows that (5.4) is necessary in order to have $\sum_{k=1}^{n}P(z_k,\zeta)\le n+1$. $\square$

Condition (5.4) turns out to be more restrictive than (5.1). Indeed, the inequality between arithmetic and harmonic means [8, Ch. 2] implies

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1-|z_k|}{1+|z_k|} \geq \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1+|z_k|}{1-|z_k|}\right)^{-1}.$$

If (5.4) holds, then

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1-|z_k|}{1+|z_k|} \geq \frac{n}{n+1} > \frac{n-1}{n},$$

which implies (5.1).

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