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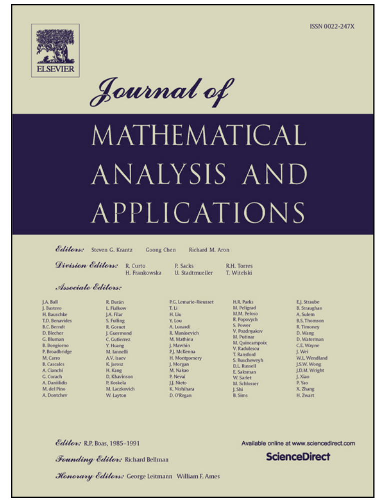

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Existence and Stability of the Doubly Nonlinear Anisotropic Parabolic Equation

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Abstract

In this paper, we are concerned with a doubly nonlinear anisotropic parabolic equation, in which the diffusion coefficient and the variable exponent depend on the time variable t. Under certain conditions, the existence of weak solution is proved by applying the parabolically regularized method. Based on a partial boundary value condition, the stability of weak solution is also investigated.

Keywords: anisotropic parabolic equation; partial boundary value condition; weak solution; characteristic function method.

2010 MSC: 35B35; 35D30; 35K55.

1 Introduction

Since the significant disruption that is being caused by the coronavirus pandemic, we are aware that all communities must resolutely work together to battle the pandemic amid globalization. A growing number of mathematical models have been developed by health care systems, academic institutions and others to help forecast coronavirus spread, deaths, and medical supply needs, including ventilators, hospital beds and intensive care units, timing of patient surges and more. Mathematically, a model of infectious disease can be regarded as a special reaction-diffusion process. Motivated by this fact, in this study we consider a kind of reaction-diffusion equation, namely, a doubly nonlinear parabolic anisotropic equation:

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\right) + \sum_{i=1}^{N} g^i(x,t)\frac{\partial B(u)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1}$$

where $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $0 \le a_i(x,t) \in C^1(\overline{Q}_T)$, $1 < p_i(x,t) \in C^1(\overline{Q}_T)$, $g^i(x,t) \in C^1(\overline{Q}_T)$, $B'(u) = b(u) \ge 0$ and $B(0) = 0$. Equation (1) arises from physics, fluid mechanics, as well as from the epidemic model of diseases in biology and ecology [27]. Compared with the isotropic-type equations, equation (1) is much closer to a diffusion process such as the epidemic of coronavirus disease. If

$$a_i(x,t) > 0,\ (x,t) \in \Omega \times [0,T] \text{ and } a_i(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,T],\ i = 1,2,\cdots,N, \tag{2}$$

we conjecture that it inevitably leads to

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,T),$$

which was partially proved in [25].

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The biological explanation of condition (2) lies in the fact that if $u(x)$ represents the velocity of spreading progress of an infectious disease such as coronavirus disease, condition (2) implies that the virus (or disease) can not transmit across $\partial\Omega$, when the region remains under lockdown.

A special case of equation (1) is the so-called evolutionary $p(x)-$Laplacian equation, which takes the form:

$$u_t = \mathrm{div}(|\nabla u|^{p(x,t)}\nabla u),\ \ (x,t) \in Q_T,$$

and has been extensively studied in the past decades [1, 2, 5, 6, 9, 18, 20, 26] etc. Equation (1) can also be regarded as a generalized version of the polytropic infiltration equation:

$$u_t = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m),\ \ (x,t) \in Q_T, \tag{3}$$

where $m > 0$ and $p > 1$. For more details and recent results on equation (3), we refer the reader to [4, 11, 13, 22, 23, 32] and the references therein.

Recently, a number of issues considered the anisotropic equation [7, 8]:

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,u,\nabla u),\ \ (x,t) \in Q_T,$$

with the initial-boundary value conditions

$$u(x,t) = u_0(x),\ \ x \in \Omega, \tag{4}$$

$$u(x,t) = 0,\ \ (x,t) \in \partial\Omega \times [0,T).$$

A more general anisotropic equation [10, 17, 24, 28, 29, 30]:

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right) + f(x,t,u,\nabla u),\ \ (x,t) \in Q_T, \tag{5}$$

was studied on the stability and the well-posedness. Here what interests us most is that if we take the diffusion coefficient $a_i(x)\mid_{x\in\partial\Omega}= 0$ and consider a partial boundary value condition

$$u(x,t) = 0,\ \ (x,t) \in \Sigma_p \subseteq \partial\Omega \times [0,T), \tag{6}$$

can the stability of equation (1) be achieved too? From [28, 29, 30], we know that when $\Sigma_p = \Sigma_1 \times (0,T)$, where Σ_1 is a submanifold of $\partial\Omega$ (or $\Sigma_1 = \emptyset$), the stability of weak solution of equation (5) can be true. For equation (1), the diffusion coefficient $a_i(x,t)$, the variable exponent $p_i(x,t)$ and the convection coefficient $g^i(x,t)$ are all dependent on the time variable t. Distinguished from [28, 29, 30], we will show that Σ_p is a submanifold of $\partial\Omega \times (0,T)$ and is generally not a cylinder as $\Sigma_1 \times (0,T)$.

Assume that $B(u)$ is a strictly increasing function. For examples, $B(u)$ can been chosen as u^m, $e^u - 1$, $\ln(1+u)$ and

$$B(u) = \begin{cases} u^{m_1}, & \text{if } 0 \le u < 1, \\ u^{m_2}, & \text{if } u \ge 1 \end{cases}$$

with $m_1 \ne m_2$. For convenience, we denote

$$p_- = \min_{(x,t)\in\overline{Q_T}}\{p_1(x,t), p_2(x,t), \cdots, p_{N-1}(x,t), p_N(x,t)\},\ \ p_- > 1,$$

$$p_+ = \max_{(x,t)\in\overline{Q_T}}\{p_1(x,t), p_2(x,t), \cdots, p_{N-1}(x,t), p_N(x,t)\}.$$

Definition 1 *We say that $u(x,t)$ is a weak solution of equation (1), if*

$$u \in L^\infty(Q_T),\ \frac{\partial}{\partial t}\int_0^u \sqrt{b(s)}ds \in L^2(Q_T),\ \ a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)} \in L^1(Q_T),\ \ i = 1,2,\cdots,N, \tag{7}$$

and for any function $\varphi \in C(0,T;W_0^{1,p_+}(\Omega))$ *there holds*

$$\begin{aligned}&\iint_{Q_T}\left[\frac{\partial u}{\partial t}\varphi(x,t)+\sum_{i=1}^{N}a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\right]dxdt\\&=\sum_{i=1}^{N}\iint_{Q_T}g^i(x,t)\frac{\partial B(u)}{\partial x_i}\varphi(x,t)dxdt.\end{aligned} \tag{8}$$

The initial value condition (4) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega\left|\int_0^{u(x,t)}\sqrt{b(s)}-\int_0^{u_0(x)}\sqrt{b(s)}ds\right|dx=0, \tag{9}$$

and the partial boundary value condition (6) is true in the sense of trace.

Let us summarize our main results as follows. For convenience, we use c to represent a constant that may change from line to line throughout the whole paper,

Theorem 2 *Suppose that* $p_- \geq 2$ *and*

$$\frac{\partial p_i(x,t)}{\partial t}\leq 0,\ i=1,2,\cdots,N. \tag{10}$$

Suppose that $a_i(x,t)$ *satisfies condition (2) and one of the following conditions:*

$$(i).\ \left|\frac{\partial a_i(x,t)}{\partial t}\right|\leq ca_i(x,t),\ i=1,2,\cdots,N. \tag{11}$$

$$(ii).\ \frac{\partial a_i(x,t)}{\partial t}\leq 0,\ i=1,2,\cdots,N. \tag{12}$$

Suppose that $u_0(x)\geq 0$ *satisfies*

$$u_0\in L^\infty(\Omega),\ a_i(x,0)u_0(x)\in W^{1,p_i(x,0)}(\Omega),\ i=1,2,\cdots,N. \tag{13}$$

Then there is a nonnegative solution of equation (1) under condition (4).

Theorem 3 *Suppose that for* $i=1,2,\cdots,N$, $a_i(x,t)\equiv a(x,t)$ *satisfies condition (2) and for the large* n *there holds*

$$\int_0^T n^{1-\frac{1}{p_{it}^+}}\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}\left|\frac{\partial a(x,t)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{p_{it}^+}}dt\leq c, \tag{14}$$

$$\int_0^T\int_\Omega g^i(x,t)^{q_i(x,t)}a(x,t)^{-\frac{1}{p_i(x,t)-1}}dx\leq c, \tag{15}$$

where $q_i(x,t)=\frac{p_i(x,t)}{p_i(x,t)-1}$, $p_{it}^+=\max_{x\in\overline{\Omega}}p_i(x,t)$, $q_{it}^+=\max_{x\in\overline{\Omega}}q_i(x,t)$ *and*

$$\Omega_{\frac{1}{n}t}=\left\{x\in\partial\Omega: a(x,t)>\frac{1}{n}\right\},\ \ t\in[0,T).$$

Suppose that $u(x,t)$ *and* $v(x,t)$ *are two weak solutions of equation (1) with the initial values* $u_0(x)$ *and* $v_0(x)$ *respectively, and with a partial homogeneous boundary value condition*

$$u(x,t)=v(x,t)=0,\ \ (x,t)\in\Sigma_p=\left\{(x,t)\in\partial\Omega\times(0,T):\sum_{i=1}^{N}g^i(x,t)\frac{\partial a(x,t)}{\partial x_i}\neq 0\right\}. \tag{16}$$

Then we have

$$\int_\Omega|u(x,t)-v(x,t)|dx\leq c\int_\Omega|u_0(x)-v_0(x)|dx,\ a.e.\ t\in[0,T). \tag{17}$$

It is remarkable that Theorem 3 can be generalized to the case of $a_i(x,t) \neq a_j(x,t)$ as $i \neq j$. The proof can be processed in an analogous manner.

The rest of the paper is organized as follows. Proofs of Theorems 2 and 3 are presented in Sections 2 and 3, respectively. The characteristic function method is introduced in Section 4. We show that this method can also be applied to study the stability for other degenerate parabolic equations. A brief conclusion is given in Section 5.

2 Proof of Theorem 2

For simplicity, we assume that $B(u)$ is a C^1 strictly monotone increasing function. We prove Theorem 2 by starting to consider a parabolically regularized system:

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left((a_i(x,t)+\varepsilon)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\right) + \sum_{i=1}^{N} g^i(x,t)\frac{\partial B(u)}{\partial x_i},\ (x,t) \in\ Q_T, \tag{18}$$

$$u(x,0) = u_0(x) + \varepsilon,\ x \in \Omega, \tag{19}$$

$$u(x,t) = \varepsilon,\ (x,t) \in \partial\Omega \times (0,T). \tag{20}$$

Proof of Theorem 2. Since $u_0(x) \geq 0$ satisfies (13), similar to the evolutionary $p-$Laplacian equation [27], by using the monotone convergence method, we can prove that there exists a constant M such that the solution $u_\varepsilon \in L^1(0,T : W^{1,p(x)}(\Omega))$ of the initial-boundary value problem (18)-(20) satisfies

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leqslant M. \tag{21}$$

For more results on the existence of weak solutions to the initial-boundary value problem (18)-(20), we refer to [5, 6].

Denote

$$\int_0^r B(s)ds = \mathbb{B}(r).$$

Multiplying both sides of (18) by $B(u_\varepsilon) - B(\varepsilon)$ yields

$$\begin{aligned}&\int_\Omega \mathbb{B}(u_\varepsilon(x,t))dx + \sum_{i=1}^{N}\iint_{Q_t}(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt\\ =&\int_\Omega \mathbb{B}(u_0(x))dx + B(\varepsilon)\int_\Omega[u(x,t)-u_0(x)]dx\\ &+\sum_{i=1}^{N}\iint_{Q_t} g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}[B(u_\varepsilon)-B(\varepsilon)]dxdt,\end{aligned} \tag{22}$$

where $Q_t = \Omega \times (0,t)$ for any $t \in [0,T)$.

Since

$$\begin{aligned}&\iint_{Q_t} g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}[B(u_\varepsilon)-B(\varepsilon)]dxdt\\ =&-\frac{1}{2}\iint_{Q_t}\frac{\partial g^i(x,t)}{\partial x_i}[B(u_\varepsilon)-B(\varepsilon)]^2dxdt,\end{aligned}$$

from (22) we have

$$\begin{aligned}&\sum_{i=1}^{N}\iint_{Q_T} a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt\\ \leq& c\iint_{Q_T}(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt\\ \leq& c.\end{aligned} \tag{23}$$

Multiplying both sides of (18) by $[B(u_\varepsilon)-B(\varepsilon)]_t$ and integrating it over Q_t gives

$$\begin{aligned}
&\iint_{Q_t}[B(u_\varepsilon)-B(\varepsilon)]_t u_{\varepsilon t}dxdt\\
=&-\sum_{i=1}^{N}\iint_{Q_t}(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p(x,t)-2}\frac{\partial}{\partial x_i}B(u_\varepsilon)\frac{\partial}{\partial x_i}B(u_\varepsilon)_t dxdt\\
&+\sum_{i=1}^{N}\iint_{Q_t}g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}[B(u_\varepsilon)-B(\varepsilon)]_t dxdt.
\end{aligned}\tag{24}$$

Note that

$$\begin{aligned}
&\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial}{\partial x_i}B(u_\varepsilon)_t\\
=&\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds-\frac{1}{2}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}\frac{\partial}{\partial t}s^{\frac{p_i(x,t)-2}{2}}ds\\
=&\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds-\frac{1}{4}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}\ln s\frac{\partial p_i}{\partial t}ds\\
=&\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds-\frac{1}{p_i(x,t)}\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|\frac{\partial p_i}{\partial t}ds\\
&+\frac{2}{p_i(x,t)}\frac{\partial p_i}{\partial t}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds.
\end{aligned}\tag{25}$$

Then, we have

$$\begin{aligned}
&-\iint_{Q_t}(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial}{\partial x_i}B(u_\varepsilon)_t dxdt\\
=&-\frac{1}{2}\iint_{Q_t}\frac{\partial}{\partial t}\left[(a_i(x,t)+\varepsilon)\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds\right]dxdt\\
&+\frac{1}{2}\iint_{Q_t}\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds\frac{\partial a_i(x,t)}{\partial t}dxdt\\
&+\frac{1}{4}\iint_{Q_t}[a_i(x,t)+\varepsilon]\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2}s^{\frac{p_i(x,t)-2}{2}}\ln s\frac{\partial p_i}{\partial t}dsdxdt\\
=&\frac{1}{2}\int_\Omega\frac{2}{p_i(x,t)}\left[(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}-(a_i(x,0)+\varepsilon)\left|\frac{\partial B(u_0)}{\partial x_i}\right|^{p_i(x,0)}\right]dx\\
&+\frac{1}{2}\iint_{Q_t}\int_0^{\left|\frac{\partial}{\partial x_i}B(u_\varepsilon)\right|^2}s^{\frac{p_i(x,t)-2}{2}}ds\frac{\partial a_i(x,t)}{\partial t}dxdt\\
&+\iint_{Q_t}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{1}{p_i(x,t)}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|dsdxdt\\
&-\iint_{Q_t}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{2}{p_i(x,t)}\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2}s^{\frac{p_i(x,t)-2}{2}}dsdxdt\\
\le&c.
\end{aligned}\tag{26}$$

To derive (26) from (10)-(12), we have used the following facts. From condition (i), i.e. $\left|\frac{\partial a_i(x,t)}{\partial t}\right|\le$

$ca_i(x,t)$, we get

$$\iint_{Q_t}\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2} s^{\frac{p_i(x,t)-2}{2}}ds\frac{\partial a_i(x,t)}{\partial t}dxdt \le c\iint_{Q_t} a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt \le c.$$

From condition (ii), i.e. $\frac{\partial a_i(x,t)}{\partial t} \le 0$, we have

$$\iint_{Q_t}\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2} s^{\frac{p_i(x,t)-2}{2}}ds\frac{\partial a_i(x,t)}{\partial t}dxdt \le 0.$$

Let

$$Q_1 = \left\{(x,t)\in Q_t : \left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right| < 1\right\},\ \ Q_2 = Q_t \setminus Q_1.$$

In view of $\frac{\partial p_i}{\partial t} \le 0$, we deduce

$$\begin{aligned}
&\iint_{Q_t}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{1}{p_i(x,t)}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|dsdxdt\\
\le&\iint_{Q_1}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{1}{p_i(x,t)}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|dsdxdt\\
&+\iint_{Q_2}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{1}{p_i(x,t)}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|dsdxdt\\
\le& c+\iint_{Q_2}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{1}{p_i(x,t)}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}\ln\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|dsdxdt\\
\le& c
\end{aligned}$$

and

$$\begin{aligned}
&-\iint_{Q_t}\frac{\partial p_i}{\partial t}[a_i(x,t)+\varepsilon]\frac{2}{p_i(x,t)}\int_0^{\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2} s^{\frac{p_i(x,t)-2}{2}}dsdxdt\\
\le& c\iint_{Q_t}[a_i(x,t)+\varepsilon]\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt\\
\le& c.
\end{aligned}$$

It follows from $p_i(x,t)\ge p_- \ge 2$ that

$$\begin{aligned}
&\iint_{Q_t} g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}[B(u_\varepsilon)-B(\varepsilon)]_t dxdt\\
\le&\iint_{Q_t}\left[c(\delta)\left|g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^2+\delta|B(u_\varepsilon)_t|^2\right]dxdt\\
\le& c(\delta)\left(\iint_{Q_t}\left|g^i(x,t)a_i(x,t)^{-\frac{2}{p_i(x,t)}}\right|^{\frac{p_i(x,t)}{p_i(x,t)-2}}dxdt\right)^{\frac{1}{p_{i2}}}\left(\iint_{Q_t}a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}dxdt\right)^{\frac{1}{p_{i1}}}\\
&+\iint_{Q_t}\delta|B'(u_\varepsilon)u_{\varepsilon t}|^2dxdt\\
\le& c+\frac{1}{2}\iint_{Q_t}b(u_\varepsilon)|u_{\varepsilon t}|^2dxdt,
\end{aligned}\tag{27}$$

where the small constant δ satisfies $\delta b(M)\le\frac{1}{2}$.

From (24)-(27), we can derive

$$\iint_{Q_t} (B(u_\varepsilon))_t u_{\varepsilon t} dxdt = \iint_{Q_t} b(u_\varepsilon)|u_{\varepsilon t}|^2 dxdt \leqslant c \tag{28}$$

and

$$\frac{\partial}{\partial t}\int_0^{u_\varepsilon} \sqrt{b(s)}ds \rightharpoonup \frac{\partial}{\partial t}\int_0^{u} \sqrt{b(s)}ds, \text{ in } L^2(Q_T). \tag{29}$$

From inequalities (21), (23) and (28), it implies

$$u_\varepsilon \rightharpoonup u, \text{ weakly star in } L^\infty(Q_T),$$

$$u_\varepsilon \to u, \text{ a.e. in } Q_T,$$

and there exists an $N-$dimensional vector $\overrightarrow{\zeta} = (\zeta_1, \cdots, \zeta_N)$ satisfying

$$\left|\overrightarrow{\zeta}\right| \in L^1\left(0, T; L^{\frac{p(x)}{p(x)-1}}(\Omega)\right)$$

such that

$$a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i} \rightharpoonup \zeta_i \ \ in\ L^1(Q_T), \ \ i = 1, 2, \cdots, N.$$

In order to prove u to be the solution of equation (1), we shall prove that

$$\sum_{i=1}^{N}\iint_{Q_T} a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\varphi_{x_i} dxdt = \iint_{Q_T}\overrightarrow{\zeta}\cdot\nabla\varphi dxdt \tag{30}$$

for any $\varphi \in C_0^1(Q_T)$.

Note that

$$\begin{aligned}&\iint_{Q_T}\left[u_{\varepsilon t}\varphi + \sum_{i=1}^{N}(a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\nabla\varphi_{x_i}\right]dxdt\\&+\sum_{i=1}^{N}\iint_{Q_T}\left[\frac{\partial\varphi(x,t)}{\partial x_i}g^i(x,t)B(u_\varepsilon) + \frac{\partial g^i(x,t)}{\partial x_i}B(u_\varepsilon)\varphi(x,t)\right]dxdt = 0.\end{aligned} \tag{31}$$

Due to $a_i(x,t)\mid_{\partial\Omega\times[0,T]}= 0$ and $a_i(x,t) > 0$ for $(x,t) \in \Omega\times[0,T]$, in view of $\varphi(x,t) \in C_0^1(Q_T)$, we obtain $\max\limits_{supp\varphi}\frac{|\varphi_{x_i}(x,t)|}{a_i(x,t)} \geq c > 0$, and

$$\begin{aligned}&\varepsilon\left|\iint_{Q_T}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_i}dxdt\right|\\ \leq&\varepsilon c\sup_{supp\varphi}\frac{|\varphi_{x_i}|}{a_i(x,t)}\iint_{Q_T} b_i(x,t)\left(\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)}+1\right)dxdt\\ \to&0, \text{ as } \varepsilon\to 0.\end{aligned}$$

This further leads to

$$\begin{aligned}\iint_{Q_T}\vec{\zeta}\cdot\nabla\varphi dxdt =&\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\iint_{Q_T} a_i(x,t)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial\varphi(u_\varepsilon)}{\partial x_i}dxdt\\ =&\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\iint_{Q_T}[a_i(x,t)+\varepsilon]\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial\varphi(u_\varepsilon)}{\partial x_i}dxdt\\ &-\lim_{\varepsilon\to 0}\varepsilon\sum_{i=1}^{N}\iint_{Q_T}\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial\varphi(u_\varepsilon)}{\partial x_i}dxdt\\ =&\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\iint_{Q_T}[a_i(x,t)+\varepsilon]\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial\varphi(u_\varepsilon)}{\partial x_i}dxdt.\end{aligned}$$

Since $u_\varepsilon \to u$, we see $B(u_\varepsilon) \to B(u)$ and

$$\begin{aligned}&\iint_{Q_T} (u\varphi_t + \vec{\zeta}\cdot\nabla\varphi)dxdt \\ &+ \sum_{i=1}^N \iint_{Q_T} \left[\frac{\partial\varphi(x,t)}{\partial x_i} g^i(x,t)B(u) + \frac{\partial g^i(x,t)}{\partial x_i} B(u)\varphi(x,t)\right] dxdt = 0.\end{aligned} \tag{32}$$

Let $0 \leqslant \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ on suppφ_1. In view of $v \in L^\infty(Q_T)$ and $b_i(x,t)\left|\frac{\partial B_i(v)}{\partial x_i}\right|^{p_i(x,t)} \in L^1(Q_T)$ for $i = 1, 2, \cdots, N$, we have

$$\begin{aligned}&\iint_{Q_T} \psi a_i(x,t)\left(\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i} - \left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right) \\ &\cdot\left(\frac{\partial B(u_\varepsilon)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right) dxdt \geqslant 0.\end{aligned} \tag{33}$$

Choosing $\varphi = \psi B(u_\varepsilon)$ in (31) yields

$$\begin{aligned}&\iint_{Q_T} \left[\frac{\partial u_\varepsilon}{\partial t}\psi B(u_\varepsilon) + \sum_{i=1}^N (a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}(\psi B(u_\varepsilon))_{x_i}\right. \\ &\left. - \sum_{i=1}^N g^i(x,t)\frac{\partial B(u_\varepsilon)}{\partial x_i}\psi B(u_\varepsilon)\right] dxdt = 0.\end{aligned} \tag{34}$$

It follows from (33)-(34) that

$$\begin{aligned}&\iint_{Q_T} \psi_t \mathbb{B}(u_\varepsilon)dxdt - \sum_{i=1}^N \iint_{Q_T} (a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\psi_{x_i} B(u_\varepsilon)dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} (a_i(x,t)+\varepsilon)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\left(\frac{\partial B(u_\varepsilon)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right)\psi dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} (a_i(x,t)+\varepsilon)\left|\frac{\partial B(u_\varepsilon)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u_\varepsilon)}{\partial x_i}\frac{\partial B(v)}{\partial x_i}\psi dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} g^i(x,t)B(u_\varepsilon)\left(\frac{\partial B(u_\varepsilon)}{\partial x_i}\psi + B(u_\varepsilon)\psi_{x_i}\right) dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} \frac{\partial g^i(x,t)}{\partial x_i} B(u_\varepsilon)\psi dxdt \geq 0.\end{aligned} \tag{35}$$

Letting $\varepsilon \to 0$, we have

$$\begin{aligned}&\iint_{Q_T} \psi_t \mathbb{B}(u)dxdt - \sum_{i=1}^N \iint_{Q_T} B(u)\zeta_i\psi_{x_i} dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} a_i(x,t)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\left(\frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right) dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} a_i(x,t)\zeta_{x_i}\frac{\partial B(v)}{\partial x_i}\psi dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} g^i(x,t)B(u)\left(\frac{\partial B(u)}{\partial x_i}\psi + B(u)\psi_{x_i}\right) dxdt \\ &- \sum_{i=1}^N \iint_{Q_T} \frac{\partial g^i(x,t)}{\partial x_i} B(u)\psi dxdt \geq 0.\end{aligned} \tag{36}$$

Taking $\varphi = \psi B(u)$ in (32), we get

$$\begin{aligned}
&\iint_{Q_T} \mathbb{B}(u)\psi_t dxdt - \sum_{i=1}^N \iint_{Q_T} \psi\zeta_i \frac{\partial B(u)}{\partial x_i} dxdt - \sum_{i=1}^N \iint_{Q_T} B(u)\zeta_i \psi_{x_i} dxdt \\
&- \sum_{i=1}^N \iint_{Q_T} a_i(x,t)\zeta_{x_i} \frac{\partial B(v)}{\partial x_i}\psi dxdt \\
&- \sum_{i=1}^N \iint_{Q_T} g^i(x,t)B(u)\left(\frac{\partial B(u)}{\partial x_i}\psi + B(u)\psi_{x_i}\right) dxdt \\
&- \sum_{i=1}^N \iint_{Q_T} \frac{\partial g^i(x,t)}{\partial x_i} B(u)\psi dxdt \geq 0.
\end{aligned} \tag{37}$$

By combing (36) and (37), we have

$$\sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i}\right)\left(\frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right) dxdt \geqslant 0. \tag{38}$$

In particular, taking $v = B^{-1}(B(u) - \lambda\varphi)$ and $\lambda > 0$, we find

$$\lambda \sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial}{\partial x_i}(B(u)-\lambda\varphi)\right|^{p_i(x,t)-2} \frac{\partial}{\partial x_i}(B(u)-\lambda\varphi)\right)\frac{\partial \varphi}{\partial x_i} dxdt \geqslant 0$$

and so

$$\lambda \sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial}{\partial x_i}(B(u)-\lambda\varphi)\right|^{p_i(x,t)-2} \frac{\partial}{\partial x_i}(B(u)-\lambda\varphi)\right)\frac{\partial \varphi}{\partial x_i} dxdt \geqslant 0.$$

When λ goes to zero, we have

$$\sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i} dxdt \geqslant 0. \tag{39}$$

Similarly, when $\lambda < 0$, we get

$$\sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i} dxdt \leq 0.$$

Accordingly, we obtain

$$\sum_{i=1}^N \iint_{Q_T} \psi \left(\zeta_i - a_i(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i} dxdt = 0.$$

Since $\psi = 1$ on suppφ, we arrive at (30).

The initial value condition (4) in the sense of (9) can be derived from (29). We omit the details here. Consequently, $u(x)$ satisfies equation (1) in the sense of Definition 1. ■

3 Proof of Theorem 3

To discuss the stability of solutions of equation (1), we need to introduce the following technical lemma. Let $p(x) \in C^1(\overline{\Omega})$, and denote $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and $p^- = \max_{x\in\overline{\Omega}} p(x)$.

Lemma 4 *[12, 15] (I) The space* $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$, $\left(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)}\right)$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(II) Let $p_1(x)$ *and* $p_2(x)$ *be real functions with* $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} = 1$ *and* $p_1(x) > 1$. *Then the conjugate space of* $L^{p_1(x)}(\Omega)$ *is* $L^{p_2(x)}(\Omega)$. *And for any* $u \in L^{p_1(x)}(\Omega)$ *and* $v \in L^{p_2(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uvdx\right| \leq 2\|u\|_{L^{p_1(x)}(\Omega)}\|v\|_{L^{p_2(x)}(\Omega)}.$$

(III) If $\|u\|_{L^{p(x)}(\Omega)} = 1$, *then* $\int_\Omega \|u\|^{p(x)}dx = 1$; *if* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \int_\Omega |u|^{p(x)}dx \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$; *and if* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \int_\Omega |u|^{p(x)}dx \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

For any large integer n, we define an odd function $S_n(s)$ by

$$S_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2s^2\mathrm{e}^{1-n^2s^2}, & 0 \leq s \leqslant \frac{1}{n}, \end{cases}$$

and let

$$H_n(s) = \int_0^s S_n(s)ds.$$

Then

$$\lim_{n\to 0} S_n(s) = \operatorname{sgn}(s) \ \text{ and } \ \lim_{n\to 0} sS_n'(s) = 0, \ \ s \in (-\infty, +\infty).$$

Meanwhile, since $a_i(x,t) \equiv a(x,t) \geq 0$, for any $\lambda > 0$ we define

$$\varphi_n(x,t) = \begin{cases} 1, & \text{if } \ x \in \Omega_{\frac{2}{n}t}, \\ n\left(a(x,t) - \frac{1}{n}\right), & \text{if } x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_{\frac{1}{n}t}, \end{cases}$$

where $\Omega_{\lambda t} = \{x \in \Omega : a(x,t) > \lambda\}$.

Proof of Theorem 3. Supposed that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1). After a process of limit, we can choose $\varphi_n S_n(B(u) - B(v))$ as a test function. In view of $a_i(x,t) \equiv a(x,t)$, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \varphi_n(x,t)S_n(B(u)-B(v))\frac{\partial(u-v)}{\partial t}dxdt \\
&+\sum_{i=1}^N \int_0^t \int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i} - \left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right) \\
&\cdot\left(\frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt \\
&+\sum_{i=1}^N \int_0^t \int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i} - \left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right) \\
&\cdot S_n(B(u)-B(v))\frac{\partial \varphi_n(x,t)}{\partial x_i}dxdt \\
=& -\sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u)-B(v))\left(\frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt \\
&-\sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u)-B(v))S_n(B(u)-B(v))\frac{\partial \varphi_n(x,t)}{\partial x_i}dxdt \\
&+\sum_{i=1}^N \int_0^t \int_\Omega \frac{\partial g^i(x,t)}{\partial x_i}(B(u)-B(v))S_n(B(u)-B(v))\varphi_n(x,t)dxdt.
\end{aligned} \tag{40}$$

Note that the second term in the left hand side of (40) satisfies

$$\begin{aligned}&\sum_{i=1}^{N}\int_0^t\int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right)\\&\cdot\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S'_n(B(u)-B(v))\varphi_n(x,t)dxdt\geq 0.\end{aligned}\tag{41}$$

To evaluate the third term on the left hand side of (40), we use

$$\frac{\partial\varphi_n(x,t)}{\partial x_i}=\begin{cases}0, & \text{if } x\in\Omega_{\frac{2}{n}t},\\ n\frac{\partial a(x,t)}{\partial x_i}, & \text{if } x\in\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t},\\ 0, & \text{if } x\in\Omega\setminus\Omega_{\frac{1}{n}t}.\end{cases}$$

In view of condition (14), by the straightforward calculations we can deduce that

$$\begin{aligned}&\Bigg|\sum_{i=1}^{N}\int_0^t\int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\right)\\&\cdot\frac{\partial\varphi_n(x,t)}{\partial x_i}S_n(B(u)-B(v))dxdt\Bigg|\\
\leq&\sum_{i=1}^{N}\int_0^t n\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-1}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-1}\right)\\&\cdot\left|\frac{\partial a(x,t)}{\partial x_i}S_n(B(u)-B(v))\right|dxdt\\
\leq&c\sum_{i=1}^{N}\int_0^t\left[\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right.\\&\left.+\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right]dt\\&\cdot\int_0^t n\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial a(x,t)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{p_{it}^+}}dt\\
\leq&c\sum_{i=1}^{N}\int_0^t\left[\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right.\\&\left.+\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right]dt\\&\cdot\int_0^t n^{1-\frac{1}{p_{it}^+}}\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}\left|\frac{\partial a(x,t)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{p_{it}^+}}dt\\
\leq&c\sum_{i=1}^{N}\int_0^t\left[\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right.\\&\left.+\left(\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}a(x,t)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right]dt,\\
\to&0,\ \text{as } n\to 0.\end{aligned}\tag{42}$$

It follows from Hölder's inequality and (15) that

$$\begin{aligned}
&-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt\\
=&-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))S_n'(B(u)-B(v))\\
&\cdot a(x,t)^{-\frac{1}{p_i(x,t)}}a(x,t)^{\frac{1}{p_i(x,t)}}\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)\varphi_n(x,t)dxdt\\
\leq&\sum_{i=1}^{N}\left(\int_0^t\int_\Omega\left[g^i(x,t)(B(u)-B(v))S_n'(B(u)-B(v))a(x,t)^{-\frac{1}{p_i(x,t)}}\right]^{q_i(x,t)}dxdt\right)^{\frac{1}{q_i^+}}\\
&\cdot\left(\int_0^t\int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)^{\frac{1}{p_{1i}}}\to 0,\ \text{as } n\to 0,
\end{aligned}\tag{43}$$

where $p_{1i}=p_i^+$ or p_i^- depends on whether

$$\left(\int_0^t\int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)\leq 1$$

or

$$\left(\int_0^t\int_\Omega a(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)> 1.$$

Recall the partial boundary value condition (16), i.e.

$$u(x,t)=v(x,t)=0,\ \ (x,t)\in\Sigma_p=\left\{(x,t)\in\partial\Omega\times(0,T):\sum_{i=1}^{N}g^i(x,t)\frac{\partial a(x,t)}{\partial x_i}\neq 0\right\}.$$

Then we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))S_n(B(u)-B(v))\frac{\partial\varphi_n(x,t)}{\partial x_i}dxdt\right|\\
\leq&\lim_{n\to\infty}\int_0^t n\int_{\Omega_{\frac{1}{n}t}\setminus\Omega_{\frac{2}{n}t}}|(B(u)-B(v))S_n(B(u)-B(v))|\left|\sum_{i=1}^{N}g^i(x,t)\frac{\partial a(x,t)}{\partial x_i}\right|dxdt\\
\leq& c\int_0^t\int_{\Sigma_{1t}}|B(u)-B(v)|\,d\Sigma dt\\
=&0
\end{aligned}\tag{44}$$

and

$$\left|\sum_{i=1}^{N}\int_0^t\int_\Omega\frac{\partial g^i(x,t)}{\partial x_i}(B(u)-B(v))S_n(B(u)-B(v))\varphi_n(x,t)dxdt\right|\leq c\int_0^t\int_\Omega|u-v|dxdt.\tag{45}$$

Since $B(r)\geq 0$ is monotone, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t\int_\Omega\varphi_n(x,t)S_n(B(u)-B(v))\frac{\partial(u-v)}{\partial t}dxdt\\
=&\int_0^t\int_\Omega\operatorname{sgn}(B(u)-B(v))\frac{\partial(u-v)}{\partial t}dxdt\\
=&\int_0^t\int_\Omega\operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
=&\int_\Omega|u(x,t)-v(x,t)|dx-\int_\Omega|u_0(x)-v_0(x)|dx.
\end{aligned}\tag{46}$$

By (41)-(46), letting $n\to\infty$ in (40) yields

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \leqslant \int_\Omega |u_0(x)-v_0(x)|\,dx + c\int_0^t\int_\Omega |u-v|dxdt,\ \ t\in[0,T).$$

Using Gronwall's inequality, we obtain

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \leqslant \int_\Omega |u_0(x)-v_0(x)|\,dx,\ \ t\in[0,T).$$

■

4 Weak Characteristic Method

We can generalize the method described in the preceding section to prove the stability of weak solutions. Let $\chi(x,t)$ be a nonnegative $C^1(\overline{Q_T})$ function as

$$\chi(x,t)>0,\ \text{if}\ (x,t)\in Q_T=\Omega\times(0,T),$$

and

$$\chi(x,t)=0,\ \text{if}\ (x,t)\in \Gamma_T=\partial\Omega\times[0,T).$$

If we denote

$$\chi_t=\chi(x,t),\ x\in\Omega,$$

for $t\in[0,T)$, then χ_t is the weak characteristic function of Ω as defined in [29]. Likewise, we can simply call $\chi(x,t)$ a weak characteristic function of Q_T.

For $\lambda>0$, we define

$$\varphi_n(x,t)=\begin{cases} 1, & \text{if}\ \ x\in D_{\frac{2}{n}t},\\ n(\chi(x,t)-\frac{1}{n}), & \text{if}\ \ x\in D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t},\\ 0, & \text{if}\ \ x\in\Omega\setminus D_{\frac{1}{n}t}.\end{cases}$$

Then

$$\frac{\partial\varphi_n(x,t)}{\partial x_i}=\begin{cases} 0, & \text{if}\ \ x\in D_{\frac{2}{n}t},\\ n\frac{\partial\chi(x,t)}{\partial x_i}, & \text{if}\ \ x\in D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t},\\ 0, & \text{if}\ \ x\in\Omega\setminus D_{\frac{1}{n}t},\end{cases}$$

where $D_{\lambda t}=\{x\in\Omega:\chi(x,t)>\lambda\}$.

Theorem 5 *Suppose that there is a weak characteristic function $\chi(x,t)$ of Q_T satisfying,*

$$\int_0^T n\left(\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}} a_i(x,t)\left|\frac{\partial\chi(x,t)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{p_{it}^+}}dt\le c,\ \ i=1,2,\cdots,N, \tag{47}$$

$$\int_\Omega g^i(x,t)^{q_i(x,t)}a_i(x,t)^{-\frac{1}{p_i(x,t)-1}}dx\le c(T),\ \ i=1,2,\cdots,N, \tag{48}$$

where $q_i(x,t)$, p_{it}^+ and q_{it}^+ are the same as given in Theorem 3. Suppose that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, with a partial homogeneous boundary value condition

$$u(x,t)=v(x,t)=0,\ \ (x,t)\in\left\{x\in\partial\Omega\times(0,T):\sum_{i=1}^N g^i(x,t)\frac{\partial\chi(x,t)}{\partial x_i}\neq 0\right\}. \tag{49}$$

Then we have

$$\int_\Omega |u(x,t)-v(x,t)|dx\le c\int_\Omega |u_0(x)-v_0(x)|dx,\ \ a.e.\ t\in[0,T). \tag{50}$$

Proof of Theorem 5. Choose $\varphi_n S_n(B(u)-B(v))$ as a test function. Then we have

$$\begin{aligned}
&\int_0^t\int_\Omega \varphi_n(x,t)S_n(B(u)-B(v))\frac{\partial(u-v)}{\partial t}dxdt\\
&+\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right)\\
&\cdot\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt\\
&+\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right)\\
&\cdot S_n(B(u)-B(v))\frac{\partial\varphi_n(x,t)}{\partial x_i}dxdt\\
=&-\sum_{i=1}^N\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt\\
&-\sum_{i=1}^N\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))S_n(B(u)-B(v))\frac{\partial\varphi_n(x,t)}{\partial x_i}dxdt\\
&+\sum_{i=1}^N\int_0^t\int_\Omega \frac{\partial g^i(x,t)}{\partial x_i}(B(u)-B(v))S_n(B(u)-B(v))\varphi_n(x,t)dxdt.
\end{aligned}\tag{51}$$

As discussing in the proof of Theorem 3, we have

$$\begin{aligned}
&\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(v)}{\partial x_i}\right)\\
&\cdot\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S'_n(B(u)-B(v))\varphi_n(x,t)dxdt\geq 0.
\end{aligned}\tag{52}$$

In view of condition (47), we can deduce

$$\begin{aligned}
&\left|\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\right)\right.\\
&\left.\cdot\frac{\partial\varphi_n(x,t)}{\partial x_i}S_n(B(u)-B(v))dxdt\right|\\
=&\left|\sum_{i=1}^N\int_0^t\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}} a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}-\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial B(u)}{\partial x_i}\right)\right.\\
&\left.\cdot\frac{\partial\varphi_n(x,t)}{\partial x_i}S_n(B(u)-B(v))dxdt\right|\\
\leq&\sum_{i=1}^N\int_0^t n\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}} a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)-1}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)-1}\right)\\
&\cdot\left|\frac{\partial\chi(x,t)}{\partial x_i}S_n(B(u)-B(v))\right|dxdt\\
\leq& c\sum_{i=1}^N\int_0^t\left[\left(\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}} a(x,t)\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right.\\
&\left.+\left(\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}} a(x,t)\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}dx\right)^{\frac{1}{q_{it}^+}}\right]dt\to 0,\ \text{as } n\to 0.
\end{aligned}\tag{53}$$

Similar to the derivation of (43), using Hölder's inequality and (48), we obtain

$$\begin{aligned}
&-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)S_n'(B(u)-B(v))\varphi_n(x,t)dxdt\\
=&-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))S_n'(B(u)-B(v))\\
&\cdot a_i(x,t)^{-\frac{1}{p_i(x,t)}}a_i(x,t)^{\frac{1}{p_i(x,t)}}\left(\frac{\partial B(u)}{\partial x_i}-\frac{\partial B(v)}{\partial x_i}\right)\varphi_n(x,t)dxdt \qquad (54)\\
\le&\sum_{i=1}^{N}\left(\int_0^t\int_\Omega\left[g^i(x,t)(B(u)-B(v))S_n'(B(u)-B(v))a_i(x,t)^{-\frac{1}{p_i(x,t)}}\right]^{q_i(x,t)}dxdt\right)^{\frac{1}{q_i^+}}\\
&\cdot\left(\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)^{\frac{1}{p_{1i}}}.
\end{aligned}$$

Note that the right hand side of (54) goes to 0 as $n\to 0$. Here, $p_{1i}=p_i^+$ or p_i^- depends on whether

$$\left(\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)\le 1$$

or

$$\left(\int_0^t\int_\Omega a_i(x,t)\left(\left|\frac{\partial B(u)}{\partial x_i}\right|^{p_i(x,t)}+\left|\frac{\partial B(v)}{\partial x_i}\right|^{p_i(x,t)}\right)dxdt\right)> 1.$$

In view of (49), we get

$$\begin{aligned}
&\lim_{n\to\infty}\left|-\sum_{i=1}^{N}\int_0^t\int_\Omega g^i(x,t)(B(u)-B(v))S_n(B(u)-B(v))\frac{\partial\varphi_n(x,t)}{\partial x_i}dxdt\right|\\
\le&\lim_{n\to\infty}\int_0^t n\int_{D_{\frac{1}{n}t}\setminus D_{\frac{2}{n}t}}|(B(u)-B(v))S_n(B(u)-B(v))|\left|\sum_{i=1}^{N}g^i(x,t)\frac{\partial\chi(x,t)}{\partial x_i}\right|dxdt \qquad (55)\\
\le&c\int_0^t\int_{\Sigma_{1t}}|B(u)-B(v)|\,d\Sigma dt\\
=&0.
\end{aligned}$$

Similarly, we can deduce that both (45) and (46) hold too. From (51)-(55), we arrive at the desired result (50). ■

We can see that by choosing different appropriate characteristic function of Q_T, we can obtain the corresponding stability results under various conditions. For example,

i) If we take $\chi(x,t)=\chi_{[\tau,s]}(t)\prod_{j=1}^{N}a_j(x,t)$, where $\chi_{[\tau,s]}(t)$ is the characteristic function of $[s,t]\subset(0,T)$, then

$$\frac{\partial\chi(x,t)}{\partial x_i}=\chi_{[\tau,s]}(t)\prod_{j=1}^{N}a_j(x,t)\sum_{k=1}^{N}\frac{a_{kx_i}}{a_k(x,t)},$$

$$\left|\frac{\partial\chi(x,t)}{\partial x_i}\right|^{p_i(x,t)}=\left|\chi_{[\tau,s]}(t)\prod_{j=1}^{N}a_j(x,t)\sum_{k=1}^{N}\frac{a_{kx_i}}{a_k(x,t)}\right|^{p_i(x,t)},$$

where $a_{kx_i}=\frac{\partial a_k(x,t)}{\partial x_i}$, $k=1,\,2,\,...,N$.

By virtue of Theorem 5, we obtain

Corollary 6 *Suppose that*

$$\int_0^T n^{1-\frac{p_{it}^-}{p_{it}^+}} \left(\int_{D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}} a_i(x,t) \left| \sum_{k=1}^N \frac{a_{kx_i}}{a_k(x,t)} \right|^{p_i(x,t)} dx \right)^{\frac{1}{p_{it}^+}} dt \leq c, \ i = 1, 2, \cdots, N.$$

Suppose that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, with a partial homogeneous boundary value condition

$$u(x,t) = v(x,t) = 0, \ (x,t) \in \left\{ (x,t) \in \partial\Omega \times (0,T) : \prod_{j=1}^N a_j(x,t) \sum_{i,k=1}^N \frac{g^i a_{kx_i}}{a_k(x,t)} \neq 0 \right\}.$$

Then we have the stability of weak solution in the sense of (17).

ii) If we take $\chi(x,t) = \chi_{[\tau,s]}(t) d^\alpha(x)$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function from the boundary $\partial\Omega$ and $\alpha \geq 1$ is a constant, then

$$\frac{\partial \chi(x,t)}{\partial x_i} = \alpha \chi_{[\tau,s]}(t) d^{\alpha-1}(x), \quad \left| \frac{\partial \chi(x,t)}{\partial x_i} \right|^{p_i(x,t)} = \left| \alpha \chi_{[\tau,s]}(t) d^{\alpha-1}(x) \right|^{p_i(x,t)}.$$

According to Theorem 5, we can also obtain

Corollary 7 *Suppose that*

$$\int_0^T n^{1-\frac{(\alpha-1)p_{it}^-}{p_{it}^+}} \left(\int_{D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}} a_i(x,t) dx \right)^{\frac{1}{p_{it}^+}} dt \leq c, \ i = 1, 2, \cdots, N.$$

Suppose that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, with a partial homogeneous boundary value condition

$$u(x,t) = v(x,t) = 0, \ \ (x,t) \in \left\{ x \in \partial\Omega \times (0,T) : \sum_i^N g^i(x,t) n_i \neq 0 \right\},$$

where $n = \{n_i\}$ is the outer normal vector of Ω. Then we have the stability of weak solution in the sense of (17).

5 Conclusion

In this study, we applied an analytical method to study the stability of weak solution for a doubly nonlinear anisotropic parabolic equation, where the diffusion coefficient and the variable exponent depend on the time variable t. Under certain parametric choices, it includes the heat equation, reaction-diffusion equations, non-Newtonian fluid equation and electrorheological fluid equation and the epidemic model of diseases as particular cases.

When $a_i(x,t)|_{x\in\Omega} > 0$ and $B(u)$ is a strictly monotone increasing function, it excludes the strongly degenerate hyperbolic-parabolic equation, for which only under the entropy conditions, the uniqueness of weak solution can be guaranteed [3, 14, 31]. However, only under the condition $B'(u) = b(u) \geq 0$ or $a_i(x,t)$ is degenerate in the interior of Ω, how to prove the uniqueness of weak solution to equation (1) is still an interesting and challenging problem. In addition, if there is an external forcing term $f(u) \geq 0$ in equation (1), i.e.

$$u_t = \sum_{i=1}^N \left(a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) + \sum_{i=1}^N g^i(x,t) \frac{\partial B(u)}{\partial x_i} + f(u), \ \ (x,t) \in Q_T, \tag{56}$$

we conjecture that weak solutions may blow-up in finite time. How to show such a blow-up behavior and the long time behavior of solutions to equation (56) seems more interesting and helpful from the physical and biological point of view. We will continue to work on this problem in a subsequent work.

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