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Specification property for step skew products

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Step skew products with interval fibres and a subshift as a base are considered. It is proved that if the fibre maps are continuous, piecewise monotone, expanding and surjective and the subshift has the specification property and a periodic orbit such that the composition of the fibre maps along this orbit is mixing, then the corresponding step skew product has the specification property.

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1. Introduction

An interesting topic combining nonautonomous dynamical systems, skew products and random dynamical systems is the following. Let T_1 and T_2 be two continuous selfmaps of the interval $I = [0, 1]$. If $x_0 \in I$, we decide, by tossing a coin, whether we take $x_1 = T_1(x_0)$ or $x_1 = T_2(x_0)$. After $(n-1)$ steps arriving at x_{n-1} we again decide, tossing a coin, whether we take $x_n = T_1(x_{n-1})$ or $x_n = T_2(x_{n-1})$. We thus obtain a sequence $(x_n)_{n=0}^{\infty}$. It can be viewed as the trajectory of the point x_0 in the nonautonomous system given by the sequence of maps determined by the coin tossing, each of the maps being either T_1 or T_2. Since any choice of $\omega = \omega_0 \omega_1 \omega_2 \cdots \in \Sigma_2^+ = \{1, 2\}^{\mathbb{Z}_+}$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, yields a nonautonomous system given by the sequence of maps $T_{\omega_0}, T_{\omega_1}, T_{\omega_2}, \dots$, all such nonautonomous systems are in a sense present in the skew product $(\omega, x) \mapsto (S(\omega), T_{\omega_0}(x))$ where S is the shift transformation $\Sigma_2^+ \to \Sigma_2^+$, $(S\omega)_n = \omega_{n+1}$.

As a straightforward generalization, one can consider $\Sigma_n^+ = \{1, 2, \dots, n\}^{\mathbb{Z}_+}$ and n continuous maps $T_1, T_2, \dots, T_n$. Also, instead of the full shift Σ_n^+ one can consider a subshift $B \subseteq \Sigma_n^+$. The present paper deals with the *step skew product* $F \colon B \times I \to B \times I$ defined by

$$F(\omega, x) = (S(\omega), T_\omega(x)) \tag{1}$$

where S is the shift transformation $\Sigma_n^+ \to \Sigma_n^+$ and the continuous fibre map T_ω depends only on the first (i.e. beginning) coordinate ω_0 of ω. Clearly, F is continuous.

The dynamics of step skew products with interval fibres has been studied by many authors, usually under additional assumptions on fibre maps, see e.g. the recent papers [8,7] and references therein.

In the present paper we study the specification property of step skew products (1). The specification property was introduced by Bowen [4], a chapter on it can be found in [6]. It is a very strong property; systems with this property have a dense subset of periodic points and are topologically mixing. For continuous maps on the interval, specification is equivalent with topological mixing [3,5]. The specification property for step skew products with circle rotations in the fibres was studied in [9].

Our main result is Theorem 4.1 and Corollary 4.2. However, also Theorem 3.1 used in the proof of Theorem 4.1 and Example 3.2 are of some interest.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers. Let (Y, R) be a topological dynamical system, which means that R is a continuous transformation on the compact metric space Y with metric d. The dynamical system (Y, R) has the *specification property*, if for every $\varepsilon > 0$ there is an integer $M = M(\varepsilon)$ with the following property: For any integer $k \geq 2$, and for any $(y_1, n_1), (y_2, n_2), \ldots, (y_k, n_k) \in Y \times \mathbb{N}$ there exists a point $u \in Y$ with $R^{r_k}(u) = u$ such that

$$d(R^i(y_j), R^{r_{j-1}+i}(u)) \leq \varepsilon \quad \text{for} \quad 0 \leq i < n_j \quad \text{and} \quad 1 \leq j \leq k$$

where $r_0 = 0$ and $r_j = n_1 + n_2 + \ldots + n_j + jM$ for $1 \leq j \leq k$. We call M the *gap length* for the given ε.[2] We also say that the map R itself has the specification property.

Recall that a dynamical system (X, T) or the map T itself is called *(topologically) mixing* if for every pair of nonempty open sets U and V there is a positive integer N such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. It is called *locally eventually onto* (or *topologically exact*) if for every nonempty open set U there is a positive integer n such that $T^n(U) = X$.

By an interval we mean a nondegenerate interval. The length of an interval J is denoted by $|J|$. A continuous map $T \colon [0, 1] \to [0, 1]$ is called *piecewise monotone*, if there is a finite partition $\mathcal{P}$ of $[0, 1]$ into intervals, such that $T|_P$ is monotone for all $P \in \mathcal{P}$. The endpoints of the intervals in $\mathcal{P}$ are called *critical points*. We say that T is *expanding*, if there is $\alpha > 1$ such that $|T(x) - T(y)| \geq \alpha|x - y|$ holds for all x, y which are in the same element of $\mathcal{P}$. In such a case we call α the *expansion rate*. Recall also that a continuous piecewise monotone map $T \colon [0, 1] \to [0, 1]$ is mixing if and only if it is locally eventually onto, see e.g. [2, p. 158].

Lemma 2.1. *Let T be a mixing, piecewise monotone, continuous transformation on $[0, 1]$. For every $\gamma > 0$ there is an integer m such that $T^m(U) = [0, 1]$ holds for all intervals U with $|U| \geq \gamma$.*

Proof. Let $\{U_1, \ldots, U_k\}$ be a partition of $[0, 1]$ into intervals whose lengths are smaller than $\gamma/2$. Since T is locally eventually onto, for $i = 1, \ldots, k$ there is a positive integer m_i with $f^{m_i}(U_i) = [0, 1]$. Put

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[2] Thus, $M = M(\varepsilon)$ is such that for every finite family of orbit segments, if all the gap lengths are *prescribed to be equal to M*, an ε-tracing periodic point u does exist. This is equivalent with the definition, also often used, in which $M = M(\varepsilon)$ is such that for every finite family of orbit segments, if all the gap lengths are *prescribed and greater than or equal to M*, an ε-tracing periodic point u still does exist. One implication is trivial. To prove the other one, let a system have the specification property according to the former definition, with $M = M(\varepsilon)$. To show that it has the specification property according to the latter definition, let $\varepsilon > 0$ be given. We claim that the same $M = M(\varepsilon)$ works. Indeed, let us have a finite family of orbit segments with lengths n_i and consider any prescribed gap lengths $M + L_i$ with $L_i \geq 0$. We replace the orbit segments of points y_i with lengths n_i by the orbit segments of the same points y_i with lengths $n_i + L_i$. Then, since M was taken from the former definition, an ε-tracing periodic point u exists for this new system of (longer) orbit segments and with all gap lengths equal to M. This point ε-traces also the original (shorter) orbit segments, with the gap lengths $M + L_i$, as required.

$m := \max\{m_1, \ldots, m_k\}$ and realize that if $|U| \geq \gamma$ then the interval U contains at least one of the intervals U_i. $\quad \square$

A sequence $(f_i)_{i=0}^\infty$ of maps from $[0,1]$ to $[0,1]$ is called a *nonautonomous system*. It is called *finite*, if only finitely many different maps occur in this sequence. For $i \geq 1$ and $j \geq 0$ we set $f_j^i = f_{j+i-1} \circ \cdots \circ f_{j+1} \circ f_j$ and f_j^0 denotes the identity.

Let $\Sigma_n^+ = \{1, 2, \ldots, n\}^{\mathbb{Z}^+}$ be the full n-shift with product topology and shift transformation S. Let ϱ be a metric on Σ_n^+ which generates the product topology, such that

$$\varrho(\omega, \eta) < 1 \text{ implies that } \omega \text{ and } \eta \text{ have the same first symbol.} \tag{2}$$

We define then a metric on $X = \Sigma_n^+ \times [0,1]$ by $d((\omega, x), (\eta, y)) = \max(\varrho(\omega, \eta), |x - y|)$.

Any string $w = w_1 w_2 \ldots w_k$ of elements from $\{1, 2, \ldots, n\}$ is called a *block* or a *word* of length k. Concatenation of blocks is indicated by juxtaposition; if $u = u_1 \ldots u_m$ and $v = v_1 \ldots v_n$ then $uv = u_1 \ldots u_m v_1 \ldots v_n$. We will also use the notation $w^n = w \ldots w$ (n-times) and $w^\infty = www \ldots$. Clearly, a point $\alpha \in \Sigma_n^+$ is periodic for the shift transformation if and only if $\alpha = (\alpha_0 \alpha_1 \ldots \alpha_{p-1})^\infty$ for some block $\alpha_0 \alpha_1 \ldots \alpha_{p-1}$.

If $B \subseteq \Sigma_n^+$ is nonempty, closed and invariant, i.e. $S(B) \subseteq B$, then B with S restricted to B is again a dynamical system, called a *subshift*. If no confusion can arise, the phase space B itself is called a subshift. If $w_0 w_1 w_2 \ldots$ is an element of the subshift B, then every word $w_n w_{n+1} \ldots w_{n+k}$ ($n, k \geq 0$) is called a *B-word*.

3. Finite nonautonomous systems and non-shrinking of intervals

The following theorem will be used in the proof of our main result.

Theorem 3.1. *Let finitely many expanding, piecewise monotone, continuous maps $T_1, T_2, \ldots, T_n$ from $[0,1]$ to $[0,1]$ be given. Then for every $\varepsilon > 0$ there is $\gamma > 0$ such that for any finite nonautonomous system $(f_i)_{i=0}^\infty$ containing only maps from the given finite family, and for any interval U with $|U| \geq \varepsilon$ we have $\inf_{i \geq 0} |f_0^i(U)| \geq \gamma$.*

Proof. Fix $\varepsilon > 0$. We are going to choose a suitable $\gamma > 0$. For $1 \leq j \leq n$ let $\alpha_j > 1$ be the expansion rate of the expanding map T_j. Set $\alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We have $\alpha > 1$. We fix a positive integer m such that $\alpha^m > 2$.

Set $\mathcal{F} = \{T_1, T_2, \ldots, T_n\}^m$. For any $\Delta = (H_1, H_2, \ldots, H_m) \in \mathcal{F}$ we find a number $\beta_\Delta > 0$ such that any interval $U \subseteq [0,1]$ of length $\leq \beta_\Delta$, which has a critical point of H_1 as endpoint, satisfies the condition that

the interval $G_{j-1}(U)$ has no critical point of H_j in its interior for $1 \leq j \leq m$,

where G_0 is the identity and $G_i = H_i \circ \cdots \circ H_2 \circ H_1$ for $1 \leq i \leq m$. Now we set $\beta = \min_{\Delta \in \mathcal{F}} \beta_\Delta$. Since $\mathcal{F}$ is finite we have $\beta > 0$. Finally we choose $\gamma = \min(\frac{\varepsilon}{2}, \beta)$.

Now let $(f_i)_{i=0}^\infty$ be a finite nonautonomous system consisting of the maps $T_1, T_2, \ldots, T_n$ and let U be an interval with $|U| \geq \varepsilon$. We have to show that $|f_0^i(U)| \geq \gamma$ for all $i \geq 0$.

Let $j_1 \geq 0$ be minimal such that the interval $f_0^{j_1}(U)$ contains a critical point of f_{j_1} in its interior. Since $f_0^i(U)$ does not contain a critical point of f_i in its interior for $0 \leq i < j_1$ and all maps are expanding, we get

$$|f_0^i(U)| \geq \varepsilon \geq \gamma \quad \text{for} \quad 0 \leq i \leq j_1.$$

The interval $f_0^{j_1}(U)$ has length $\geq \varepsilon$ and contains a critical point of f_{j_1} in its interior. Because of $\gamma \leq \frac{\varepsilon}{2}$ we find an interval $V_1 \subseteq f_0^{j_1}(U)$, which has a critical point of f_{j_1} as endpoint and satisfies $|V_1| = \gamma$. Let $j_2 \geq 0$ be minimal such that the interval $f_{j_1}^{j_2}(V_1)$ contains a critical point of f_{s_2} in its interior, where $s_2 = j_1 + j_2$.

Since $f_{j_1}^i(V_1)$ does not contain a critical point of f_{j_1+i} in its interior for $0 \le i < j_2$ and all maps are expanding with expansion rate at least α, we get

$$|f_0^{j_1+i}(U)| \ge |f_{j_1}^i(V_1)| \ge \alpha^i \gamma \ge \gamma \quad \text{for} \quad 0 \le i \le j_2.$$

By the choice of β and because of $\gamma \le \beta$ we get $j_2 \ge m$. This implies $|f_{j_1}^{j_2}(V_1)| \ge \alpha^m \gamma > 2\gamma$. Since the interval $f_{j_1}^{j_2}(V_1)$ contains a critical point of f_{s_2} in its interior, we find an interval $V_2 \subseteq f_{j_1}^{j_2}(V_1)$, which has a critical point of f_{s_2} as endpoint and satisfies $|V_2| = \gamma$. Now we can continue as above. Let $j_3 \ge 0$ be minimal such that the interval $f_{s_2}^{j_3}(V_2)$ contains a critical point of f_{s_3} in its interior, where $s_3 = s_2 + j_3$. Since $f_{s_2}^i(V_2)$ does not contain a critical point of f_{s_2+i} in its interior for $0 \le i < j_3$, we get as above

$$|f_0^{s_2+i}(U)| \ge |f_{s_2}^i(V_2)| \ge \alpha^i \gamma \ge \gamma \quad \text{for} \quad 0 \le i \le j_3.$$

Again we get $j_3 \ge m$ and $|f_{s_2}^{j_3}(V_2)| \ge \alpha^m \gamma > 2\gamma$. And we find an interval $V_3 \subseteq f_{s_2}^{j_3}(V_2)$, which has a critical point of f_{s_3} as endpoint and satisfies $|V_3| = \gamma$.

Repeating this procedure we finally get $|f_0^i(U)| \ge \gamma$ for all $i \ge 0$. Therefore, the theorem is proved. $\quad \square$

In Theorem 3.1 we assume that the piecewise monotone maps $T_1, \ldots, T_n$ are expanding, i.e. they have expansion rates $\alpha_j > 1$, $j = 1, \ldots, n$. We are interested in whether it works under the weaker assumption that $|T_j(x) - T_j(y)| \ge |x - y|$ holds for every $j = 1, \ldots, n$ and for all x, y which are in the same interval of monotonicity of T_j. The next example shows that this is not the case, even if the maps are surjective and piecewise linear.

Example 3.2 (*Theorem 3.1 does not work if $\alpha_j \ge 1$ instead of $\alpha_j > 1$, $j = 1, \ldots, n$*). We are going to find a finite nonautonomous system $(f_i)_{i=0}^\infty$ on $[0, 1]$ made of three piecewise linear, surjective, continuous maps φ, f and g with slopes (in absolute value) ≥ 1 such that for some nondegenerate interval J, $\lim_{n \to \infty} |f_0^n(J)| = 0$.

We will use the following notation for "connect the dots" maps. Let $0 = a_0 < a_1 < \cdots < a_n = 1$ and $b_i \in I = [0, 1]$, $i = 0, 1, \ldots, n$. Then by $\langle (a_0, b_0), \ldots, (a_n, b_n) \rangle$ we will denote the map $I \to I$ which sends a_i to b_i, $i = 0, 1, \ldots, n$ and is linear on each of the intervals $[a_i, a_{i+1}]$, $i = 0, 1, \ldots, n - 1$.

To construct the system, fix a small positive irrational number ξ; in fact any irrational ξ with $0 < \xi < 1/4$ is good for our purposes. Consider the following piecewise linear maps:

$$\varphi = \langle (0, \xi), (1 - \xi, 1), (1, 0) \rangle,$$
$$f = \langle (0, 1), (1 - 2\xi, 0), (1, 2\xi) \rangle,$$
$$g = \langle (0, 1), (1/4, 1/4), (1/2, 0), (1, 1/2) \rangle .$$

All the slopes of φ, f and g are (in absolute value) ≥ 1. Put $J = [0, \xi]$. The system $(f_i)_{i=0}^\infty$ will have the form

$$(f_i)_{i=0}^\infty = \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_1}, \psi_1, \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_2}, \psi_2, \underbrace{\varphi, \varphi, \ldots, \varphi}_{k_3}, \psi_3, \ldots \tag{3}$$

where each ψ_i is either f or g. We choose

$$k_1 = \min\{k : \varphi^k(J) \text{ contains } 1/2 \text{ in its closed middle third or intersects } [1 - \xi, 1]\} .$$

Note that the interval $\varphi^{k_1}(J)$ is obtained from J by translation by $k_1 \xi$ and so has length ξ. If $\varphi^{k_1}(J)$ contains $1/2$ in its closed middle third, we choose $\psi_1 = g$ and then $\psi_1(\varphi^{k_1}(J))$ is an interval whose left endpoint is

0 and has length $\leq (2/3)\xi = (2/3)|J|$. Otherwise the interval $\varphi^{k_1}(J)$ intersects $[1 - \xi, 1]$ and thus it is a subset of $[1 - 2\xi, 1]$. We choose $\psi_1 = f$. Then $\psi_1(\varphi^{k_1}(J))$ is a subinterval of $[0, 2\xi]$ and has length ξ; it is obtained from $\varphi^{k_1}(J)$ by translation by $2\xi - 1$.

In either case we have that $J_1 = \psi_1(\varphi^{k_1}(J))$ is a subinterval of $[0, 2\xi]$ and we choose

$$k_2 = \min\{k : \varphi^k(J_1) \text{ contains } 1/2 \text{ in its closed middle third or intersects } [1 - \xi, 1]\} .$$

Again, if $\varphi^{k_2}(J_1)$ contains $1/2$ in its closed middle third, we choose $\psi_2 = g$ and then

$$\left|\psi_2(\varphi^{k_2}(J_1))\right| \leq \frac{2}{3}\left|\varphi^{k_2}(J_1)\right| = \frac{2}{3}|J_1| .$$

Otherwise the interval $\varphi^{k_2}(J_1)$ intersects $[1 - \xi, 1]$ and we choose $\psi_2 = f$. Then the interval $\psi_2(\varphi^{k_2}(J_1)) \subseteq [0, 2\xi]$, obtained from $\varphi^{k_2}(J_1)$ by translation by $2\xi - 1$, has the same length as J_1. We continue in the same way with the interval $J_2 = \psi_2(\varphi^{k_2}(J_1))$, etc. By induction we get the system (3).

Note that when we iterate the interval $J = [0, \xi]$ under this system, its length does not change except of the moments when we apply $\psi_i = g$, when the length decreases at least by factor $2/3$. We claim that $\psi_i = g$ for infinitely many i's, which immediately implies that $\lim_{n \to \infty} |f_0^n(J)| = 0$.

Suppose, on the contrary, that $\psi_i = g$ only for finitely many i's. Then for large enough m the interval J_m (which is a subset of $[0, 2\xi]$ and has length at most ξ) has the following property: We have $J_m = f_0^n(J)$ for some $n = n(m)$ and the middle thirds of the (nondegenerate) intervals

$$J_m = f_0^n(J), \ f_0^{n+1}(J), \ f_0^{n+2}(J), \ \ldots \tag{4}$$

do not contain $1/2$. Therefore each of the intervals $f_0^{n+i}(J)$, $i = 1, 2, \ldots$, is obtained from the previous interval in (4) by translation, either by ξ or by $2\xi - 1$. By identifying 0 and 1 in $[0, 1]$, we get a circle. Then our iterative process, assigning the sequence (4) to J_m, corresponds to the iterating J_m by the irrational circle rotation by the angle $2\pi\xi$ except of the moments when our interval is ξ-close to 1 "from the left", when we use the rotation by $2\pi \cdot 2\xi$. However, the rotation by $2\pi \cdot 2\xi$ is the same as to use the rotation by $2\pi\xi$ twice. Since $\xi \in (0, 1/4)$, during the first of these two applications of the rotation by $2\pi\xi$ our interval does not hit $1/2$. Therefore, using the fact that the middle thirds of the intervals in (4) do not contain $1/2$, we conclude that the middle third of J_m never hits $1/2$ under the circle rotation by the angle $2\pi\xi$. This contradicts the irrationality of ξ (irrational rotations are minimal). $\quad\blacksquare$

4. Main result

Our main result is the following sufficient condition for the step skew product (1) to have the specification property.

Theorem 4.1. *Let $T_1, T_2, \ldots, T_n$ be piecewise monotone, continuous maps on $[0, 1]$, which are expanding and surjective. Suppose that $B \subseteq \Sigma_n^+$ is a subshift which has the specification property and contains a periodic point $\alpha = (\alpha_0 \alpha_1 \ldots \alpha_{p-1})^\infty$ such that $T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0}$ is mixing. Let $F : X \to X$ be the step skew product with $X = B \times [0, 1]$ and $F(\omega, x) = (S(\omega), T_\omega(x))$, where $S : B \to B$ is the shift transformation and $T_\omega = T_q$, if q is the first symbol of ω. Then (X, F) has the specification property.*

Proof. Fix $\varepsilon \in (0, 1)$. We have to show that there is an integer M which has the following property: For any integer $k \geq 2$, and for any

$$(\omega_1, x_1, n_1), (\omega_2, x_2, n_2), \ldots, (\omega_k, x_k, n_k) \in B \times [0, 1] \times \mathbb{N}$$

there exists a point $(\eta, z) \in B \times [0, 1]$ with $F^{r_k}(\eta, z) = (\eta, z)$ such that

$$d(F^i(\omega_j, x_j), F^{r_{j-1}+i}(\eta, z)) \le \varepsilon \quad \text{for } 0 \le i < n_j \text{ and } 1 \le j \le k$$

where $r_0 = 0$ and $r_j = n_1 + n_2 + \cdots + n_j + jM$ for $1 \le j \le k$.

In order to show this, we first choose M, which depends only on the fixed $\varepsilon > 0$. For the maps $T_1, T_2, \ldots, T_n$ and the fixed ε let $\gamma > 0$ be as in Theorem 3.1. For this γ and for the transformation $T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0}$ choose m as in Lemma 2.1. For the given ε let K be the gap length of the dynamical system (B, S) which has the specification property. Then set $M = mp + 2K$.

Since the dynamical system (B, S) has specification property (with K being the gap length for the given ε), for

$$(\omega_1, n_1), (\alpha, mp), (\omega_2, n_2), (\alpha, mp), \ldots, (\omega_k, n_k), (\alpha, mp) \in B \times \mathbb{N}$$

there exists a point $\eta \in B$ such that

$$S^{r_k}(\eta) = \eta \quad \text{and} \quad \varrho(S^i(\omega_j), S^{r_{j-1}+i}(\eta)) \le \varepsilon \quad \text{for } 0 \le i < n_j \text{ and } 1 \le j \le k \tag{5}$$

where $r_0 = 0$ and $r_j = n_1 + n_2 + \cdots + n_j + jM$ for $1 \le j \le k$ are as above, and such that

$$\varrho(S^i(\alpha), S^{r_{j-1}+n_j+K+i}(\eta)) \le \varepsilon \quad \text{for } 0 \le i < mp \text{ and } 1 \le j \le k. \tag{6}$$

Therefore $\eta \in B$ is already found with the desired properties.

Now let $(f_i)_{i=0}^{\infty}$ be the nonautonomous system determined by this point $\eta \in \Sigma_n^+$, this means $f_i = T_q$, if q is the symbol in the i-th place in η (we start counting with 0). Remember that for $i \ge 1$ and $n \ge 0$ we set $f_n^i = f_{n+i-1} \circ \cdots \circ f_{n+1} \circ f_n$ and f_n^0 denotes the identity. Now set

$$\widetilde{J}_j = \{y \in [0, 1] : |f_{r_{j-1}}^i(x_j) - f_{r_{j-1}}^i(y)| \le \varepsilon \text{ for } 0 \le i < n_j\} \quad \text{for } 1 \le j \le k.$$

Let J_j be the connected component of $\widetilde{J}_j$ which contains the point x_j.

Since the maps f_n^i are continuous, for every $1 \le j \le k$ we get that the set J_j is a closed interval which is a neighbourhood of x_j (in the topology of $[0, 1]$). Furthermore, for $1 \le j \le k$ there is $i < n_j$ with $|f_{r_{j-1}}^i(J_j)| \ge \varepsilon$ (otherwise, by continuity, an interval properly containing J_j would be a component of $\widetilde{J}_j$, a contradiction). In the following we set $s_j = n_j + M$ for $1 \le j \le k$. We have then $r_0 = 0$ and $r_j = r_{j-1} + s_j$ for $1 \le j \le k$.

We start with J_1. Because of $|f_0^i(J_1)| \ge \varepsilon$ for some $i < n_1$ we have $|f_0^{n_1+K}(J_1)| \ge \gamma$ by Theorem 3.1. By (6) and (2) we have $f_{n_1+K}^{mp} = U^m$ with $U = T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0}$. By Lemma 2.1 we get then $f_{n_1+K}^{mp} \circ f_0^{n_1+K}(J_1) = [0, 1]$, which means $f_0^{n_1+K+mp}(J_1) = [0, 1]$. Since the fibre maps are surjective and we have $n_1 + K + mp < n_1 + M = s_1$, this implies $f_0^{s_1}(J_1) = [0, 1]$. We find an interval $K_1 \subseteq J_1$ with $f_0^{r_1}(K_1) = J_2$, where we have used that $s_1 = r_1$.

Now we consider J_2. In the same way as above we get $f_{r_1}^{s_2}(J_2) = [0, 1]$. Because of $f_{r_1}^{s_2} \circ f_0^{r_1} = f_0^{r_2}$ this implies $f_0^{r_2}(K_1) = [0, 1]$. We find an interval $K_2 \subseteq K_1 \subseteq J_1$ with $f_0^{r_2}(K_2) = J_3$.

Next we consider J_3. As above we get $f_{r_2}^{s_3}(J_3) = [0, 1]$. Because of $f_{r_2}^{s_3} \circ f_0^{r_2} = f_0^{r_3}$ this implies $f_0^{r_3}(K_2) = [0, 1]$. We find an interval $K_3 \subseteq K_2 \subseteq K_1 \subseteq J_1$ with $f_0^{r_3}(K_3) = J_4$.

Finally we end with intervals $K_{k-1} \subseteq K_{k-2} \subseteq \cdots \subseteq K_1 \subseteq J_1$ satisfying $f_0^{r_k}(K_{k-1}) = [0, 1]$ and $f_0^{r_{j-1}}(K_{j-1}) = J_j$ for $2 \le j \le k$. Because of $f_0^{r_k}(K_{k-1}) = [0, 1]$ we find a point $z \in K_{k-1}$ with $f_0^{r_k}(z) = z$. Furthermore, we have $z \in J_1$ and for $2 \le j \le k$ we have $f_0^{r_{j-1}}(z) \in J_j$, since $z \in K_{j-1}$ and $f_0^{r_{j-1}}(K_{j-1}) = J_j$. By the definition of the sets J_j this implies

$$|f^i_{r_{j-1}}(x_j) - f^{r_{j-1}+i}_0(z)| \leq \varepsilon \quad \text{for} \quad 0 \leq i < n_j \text{ and } 1 \leq j \leq k.$$

Together with (5) this gives using the definition of the nonautonomous system $(f_i)_{i=0}^\infty$ that $F^{r_k}(\eta, z) = (\eta, z)$ and that

$$d(F^i(\omega_j, x_j), F^{r_{j-1}+i}(\eta, z)) \leq \varepsilon \quad \text{for} \quad 0 \leq i < n_j \text{ and } 1 \leq j \leq k.$$

The theorem is proved. $\quad\square$

Corollary 4.2. *Let $T_1, T_2, \ldots, T_n$ be piecewise monotone, continuous maps on $[0, 1]$, which are expanding and mixing. Suppose that $B \subseteq \Sigma_n^+$ is a subshift which has a fixed point. Let $F : X \to X$ be the corresponding step skew product as in Theorem 4.1. Then (X, F) has the specification property if and only if the subshift (B, S) has the specification property.*

Proof. One implication is trivial, because the specification property is preserved by passing to a factor. Now assume that the subshift (B, S) has the specification property. Since we assume that the (piecewise monotone) fibre maps are mixing (hence, they are also surjective) and that S has a fixed point in B, all the assumptions of Theorem 4.1 are fulfilled. Thus (X, F) has the specification property. $\quad\square$

Remark 4.3. Recall that, by [1], a subshift B has the specification property if and only if it has a *uniform transition length*, meaning that there exists a positive integer M such that for any B-words u and v there exists a B-word w of length M such that uwv is a B-word.

For subshifts of finite type, the specification property is equivalent to mixing, see [4] or [6, Proposition 21.2 and Proposition 21.3]. For the specification property of sofic systems see [10].

Acknowledgment

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