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Approximation of a class of functional differential equations with wideband noise perturbations

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ABSTRACT

This work focuses on functional differential equations subject to wideband noise perturbations. Modeling using a white noise is often an idealization of the actual physical process, whereas a wideband noise can be easily realized in applications and well approximates a white noise. Using functional derivatives together with the combined perturbed test function methods and martingale techniques, this paper demonstrates that when a small parameter tends to zero, the underlying process converges to a limit that is the solution of a stochastic functional differential equation. To illustrate, an integro-differential system with wideband noise perturbation is examined as an example. Not only are the results interesting from a mathematical point of view, but also they are of utility to a wide range of applications.

1. Introduction and motivation

Time delay and uncertainty due to random fluctuations are unavoidable in a wide range of real-world applications such as process control, automotive systems, biomedical sciences, epidemics models, transport, communication networks, and population dynamics. When the random disturbances are modeled by white noises, the systems are often described by stochastic delay or functional differential equations, for example, [11,16,17]. Nevertheless, a white noise model is often only an idealization of the actual physical process. It is more appropriate to use a process that can be realized in applications. A wideband noise is such a process,

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whose bandwidth is "wide" and whose limit becomes the white noise. Using diffusion approximation to treat physical random processes has received a great deal of attentions; see [7,14,15,20] and also the more analytic approaches in [9,10]. When a delay system is perturbed by a wideband noise, one deals with a class of random delay or functional differential equations. Although it is more realistic, systems with wideband noise are often difficult to deal with. However, through appropriate limit procedures, we may obtain a much simpler limit dynamic system; see earlier work in [3,7,14] and the references therein. When time delay and wideband noise are considered simultaneously, one aims to show that under appropriate scaling, the underlying system converges to a delay differential equation or stochastic delay differential equations. Using these simpler limit systems as a bridge, we can proceed to design feasible procedures to treat the original systems.

In this paper, we confirm that the limit systems are stochastic functional differential equations. For a long time, there were no bona fide operators associated with stochastic delay equations driven by a Brownian motion. In addition, there were no bona fide Itô formulas either, except some convenient way of the use of formulas in a symbolic form [16], or a Banach space form of calculus [17]. However, the form of operator and the formulas in [17] are very difficult to use in any real applications. Thanks to the recent advances in stochastic calculus, a new form of functional Itô formula was obtained recently by Dupire [5], which enables us to examine stochastic delay equations from a new angle. In our recent work [22], based on Dupire's functional Itô formula, we examined functional diffusions with two-time scales in which the slow-varying process includes path-dependent functionals and the fast-varying process is a rapidly-changing diffusion; one of the motivations is gene expression of biochemical reactions occurring in living cells (see a motivational example in [23]).

The recent development on stochastic functional differential equations alleviates much of the difficulties and provides technical tools. It helps us for our study of the wideband noise systems. In this paper, we consider the following functional differential equation with the wideband noise perturbation

$$\dot{x}^{\varepsilon}(t) = \varphi(x^{\varepsilon}(t), x^{\varepsilon}_t, \xi^{\varepsilon}(t)) + \varepsilon^{-1}\psi(x^{\varepsilon}(t), x^{\varepsilon}_t, \xi^{\varepsilon}(t)), \tag{1.1}$$

with a deterministic initial value $x(0) \in \mathbb{R}^n$, where ε is a small positive parameter, and $\xi^{\varepsilon}(t)$ is the wideband noise given by

$$\xi^{\varepsilon}(t) = \xi(t/\varepsilon^2), \tag{1.2}$$

where $\xi(\cdot)$ is an m-dimensional stationary ϕ-mixing process. Assume throughout the paper that $\xi(\cdot)$ is a bounded, right continuous, and stationary ϕ-mixing process with mean $\mathbb{E}\xi(t) = 0$. More precise conditions will be given in the subsequent sections. In this paper, we denote $x^{\varepsilon}_t := \{x^{\varepsilon}(u \wedge t) : 0 \le u \le T\}$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)' : \mathbb{R}^n \times D([0,T]; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$, $\psi = (\psi_1, \psi_2, \dots, \psi_n)' : \mathbb{R}^n \times D([0,T]; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$, where z' denotes the transpose of z and $D([0,T]; \mathbb{R}^n)$ denotes the space of càdlàg (right continuous with left limits) functions on $[0,T]$ with values in $\mathbb{R}^n$ endowed with the Skorohod topology.

In the above functional differential equation, x_t is known as a path-dependent process reflecting the past dependence. It is well-known that the path-dependent functional differential equations include many important classes of delay systems such as the integro-differential equations motivated by, for example, the following Lotka-Volterra integro-differential equation

$$\dot{x}(t) = \operatorname{diag}(x_1(t), \dots, x_n(t))\Big[A(t) - D(t)x(t) - \int_0^t \kappa(t-s)x(s)ds\Big] \tag{1.3}$$

in population dynamics (see [8,21,24]), where $\operatorname{diag}(x_1, \dots, x_n)$ denotes the diagonal matrix with the given diagonal entries, $A(t)$ and $D(t)$ are continuous $n \times n$ matrix-valued functions, and $\kappa(\cdot)$ is an appropriate kernel function with compatible dimensions. As another class of path-dependent functional differential equations, we consider the so-called running maximum equation given by

$$\dot{x}(t) = ax(t) + b \sup_{0 \le s \le t} x(s),$$

which is used in the biological models and the financial market formulations widely.

Our main goal in this paper is to examine the asymptotic properties of system (1.1) as $\varepsilon \to 0$ and establish an approximation theorem for $x^\varepsilon(\cdot) \in D([0,T];\mathbb{R}^n)$, the space of functions that are right continuous with left limits endowed with the Skorohod topology (see [13,14] and references therein). We show that under suitable conditions, $x^\varepsilon(\cdot)$ converges weakly to a functional diffusion process. To deal with the terms involving functional dependence, this paper uses the idea of functional derivative in [1,4,5]. To the best of our knowledge, this is the first attempt to study the asymptotic averaging property of the delay or functional equations with the wideband noise by using the functional derivative. For previous works on martingale methods, weak convergence, and treatment of delay equations, we refer to [14,19,25].

The rest of the paper is arranged as follows. We begin with some notation and preliminary lemmas in the next section. In Section 3, we examine functional derivatives and establish a martingale theorem for random functional processes. Based on the functional derivatives and the martingale theorem of random functional processes, Section 4 establishes a weak convergence result for solution to (1.1) as $\varepsilon \to 0$. Section 5 applies the established theorem to the integro-differential equation with wideband noise perturbation as a specific class. This section also establishes the approximation for the integro-differential Lotka-Volterra system with wideband noise perturbation as an example.

2. Preliminaries, notation, and assumptions

Throughout the paper, unless otherwise specified, we use the following notation. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the Euclidean norm $|\cdot|$, and $\mathcal{B}(\mathbb{R}^n)$ is the Borel sets of $\mathbb{R}^n$. For each $N > 0$, let $S_N = \{x : |x| \le N\}$ be the ball with radius N centered at the origin. For a vector or matrix A, denote its transpose by A'; for a matrix A, denote its trace norm by $|A| = \sqrt{\mathrm{Tr}(A'A)}$. Denote by $C^l(\mathbb{R}^n;\mathbb{R})$ the family of real-valued functions defined on $\mathbb{R}^n$ whose partial derivatives up to the lth order are continuous, and by $C_0^l(\mathbb{R}^n;\mathbb{R})$ the family of $C^l(\mathbb{R}^n;\mathbb{R})$ functions with compact support. Throughout the paper, K denotes a generic positive constant, whose value may change for different usage. Thus, $K + K = K$ and $KK = K$ are understood in an appropriate sense. We use $\varepsilon > 0$ to represent a small parameter.

Remark 2.1. In this paper, since the stochastic process $x^\varepsilon(\cdot)$ has deterministic initial data and is driven by $\xi^\varepsilon(\cdot)$, we are dealing with the so-called exogenous noise. Thus, we denote by $\mathcal{F}_t^\varepsilon$ the σ-algebra generated by $\xi^\varepsilon(s)$ for $0 \le s \le t$. That is, $\mathcal{F}_t^\varepsilon = \mathcal{F}_t^{\xi^\varepsilon} := \sigma\{\xi^\varepsilon(s) : 0 \le s \le t\}$, which is the same as $\sigma\{\xi(s) : 0 \le s \le t/\varepsilon^2\} =: \mathcal{F}^\xi_{t/\varepsilon^2}$. In other words, $\mathcal{F}_t^\varepsilon = \mathcal{F}_t^{\xi^\varepsilon} = \mathcal{F}^\xi_{t/\varepsilon^2}$. We denote by $\mathbb{E}_t^\varepsilon$ or $\mathbb{E}_t^{\xi^\varepsilon}$, and $\mathbb{E}^\xi_{t/\varepsilon^2}$ the conditional expectations conditioned on $\mathcal{F}_t^{\xi^\varepsilon}$ and $\mathcal{F}^\xi_{t/\varepsilon^2}$, respectively.

Let $\mathscr{M}$ denote the set of real-valued progressively measurable functions that are nonzero only on a bounded t-interval and

$$\bar{\mathscr{M}}^\varepsilon = \Big\{ f \in \mathscr{M} : \sup_t \mathbb{E}|f(t)| < \infty \text{ and } f(t) \text{ is } \mathcal{F}_t^\varepsilon\text{-measurable} \Big\}. \tag{2.1}$$

Similar to [12,14], let us give the definitions of the p-lim and the infinitesimal operator $\hat{\mathcal{L}}^\varepsilon$ as follows.

Definition 2.1. Let $f, f^\delta \in \bar{\mathscr{M}}^\varepsilon$ for each $\delta > 0$. We say $f = \text{p}-\lim_\delta f^\delta$ if and only if

$$\begin{cases} \sup\limits_{t,\delta} \mathbb{E}|f^\delta(t)| < \infty, \\ \lim\limits_{\delta\to 0} \mathbb{E}|f^\delta(t) - f(t)| = 0 \quad \text{for each} \quad t. \end{cases}$$

Similarly, we say p-$\lim_\varepsilon f^\varepsilon = 0$ if $f(\cdot) = 0$ almost surely, where $f^\varepsilon \in \bar{\mathscr{M}}^\varepsilon$ for each $\varepsilon > 0$, and ε replaces δ.

Definition 2.2. We say that $f(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^\varepsilon)$, the domain of $\hat{\mathcal{L}}^\varepsilon$, and $\hat{\mathcal{L}}^\varepsilon f = g$ if $f, g \in \bar{\mathscr{M}}^\varepsilon$ and

$$\text{p-}\lim_{\delta\downarrow 0} \Big(\frac{\mathbb{E}_t^\varepsilon f(t+\delta) - f(t)}{\delta} - g(t) \Big) = 0.$$

It follows that $\hat{\mathcal{L}}^\varepsilon$ is a type of infinitesimal operator. Although the process might be non-Markovian, the following lemma was proved by Kurtz in [12] (see also [14, p.39]).

Lemma 2.2. *If* $f \in \mathscr{D}(\hat{\mathcal{L}}^\varepsilon)$, *then*

$$M_f^\varepsilon(t) = f(t) - \int_0^t \hat{\mathcal{L}}^\varepsilon f(u)du$$

is a martingale, and

$$\mathbb{E}_t^\varepsilon f(t+s) - f(t) = \int_t^{t+s} \mathbb{E}_t^\varepsilon \hat{\mathcal{L}}^\varepsilon f(u)du \quad \textit{w.p.1.}$$

In our setup, the noise process in (1.1) is wideband. It is known that a wideband noise is one such that it approximates the "white noise". In fact, ϕ-mixing process is a large class of such process. We recall the definition next.

Definition 2.3. Let $\mathcal{F}^{\xi_t^s}$ denote the smallest σ-algebra that measures $\{\xi(u) : t \le u \le s\}$. If there is a function $\phi(t) \to 0$ as $t \to \infty$ such that

$$\sup_{A\in\mathcal{F}^{\xi_{t+s}^\infty}, B\in\mathcal{F}^{\xi_0^t}} |\mathbb{P}(A|B) - \mathbb{P}(A)| \le \phi(s),$$

then $\xi(\cdot)$ is said to be ϕ-mixing with mixing rate $\phi(\cdot)$.

In this paper, we assume that $\xi(\cdot)$ is a ϕ-mixing process with mixing rate $\phi(\cdot)$ satisfying (e.g., [2,14])

$$\int_0^\infty \phi^{1/2}(t)dt < \infty.$$

Now let us recall certain properties for ϕ-mixing processes, which constitute a large class of processes having *decreasing dependence* property. The following lemma is a modified version of [14, Chapter 4, Lemma 4].

Lemma 2.3. *Let $\xi(\cdot)$ be a ϕ-mixing process with mixing rate $\phi(\cdot)$, and $h(\cdot)$, $h_1(\cdot)$, and $h_2(\cdot)$ be functions of ξ, which are bounded and measurable on $\mathcal{F}^{\xi_t^\infty} = \sigma\{\xi(\tau), t \le \tau < \infty\}$. Then there exist κ_i for $i = 1, 2, 3$ such that*

$$|\mathbb{E}(h(\xi(t+s))|\mathcal{F}^{\xi_0^t}) - \mathbb{E}h(\xi(t+s))| \le \kappa_1\phi(s), \tag{2.2}$$

$$|\mathbb{E}(h_1(\xi(u))h_2(\xi(v))|\mathcal{F}^{\xi_0^t}) - \mathbb{E}h_1(\xi(u))h_2(\xi(v))| \le \begin{cases} \kappa_2\phi(v-u) \\ \kappa_3\phi(u-t) \end{cases} \quad \text{for } \ t < u < v, \tag{2.3}$$

where $\mathcal{F}^{\xi_0^t} = \sigma\{\xi(s); 0 \le s \le t\}$. Hence, the right-hand side of (2.3) is bounded above by

$$(\kappa_2 \vee \kappa_3)\phi^{1/2}(v-u)\phi^{1/2}(u-t).$$

Proof. The proof of inequality (2.2) is as in [14, Chepter 4, Lemma 4], and (2.3) improves the corresponding result of the aforementioned lemma. Noting that $h_i(\cdot)$ are bounded,

$$\begin{aligned}
&|\mathbb{E}(h_1(\xi(u))h_2(\xi(v))|\mathcal{F}^{\xi_0^t}) - \mathbb{E}h_1(\xi(u))h_2(\xi(v))| \\
&\quad \le |\mathbb{E}[h_1(\xi(u))(\mathbb{E}(h_2(\xi(v))|\mathcal{F}^{\xi_0^u}) - \mathbb{E}h_2(\xi(v)))|\mathcal{F}^{\xi_0^t}]| \\
&\qquad + |\mathbb{E}h_1(\xi(u))(\mathbb{E}(h_2(\xi(v))|\mathcal{F}^{\xi_0^u}) - \mathbb{E}h_2(\xi(v)))| \\
&\quad \le 2K_1\kappa_1\phi(v-u),
\end{aligned}$$

where K_1 is the bound of $h_1(\cdot)$. On the other hand, define $g(\xi(u)) = h_1(\xi(u))\mathbb{E}(h_2(\xi(v))|\mathcal{F}^{\xi_0^t})$. Then $g(\cdot)$ is also a function with the $h(\cdot)$'s properties. Applying the inequality (2.2) yields that there exists a κ_3 such that

$$\begin{aligned}
&|\mathbb{E}(h_1(\xi(u))h_2(\xi(v))|\mathcal{F}^{\xi_0^t}) - \mathbb{E}h_1(\xi(u))h_2(\xi(v))| \\
&\quad = |\mathbb{E}[h_1(\xi(u))(\mathbb{E}(h_2(\xi(v))|\mathcal{F}^{\xi_0^u}))|\mathcal{F}^{\xi_0^t}] - \mathbb{E}[h_1(\xi(u))(\mathbb{E}(h_2(\xi(v))|\mathcal{F}^{\xi_0^u}))]| \\
&\quad = |\mathbb{E}(g(\xi(u))|\mathcal{F}^{\xi_0^t}) - \mathbb{E}g(\xi(u))| \\
&\quad \le \kappa_3\phi(u-t).
\end{aligned}$$

The proof is completed by choosing $\kappa_2 = 2K\kappa_1$. □

3. Functional derivative

To examine the weak convergence using the martingale averaging method, we need to consider $\hat{\mathcal{L}}^\varepsilon f$ for $f(\cdot)$ with appropriate properties. Since functionals are considered, it is necessary to consider the derivative of the functionals in the form of $\overline{V}^\varepsilon(t) = V(t, x^\varepsilon(t), x_t^\varepsilon)$. To proceed, we need to examine the derivative for the functional $V(t, x, y)$ on $[0, T] \times \mathbb{R}^n \times D([0, T]; \mathbb{R}^n)$. Now let us define continuity for functionals first; see [1]. We use $\|x_t\|_\infty := \sup_{u\in[0,T]}\{|x(t \wedge u)| : 0 \le u \le T\}$; see for example, [18, Chapter V].

Definition 3.1 *(Joint continuity in (t, x, y)).* A continuous functional is a continuous map $V : [0, T] \times \mathbb{R}^n \times D([0, T]; \mathbb{R}^n) \to \mathbb{R}$ if, for any $(t, x, y) \in [0, T] \times \mathbb{R}^n \times D([0, T]; \mathbb{R}^n)$ and any $\Delta > 0$, there exists an $\eta > 0$ such that for any $(\widetilde{t}, \widetilde{x}, \widetilde{y}) \in [0, T] \times \mathbb{R}^n \times D([0, T]; \mathbb{R}^n)$ satisfying

$$d_\infty((t, x, y), (\widetilde{t}, \widetilde{x}, \widetilde{y})) = |t - \widetilde{t}| + |x - \widetilde{x}| + \|y - \widetilde{y}\|_\infty < \eta,$$

we have

$$|V(t,x,y) - V(\widetilde{t},\widetilde{x},\widetilde{y})| < \Delta.$$

The set of continuous functionals is denoted by $\mathbb{C}^{0,0,0}([0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n);\mathbb{R})$.

Next, we introduce the "local boundedness" for functionals. We call a functional V "boundedness preserving" if it is bounded on each bounded set of paths [1,4]. The precise definition is given below.

Definition 3.2. A functional $V : [0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n) \to \mathbb{R}$ is said to be boundedness preserving if for any compact $G \in \mathbb{R}^n$ and $t_0 < T$, there exists a $K_{G,t_0} > 0$, such that for all $t \leq t_0$, $x \in G$ and $y \in D([0,T];G)$, we have $|V(t,x,y)| \leq K_{G,t_0}$.

Following [1,4], let us give the definitions of *horizontal* and *vertical* derivatives. Denote $(e_i, i = 1,\cdots,n)$ the canonical basis in $\mathbb{R}^n$.

Definition 3.3. A non-anticipative functional $V : [0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n) \to \mathbb{R}$ is said to be horizontally differentiable at $(t,x,y) \in [0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n)$ if the limit

$$\mathcal{D}V(t,x,y) = \lim_{\delta \downarrow 0} \frac{V(t+\delta,x,y) - V(t,x,y)}{\delta}$$

exists. In such a case, $\mathcal{D}V(t,y,x)$ is called the horizontal derivative of V at (t,x,y).

Definition 3.4. For $x(t) \in \mathbb{R}^n$ and $x_t = \{x(u \wedge t) : 0 \leq u \leq T\}$, a non-anticipative functional $V : [0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n) \to \mathbb{R}$ is said to be vertically differentiable at $(t,x(t),x_t) \in [0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n)$ if the functional map

$$\begin{aligned}&\mathbb{R}^n \to \mathbb{R},\\ &e \mapsto V(t,x(t)+e,x_t+e\mathbf{1}_{[t,T]})\end{aligned}$$

is differentiable at 0. Its gradient at 0 is called the vertical derivative of V at $(t,x(t),x_t)$:

$$\nabla V(t,x(t),x_t) = (\nabla_1 V(t,x(t),x_t), \nabla_2 V(t,x(t),x_t), \dots, \nabla_n V(t,x(t),x_t)),$$

where

$$\nabla_i V(t,x(t),x_t) = \lim_{h \to 0} \frac{V(t,x(t)+he_i,x_t+he_i\mathbf{1}_{[t,T]}) - V(t,x(t),x_t)}{h}.$$

Remark 3.1. In view of the definitions above, although x_t may be a continuous function if $x(t)$ is a continuous process, it is obvious that $x_t + e\mathbf{1}_{[t,T]}$ is right continuous and has the left limit, that it, it is in $D([0,T];\mathbb{R}^n)$. Thus we need to have V be defined on $[0,T] \times \mathbb{R}^n \times D([0,T];\mathbb{R}^n)$.

Let us define

$$\begin{aligned}V_{x_i}(t,x(t),x_t) &= \lim_{h \to 0} \frac{V(t,x(t)+he_i,x_t) - V(t,x(t),x_t)}{h},\\ \partial_i V(t,x(t),x_t) &= \lim_{h \to 0} \frac{V(t,x(t),x_t+he_i\mathbf{1}_{[t,T]}) - V(t,x(t),x_t)}{h}\end{aligned}$$

and

$$V_x(\cdot) = (V_{x_1}(\cdot), V_{x_2}(\cdot), \dots, V_{x_n}(\cdot)) \quad \text{and} \quad \partial V(\cdot) = (\partial_1 V(\cdot), \partial_2 V(\cdot), \dots, \partial_n V(\cdot)).$$

In fact, $V_x(\cdot)$ is the common derivative of V with respect to the second variable and $\partial V(\cdot)$ is the functional derivative with respect to the third variable. If V_x and ∂V exist, it is clear that one can compute that

$$\nabla V(t,x,y) = V_x(t,x,y) + \partial V(t,x,y). \tag{3.1}$$

Repeating the above procedure leads to the second vertical derivative $\nabla^2 V(t,x,y)$ as the derivative of the gradient at 0 (if it exists) of the map

$$e \mapsto \nabla V(t, x+e, y+e\mathbf{1}_{[t,T]}).$$

It is clear that

$$\nabla^2 V(t,x,y) = V_{xx}(t,x,y) + 2\partial V_x(t,x,y) + \partial^2 V(t,x,y). \tag{3.2}$$

Furthermore, let us define $\mathbb{C}^{1,1,1}$ functionals.

Definition 3.5 *($\mathbb{C}^{1,1,1}$ functionals).* Define $\mathbb{C}^{1,1,1}([0,T]\times\mathbb{R}^n\times D([0,T];\mathbb{R}^n);\mathbb{R})$ as the family of the jointly continuous non-anticipative functional $V \in \mathbb{C}^{0,0,0}([0,T]\times\mathbb{R}^n\times D([0,T];\mathbb{R}^n);\mathbb{R})$ such that

(i) V admits a horizontal derivative $\mathcal{D}V(t,x,y)$ for all $(t,x,y)\in[0,T]\times\mathbb{R}^n\times D([0,T];\mathbb{R}^n)$, and $\mathcal{D}V(t,\cdot,\cdot)$ is continuous for any $t\in[0,T)$;
(ii) both V_x and ∂V are jointly continuous; and
(iii) $\mathcal{D}V$, ∂V and V_x are boundedness preserving.

To obtain the desired weak convergence results, tightness has to be proved first. Therefore one needs to verify the following

$$\lim_{N_0\to\infty} \mathbb{P}\Big(\sup_{t\le T}|x^\varepsilon(t)| \ge N_0\Big) = 0 \quad \text{for each} \quad T<\infty, \tag{3.3}$$

where $\mathbb{P}(A)$ denotes the probability of A. The verification of (3.3) is usually quite involved, and requires complicated calculations. To circumvent the difficulties, we use the truncation technique as follows. For each $N>0$ sufficiently large satisfying $|x(0)|\le N$, consider

$$\dot{x}^{\varepsilon,N}(t) = \varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)), \tag{3.4}$$

where $x_t^{\varepsilon,N} = \{x^{\varepsilon,N}(t\wedge u): 0\le u\le T\}$, $\varphi^N(x,y,\xi) = \varphi(x,y,\xi)q(x)$, $\psi^N(x,y,\xi)=\psi(x,y,\xi)q(x)$, in which $q(x)$ is a nonnegative and smooth function satisfying

$$q(x) = \begin{cases} 1, & \text{when} \ \ x\in S_N, \\ 0, & \text{when} \ \ x\in\mathbb{R}^n - S_{N+1}. \end{cases}$$

From this truncation technique, it can be seen that $x^{\varepsilon,N}(t) = x^\varepsilon(t)$ up until the first exit time from $S_N = \{x : |x|\le N\}$. Then $x^{\varepsilon,N}(t)$ is said to be an N-truncation of $x^\varepsilon(t)$. For this truncated process and for any $T<\infty$,

$$\lim_{N_0\to\infty} \mathbb{P}\Big(\sup_{t\le T}|x^{\varepsilon,N}(t)| \ge N_0\Big) = 0. \tag{3.5}$$

According to (3.5), it is easily seen that for any $T<\infty$,

$$\lim_{N_0\to\infty} \mathbb{P}\left(\sup_{t\le T} \|x_t^{\varepsilon,N}\|_\infty \ge N_0\right) = 0. \tag{3.6}$$

To establish the chain rule of the functional derivative with respect to the functional differential equation, let us impose the following assumption on the coefficients.

(A1) $\varphi(\cdot,\cdot,\xi)$, $\psi(\cdot,\cdot,\xi)$, $\psi_x(\cdot,\cdot,\xi)$, $\partial\psi(\cdot,\cdot,\xi)$, $\psi_{xx}(\cdot,\cdot,\xi)$, $\partial\psi_x(\cdot,\cdot,\xi)$ and $\partial^2\psi(\cdot,\cdot,\xi)$ are boundedness preserving for any $\xi\in\mathbb{R}^m$, and $\varphi(\cdot)$, $\psi(\cdot)$, $\psi_x(\cdot)$, $\partial\psi(\cdot)$ are jointly continuous.

Working with the N-truncated process, we have the operator $\hat{\mathcal{L}}^{\varepsilon,N}$ corresponding to $\hat{\mathcal{L}}^{\varepsilon}$. To proceed, let us establish the following functional derivatives of the solution of the truncated functional differential equation (3.4).

Theorem 3.2 *(Functional derivative). Let $x^{\varepsilon,N}(t)$ be the solution of (3.4) satisfying assumption (A1). For any $V\in\mathbb{C}^{1,1,1}([0,T]\times\mathbb{R}^n\times D([0,T];\mathbb{R}^n);\mathbb{R})$, put $\overline{V}^{\varepsilon,N}(t)=V(t,x^{\varepsilon,N}(t),\, x_t^{\varepsilon,N})$. Then*

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(t) &= \mathcal{D}V(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})\\ &\quad+\nabla V(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})[\varphi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))+\varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))],\end{aligned} \tag{3.7}$$

and as a result, $\overline{V}^{\varepsilon,N}(t)$ is a continuous semimartingale and

$$M_V^{\varepsilon,N}(t) := \overline{V}^{\varepsilon,N}(t)-\overline{V}^{\varepsilon,N}(0)-\int_0^t \hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(u)du$$

is a martingale.

Proof. Since $\overline{V}^{\varepsilon,N}(t)=V(t,x^{\varepsilon,N}(t),\, x_t^{\varepsilon,N})$ and $\xi^\varepsilon(t)$ is the only driving stochastic process for $x^{\varepsilon,N}(t)$, we can use the conditional expectation $\mathbb{E}_t^{\xi^\varepsilon}$ with respect to $\mathcal{F}_t^{\xi^\varepsilon}$ when $\hat{\mathcal{L}}^{\varepsilon,N}$ is considered. According to the definition of $\hat{\mathcal{L}}^{\varepsilon,N}$,

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(t) &= \text{p-}\lim_{\delta\downarrow 0}\frac{\mathbb{E}_t^{\xi^\varepsilon}\overline{V}^{\varepsilon,N}(t+\delta)-\overline{V}^{\varepsilon,N}(t)}{\delta}\\ &= \text{p-}\lim_{\delta\downarrow 0}\frac{\mathbb{E}_t^{\xi^\varepsilon}V(t+\delta,x^{\varepsilon,N}(t+\delta),x_{t+\delta}^{\varepsilon,N})-V(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})}{\delta}.\end{aligned}$$

Define $h_\delta=x^{\varepsilon,N}(t+\delta)-x^{\varepsilon,N}(t)$. According to the definition of $x_t^{\varepsilon,N}$,

$$x_{t+\delta}^{\varepsilon,N}(\cdot)=x_t^{\varepsilon,N}(\cdot)+h_{\cdot-t}\mathbf{1}_{[t,t+\delta)}(\cdot)+h_\delta\mathbf{1}_{[t+\delta,T]}(\cdot)=: x_t^{\varepsilon,N}(\cdot)+h_\delta^{\cdot}\mathbf{1}_{[t,T]}(\cdot),$$

where

$$h_\delta^u=\begin{cases}0, & \text{for } u\in[0,t),\\ h_{u-t}, & \text{for } u\in[t,t+\delta),\\ h_\delta, & \text{for } u\in[t+\delta,T].\end{cases}$$

$\hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(t)$ can therefore be rewritten as

$$\hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(t)=\text{p-}\lim_{\delta\downarrow 0}\frac{\mathbb{E}_t^{\xi^\varepsilon}V(t+\delta,x^{\varepsilon,N}(t)+h_\delta,x_t^{\varepsilon,N}+h_\delta^u\mathbf{1}_{[t,T]})-V(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})}{\delta}.$$

Using the definitions of horizontal and vertical derivatives, and applying the first-order Taylor expansion yield

$$\begin{aligned}
&V(t+\delta, x^{\varepsilon,N}(t)+h_\delta, x_t^{\varepsilon,N}+h_\delta^u \mathbf{1}_{[t,T]}) - V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) \\
&= V(t+\delta, x^{\varepsilon,N}(t)+h_\delta, x_t^{\varepsilon,N}+h_\delta \mathbf{1}_{[t,T)}) - V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) \\
&\quad + V(t+\delta, x^{\varepsilon,N}(t)+h_\delta, x_t^{\varepsilon,N}+h_\delta^u \mathbf{1}_{[t,T]}) - V(t+\delta, x^{\varepsilon,N}(t)+h_\delta, x_t^{\varepsilon,N}+h_\delta \mathbf{1}_{[t,T)}) \\
&= \mathcal{D}V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})\delta + \nabla V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})h_\delta + R,
\end{aligned}$$

where R is the remainder term which can be expressed as

$$R = R_1(\delta^*)\delta^2 + R_2(\delta^*)\delta h_\delta + h_\delta' R_3(\delta^*)h_\delta + R_4(\delta^*)(h_\delta^u - h_\delta),$$

with $\delta^* \in [0,\delta]$ and

$$\begin{aligned}
R_1(\delta^*) &= \mathcal{D}V(t+\delta^*, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) - \mathcal{D}V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}), \\
R_2(\delta^*) &= \mathcal{D}V(t, x^{\varepsilon,N}(t+\delta^*), x_{t+\delta^*}^{\varepsilon,N}) - \mathcal{D}V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) \\
&\quad + \nabla V(t+\delta^*, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) - \nabla V(t+\delta^*, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}), \\
R_3(\delta^*) &= \nabla V(t, x^{\varepsilon,N}(t+\delta^*), x_{t+\delta^*}^{\varepsilon,N}) - \nabla V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}), \\
R_4(\delta^*) &= \partial V(t+\delta^*, x^{\varepsilon,N}(t+\delta^*), x_{t+\delta^*}^{\varepsilon,N}).
\end{aligned}$$

According to the definition of $\mathbb{C}^{1,1,1}$, $\mathcal{D}V$, ∂V, and V_y are boundedness preserving. These facts, together with the boundedness of $x^{\varepsilon,N}(t)$ and $x_t^{\varepsilon,N}$ due to the truncation technique, yield that both $R_1(\delta^*)$, $R_2(\delta^*)$ and $R_3(\delta^*)$ are bounded, which shows that

$$|R| \le K(\delta^2 + \delta|h_\delta| + |h_\delta|^2) + K|h_{u-t} - h_\delta|\mathbf{1}_{[t,t+\delta)}(u).$$

Note that

$$h_\delta = \int_t^{t+\delta} \varphi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s))ds.$$

Assumption (A1) shows that $\varphi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s))$ is bounded for any given $\varepsilon > 0$, which implies that $|h_\delta| \le K\delta$. This verifies that $|R| \le K\delta^2 + K\delta\mathbf{1}_{[t,t+\delta)}(u)$, which implies that $\text{p-}\lim_{\delta\downarrow 0} |R|/\delta \to 0$ as $\delta \to 0$. Therefore,

$$\hat{\mathcal{L}}^{\varepsilon,N}\overline{V}^{\varepsilon,N}(t) = \mathcal{D}V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) + \nabla V(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})\text{p-}\lim_{\delta\downarrow 0} \frac{\mathbb{E}_t^{\xi^\varepsilon} h_\delta}{\delta}. \tag{3.8}$$

In accordance with the definition of p-lim,

$$\mathbb{E}_t^{\xi^\varepsilon} h_\delta = \mathbb{E}_t^{\xi^\varepsilon} \int_t^{t+\delta} \varphi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s))ds,$$

and $\varphi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(s), x_s^{\varepsilon,N}, \xi^\varepsilon(s))$ is bounded from Assumption (A1), so applying the Lebesgue Dominated Convergence Theorem,

$$\text{p-}\lim_{\delta\downarrow 0}\frac{\mathbb{E}_t^{\xi^\varepsilon}h_\delta}{\delta}=\varphi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))+\varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t)). \tag{3.9}$$

Substituting (3.9) into (3.8) gives the desired (3.7). Applying Lemma 2.2 yields that $M_V^{\varepsilon,N}(t)$ is a martingale. This completes the proof. □

4. Weak convergence and averaged system

To proceed, the following assumptions are needed.

(A2) For any $x\in\mathbb{R}^n$ and $y\in D([0,T];\mathbb{R}^n)$ as parameters, assume that

$$\mathbb{E}\psi(x,y,\xi(t))=0, \tag{4.1a}$$

$$\mathbb{E}\varphi(x,y,\xi(t))=\bar{\varphi}(x,y). \tag{4.1b}$$

Remark 4.1. Lemma 2.3 shows that

$$|\mathbb{E}h_1(u)h_2(v)-\mathbb{E}h_1(u)\mathbb{E}h_2(v)|\le\kappa_4\phi(v-u)$$

for $u<v$, where κ_4 is a constant. This, together with $\mathbb{E}\psi_i(\cdot,\xi(t))=\mathbb{E}\psi_j(\cdot,\xi(t))=0$ in Assumption (A2), shows that for each $x\in\mathbb{R}^n$ and $y\in D([0,T];\mathbb{R}^n)$,

$$\int_0^\infty\mathbb{E}\psi_i(x,y,\xi(t))\psi_j(x,y,\xi(0))dt\le\kappa_4\int_0^\infty\phi(t)dt\le\kappa_4\max_{0\le s<\infty}\phi^{1/2}(s)\int_0^\infty\phi^{1/2}(t)dt<\infty.$$

Similarly,

$$\int_0^\infty\mathbb{E}\nabla\psi(x,y,\xi(t))\psi(x,y,\xi(0))dt<\infty.$$

For each $x\in\mathbb{R}^n$ and $y\in D([0,T];\mathbb{R}^n)$, we can therefore define

$$\frac{1}{2}S^0_{ij}(x,y)=\int_0^\infty\mathbb{E}\psi_i(x,y,\xi(t))\psi_j(x,y,\xi(0))dt, \tag{4.2a}$$

$$\bar{\psi}(x,y)=\int_0^\infty\mathbb{E}\nabla\psi(x,y,\xi(t))\psi(x,y,\xi(0))dt \tag{4.2b}$$

where $\psi_i(\cdot)$, S^0_{ij} denote the ith component and the ijth entry of $\psi(\cdot)$ and $S^0(\cdot)$, respectively. Define

$$S(x,y)=\frac{1}{2}[S^0(x,y)+(S^0(x,y))'],$$

denote by $\rho(x,y)$ its square root, that is,

$$S(x,y)=\rho(x,y)\rho'(x,y).$$

(A3) The following equation

$$dx(t) = [\bar{\varphi}(x(t), x_t) + \bar{\psi}(x(t), x_t)]dt + \rho(x(t), x_t)dB(t) \tag{4.3}$$

has a unique weak solution (uniqueness in the sense of in distribution) on $[0, T]$ for each continuous deterministic initial value $x(0)$, where $B(t)$ is a standard Brownian motion.

Remark 4.2. Nowadays, Assumption (A3) is standard; see [14]. It is a simple way of division of labors. The coefficients of SDE (4.3) are concerned with $\psi(\cdot)$ and $\varphi(\cdot)$ of (1.1) and their derivatives (4.1b), (4.2a), and (4.2b). For general nonlinear systems, it is difficult to give conditions for coefficients of (1.1) to guarantee the existence of unique weak solution to (4.3). When $\varphi(\cdot)$ and $\psi(\cdot)$ have some special forms, for example, $\varphi(x, y, \xi) = \hat{\varphi}(x, y)\xi$ and $\psi(x, y, \xi) = \hat{\psi}(x, y)\xi$, if $\xi(t)$ is a scalar ϕ-mixing process, then

$$\begin{aligned}
\bar{\varphi}(x, y) &= \hat{\varphi}(x, y)\mathbb{E}\xi(t) = 0,\\
S(x, y) &= \hat{\psi}(x, y)\hat{\psi}'(x, y)\widetilde{\Sigma},\\
\bar{\psi}(x, y) &= \nabla\hat{\psi}(x, y)\hat{\psi}(x, y)\widetilde{\Sigma},
\end{aligned}$$

where $\widetilde{\Sigma} = \mathbb{E}\int_0^\infty \xi(u)\xi(0)du$. Choose $\rho(x, y) = \hat{\psi}(x, y)\sqrt{\widetilde{\Sigma}}$. We can impose conditions on $\hat{\psi}(x, y)$ to ensure the weak existence and uniqueness of the equation

$$dx(t) = \nabla\hat{\psi}(x(t), x_t)\hat{\psi}(x(t), x_t)\widetilde{\Sigma}dt + \hat{\psi}(x(t), x_t)\sqrt{\widetilde{\Sigma}}dB(t).$$

Similar comments apply when ξ appeared additively in the underlying functions such as $\varphi(x, y, \xi) = \hat{\varphi}(x, y) + \xi$. However, in general, there are numerous possibilities. It is more convenient to pose a condition as in the current form of (A3).

To proceed, for any function $v \in C_0^2(\mathbb{R}^n; \mathbb{R})$, let us define the operator L from $\mathbb{R}^n \times D([0, T]; \mathbb{R}^n)$ to $\mathbb{R}$ such that

$$L(x, y)v(x) = v_x(x)[\bar{\varphi}(x, y) + \bar{\psi}(x, y)] + \frac{1}{2}\mathrm{Tr}[\rho'(x, y)v_{xx}(x)\rho(x, y)]. \tag{4.4}$$

Let $x(t)$ be the solution of (4.3). Applying the Itô formula to $v(x(t))$ yields (see [16, Chapter 5])

$$M_v(t) = v(x(t)) - v(x(0)) - \int_0^t L(x(s), x_s)v(x(s))ds = \int_0^t v_x(x(s))\rho(x(s), x_s)dB(s) \tag{4.5}$$

is a martingale. We say that $x(\cdot)$ solves the martingale problem with operator $L(\cdot)$ if (4.5) holds for any $v \in C_0^2(\mathbb{R}^n; \mathbb{R})$. As was mentioned before, when we work with the N-truncated process, we rewrite the operator $L(x, y)$ as $L^N(x, y)$. We proceed with the following theorem.

Theorem 4.3. *Under assumptions* (A1) *and* (A2)*, for any $N > 0$, $\{x^{\varepsilon,N}(\cdot)\}$ is tight in $D([0, T]; \mathbb{R}^n)$.*

To prove this theorem, we need the following lemma (see [14, Theorem 4, p.48] or [13] for a proof), which uses perturbed test function methods.

Lemma 4.4. *Let $\{X^\varepsilon(\cdot)\}$ be a sequence of $\mathcal{F}_t^\varepsilon$-measurable processes with paths in $D([0, T]; \mathbb{R}^n)$ satisfying*

$$\lim_{N_0\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\Big\{\sup_{t\le T} |X^\varepsilon(t)| \ge N_0\Big\} = 0 \tag{4.6}$$

for each $T < \infty$. For each $f(\cdot) \in C_0^3(\mathbb{R}^n;\mathbb{R})$, let there be a sequence $\{f^\varepsilon(\cdot)\}$ such that $f^\varepsilon(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^\varepsilon)$ and that $\{\hat{\mathcal{L}}^\varepsilon f^\varepsilon(t); \varepsilon > 0, t \le T\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big\{ \sup_{t\le T} |f^\varepsilon(t) - f(X^\varepsilon(t))| \ge r \Big\} = 0 \quad \textit{for each} \quad r > 0. \tag{4.7}$$

Then $X^\varepsilon(\cdot)$ is tight in $D([0,T];\mathbb{R}^n)$.

Proof of Theorem 4.3. To prove the tightness of $\{x^{\varepsilon,N}(\cdot)\}$, we need only verify that the conditions in Lemma 4.4 are satisfied. In fact, under the truncation techniques, (4.6) holds. Hence we only need to show that for any $f(\cdot) \in C_0^3(\mathbb{R}^n;\mathbb{R})$, there exists $\{f^{\varepsilon,N}(\cdot)\} \in \mathscr{D}(\hat{\mathcal{L}}^{\varepsilon,N})$ such that $\{\hat{\mathcal{L}}^{\varepsilon,N} f^{\varepsilon,N}(\cdot)\}$ is uniformly integrable and for each $r > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{t\le T} |f^{\varepsilon,N}(t) - f(x^{\varepsilon,N}(t))| \ge r\} = 0$, that is, (4.7) holds for the truncated process $x^{\varepsilon,N}(t)$.

Recalling the definition of σ algebras and the corresponding conditional expectations in Remark 2.1, for any $f(\cdot) \in C_0^3(\mathbb{R}^n;\mathbb{R})$, let us define

$$f_1^{\varepsilon,N}(t) := V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) = \varepsilon^{-1} \int_t^T f_x(x^{\varepsilon,N}(t)) \mathbb{E}_t^{\xi^\varepsilon} \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(u)) du \tag{4.8}$$

and $f^{\varepsilon,N}(t) = f(x^{\varepsilon,N}(t)) + f_1^{\varepsilon,N}(t)$. Making change of variable u/ε^2 to u implies that

$$f_1^{\varepsilon,N}(t) = \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t)) \mathbb{E}_{t/\varepsilon^2}^{\xi} \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u)) du. \tag{4.9}$$

Note that $f_x(\cdot)$ is bounded since $f \in C_0^3(\mathbb{R}^n;\mathbb{R})$. By Assumption (A1), for any $(x,y) \in S_N \times D([0,T];S_N)$, $\psi^N(x,y,\xi)$ is bounded for any $\xi \in \mathbb{R}^m$. Note that $\mathbb{E}\psi(x,y,\xi(u)) = 0$ for $x \in \mathbb{R}^n$ and $y \in D([0,T];\mathbb{R}^n)$ in (A2) imply that $\mathbb{E}\psi^N(x,y,\xi(u)) = 0$ for any $(x,y) \in S_N \times D([0,T];S_N)$. These, together with Lemma 2.3, yield that there exists constant K such that

$$\begin{aligned}
\sup_{t\le T} |f_1^{\varepsilon,N}(t)| &= \varepsilon \sup_{t\le T} \Big| \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t)) \mathbb{E}_{t/\varepsilon^2}^{\xi} \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u)) du \Big| \\
&= \varepsilon \sup_{t\le T} \Big| \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t)) [\mathbb{E}_{t/\varepsilon^2}^{\xi} \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u)) \\
&\qquad\qquad - \mathbb{E}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))] du \Big| \\
&\le \varepsilon K \sup_{t\le T} \Big[\int_{t/\varepsilon^2}^{T/\varepsilon^2} \phi\Big(u - \frac{t}{\varepsilon^2}\Big) du \Big] \\
&\le \varepsilon K \sup_{u\ge 0} \phi^{1/2}(u) \int_0^\infty \phi^{1/2}(u) du \\
&= O(\varepsilon),
\end{aligned} \tag{4.10}$$

which implies that $\lim_{\varepsilon\to 0} \mathbb{E}[\sup_{t\le T} |f_1^{\varepsilon,N}(t)|] = 0$. As a consequence of the Chebyshev inequality, (4.7) follows for the truncated process $x^{\varepsilon,N}(t)$.

Let us prove the uniform integrability of $\{\hat{\mathcal{L}}^{\varepsilon,N} f^{\varepsilon,N}(\cdot)\}$ and $\{f^{\varepsilon,N}(\cdot)\} \in \mathscr{D}(\hat{\mathcal{L}}^{\varepsilon,N})$. According to definitions of $f^{\varepsilon,N}(\cdot)$ and $\hat{\mathcal{L}}^{\varepsilon,N}$, applying Theorem 3.2 gives

$$
\begin{aligned}
\hat{\mathcal{L}}^{\varepsilon,N} f^{\varepsilon,N}(t) &= \text{p-}\lim_{\delta\to 0} \frac{\mathbb{E}_t^{\xi^\varepsilon} f^{\varepsilon,N}(t+\delta) - f^{\varepsilon,N}(t)}{\delta} \\
&= \text{p-}\lim_{\delta\to 0} \frac{\mathbb{E}_t^{\xi^\varepsilon} f(x^{\varepsilon,N}(t+\delta)) - f(x^{\varepsilon,N}(t))}{\delta} \\
&\quad + \text{p-}\lim_{\delta\to 0} \frac{\mathbb{E}_t^{\xi^\varepsilon} V_1^{\varepsilon,N}(t+\delta, x^{\varepsilon,N}(t+\delta), x_{t+\delta}^{\varepsilon,N}) - V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})}{\delta} \\
&= f_x'(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))] + \hat{\mathcal{L}}^{\varepsilon,N} f_1^{\varepsilon,N}(t) \\
&= f_x'(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))] \\
&\quad + \mathcal{D}V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})) \\
&\quad + \nabla V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))].
\end{aligned}
\tag{4.11}
$$

We have that

$$
\mathcal{D}V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})) = -\varepsilon^{-1} f_x'(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)). \tag{4.12}
$$

Now let us calculate the vertical derivative of the functional $V_1^\varepsilon(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})$. We have

$$
\begin{aligned}
\nabla V_1^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) &= \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} \nabla[f_x(x^{\varepsilon,N}(t))\mathbb{E}_{t/\varepsilon^2}^\xi \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]du \\
&= \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} [f_x(x^{\varepsilon,N}(t))\mathbb{E}_{t/\varepsilon^2}^\xi \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]_y du \\
&\quad + \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} \partial[f_x(x^{\varepsilon,N}(t))\mathbb{E}_{t/\varepsilon^2}^\xi \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]du \\
&= \sum_{i=1}^3 I_{1i}^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}),
\end{aligned}
\tag{4.13}
$$

where

$$
I_{11}^{\varepsilon,N}(t,x,y) = \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_{xx}(x)\mathbb{E}_{t/\varepsilon^2}^\xi \psi^N(x,y,\xi(u))du,
$$

$$
I_{12}^{\varepsilon,N}(t,x,y) = \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x)\mathbb{E}_{t/\varepsilon^2}^\xi \psi_x^N(x,y,\xi(u))du,
$$

$$
I_{13}^{\varepsilon,N}(t,x,y) = \varepsilon \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x)\mathbb{E}_{t/\varepsilon^2}^\xi [\partial\psi^N(x,y,\xi(u))]du.
$$

Note that $f_{xx}(x)$ is bounded since $f \in C_0^3(\mathbb{R}^n;\mathbb{R})$. The same technique as the one in the estimation of $f_1^{\varepsilon,N}(\cdot)$ in (4.10) yields

$$\begin{aligned}\sup_{t\le T}|I_{11}^{\varepsilon}(t,x,y)| &= \varepsilon\sup_{t\le T}\Big|\int_{t/\varepsilon^2}^{T/\varepsilon^2}[f_{xx}(x)\mathbb{E}^{\xi}_{t/\varepsilon^2}\psi^N(x,y,\xi(u))]'du\Big|\\ &\le \varepsilon K\sup_{t\le T}\Big[\int_{t/\varepsilon^2}^{T/\varepsilon^2}\phi\Big(u-\frac{t}{\varepsilon^2}\Big)du\Big]\\ &= O(\varepsilon). \end{aligned}\tag{4.14}$$

Note that (A1) also implies that $\psi_x^N(x,y,\xi^{\varepsilon}(t))$ and $\partial\psi^N(x,y,\xi^{\varepsilon}(t))$ are bounded. These imply that

$$\mathbb{E}\psi_x^N(x,y,\xi(t)) = [\mathbb{E}\psi^N(x,y,\xi(t))]_x = 0,$$

and

$$\mathbb{E}[\partial\psi^N(x,y,\xi(t))] = \partial[\mathbb{E}\psi^N(x,y,\xi(t))] = 0.$$

Then similar technique to (4.14) gives

$$\begin{aligned}\sup_{t\le T}|I_{12}^{\varepsilon}(t,x,y)| &= \varepsilon\sup_{t\le T}\Big|\int_{t/\varepsilon^2}^{T/\varepsilon^2}f_x(x)\mathbb{E}^{\xi}_{t/\varepsilon^2}\psi_x^N(x,y,\xi(u))du\Big|\\ &= \varepsilon\sup_{t\le T}\Big|\int_{t/\varepsilon^2}^{T/\varepsilon^2}f_x(x)[\mathbb{E}^{\xi}_{t/\varepsilon^2}\psi_x^N(x,y,\xi(u)) - \mathbb{E}\psi_x^N(x,y,\xi(u))]du\Big|\\ &\le \varepsilon K\sup_{t\le T}\Big[\int_{t/\varepsilon^2}^{T/\varepsilon^2}\phi\Big(u-\frac{t}{\varepsilon^2}\Big)du\Big]\\ &= O(\varepsilon), \end{aligned}\tag{4.15}$$

and

$$\sup_{t\le T}|I_{13}^{\varepsilon}(t,x,y)| = O(\varepsilon).\tag{4.16}$$

Substituting (4.14)–(4.16) into (4.13) yields

$$\begin{aligned}&\nabla V_1^{\varepsilon,N}(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})[\varphi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^{\varepsilon}(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^{\varepsilon}(t))]\\ &= O(\varepsilon) + \int_{t/\varepsilon^2}^{T/\varepsilon^2}\mathbb{E}^{\xi}_{t/\varepsilon^2}[\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u))]'du f_{xx}(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^{\varepsilon}(t))\\ &\quad + \int_{t/\varepsilon^2}^{T/\varepsilon^2}f_x(x)\mathbb{E}^{\xi}_{t/\varepsilon^2}\nabla\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u))du\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^{\varepsilon}(t)), \end{aligned}\tag{4.17}$$

since $\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))$ is bounded. Substituting (4.17) and (4.12) into (4.11) yields

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N} f^{\varepsilon,N}(t) &= O(\varepsilon) + f_x(x^{\varepsilon,N}(t))\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) \\ &+ \int_{t/\varepsilon^2}^{T/\varepsilon^2} \mathbb{E}^{\xi}_{t/\varepsilon^2}[\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]' du f_{xx}(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) \\ &+ \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t))\mathbb{E}^{\xi}_{t/\varepsilon^2} \nabla\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u)) du \psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)).\end{aligned} \tag{4.18}$$

According to Assumption (A1), $\psi^N(x, y, \xi)$, $\nabla\psi^N(x, y, \xi)$ are bounded for any $(x, y) \in S_N \times D([0, T]; S_N)$. The uniform integrability therefore follows. Moreover, $f^{\varepsilon,N}(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^{\varepsilon,N})$. As a consequence of Lemma 4.4, $\{x^{\varepsilon,N}(\cdot)\}$ is tight in $D([0, T]; \mathbb{R}^n)$. $\square$

Theorem 4.5. *If* (A1)-(A3) *hold, then* $\{x^\varepsilon(\cdot)\}$ *is tight in* $D([0, T]; \mathbb{R}^n)$*, and the limit of any weakly convergent subsequence satisfies equation* (4.3) *with the same initial value as* $x^\varepsilon(0) = x(0)$.

In the following, we need to establish the weak convergence and characterize the weak limit. To do this, we shall apply the following lemma (see [19,25]).

Lemma 4.6. *Let* $X^\varepsilon(\cdot)$ *be* $\mathbb{R}^n$*-valued and defined on* $[0, T]$*, with the initial value* $X(0)$ *being deterministic. Let* $\{X^\varepsilon(\cdot)\}$ *be tight on* $D([0, T]; \mathbb{R}^n)$*. Suppose* (A3) *holds and* $L(\cdot)$ *is the corresponding operator defined by* (4.4)*. For each* $f(\cdot) \in C_0^3(\mathbb{R}^n; \mathbb{R})$ *(or any dense subset of it), each* $T < \infty$*, there exists* $F^\varepsilon(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^\varepsilon)$ *such that*

$$\text{p-}\lim_{\varepsilon\to 0}[F^\varepsilon(\cdot) - f(X^\varepsilon(\cdot))] = 0, \tag{4.19}$$

and

$$\text{p-}\lim_{\varepsilon\to 0}[\hat{\mathcal{L}}^\varepsilon F^\varepsilon(\cdot) - L(\cdot, X^\varepsilon_\cdot) f(X^\varepsilon(\cdot))] = 0. \tag{4.20}$$

Then, $X^\varepsilon(\cdot) \Rightarrow x(\cdot)$*, where* $x(\cdot)$ *is the weak solution of the stochastic differential equation* (4.3).

Remark 4.7. Similar to the proof of tightness, we use the perturbed test functional method to examine the weak convergence. Introducing the perturbed test functionals allows us to eliminate the noise terms $\xi^\varepsilon(t)$ through averaging, and obtain the desired terms in the limit. A distinct feature of this averaging procedure is that the fast-changing variable $\xi^\varepsilon(t)$ is averaged out. In this procedure, the slow-changing variable $x^\varepsilon(t)$ and the corresponding functional term x_t^ε are treated as parameters.

According to the definition of p-lim, to prove (4.19) for $x^\varepsilon(t)$, for any $f(\cdot) \in C_0^3(\mathbb{R}^n; \mathbb{R})$, we need to look for function $F^\varepsilon(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^\varepsilon)$ and verify the following conditions:

$$\begin{cases} \sup_{t,\varepsilon} \mathbb{E}|F^\varepsilon(t) - f(x^\varepsilon(t))| < \infty, \\ \lim_{\varepsilon\to 0} \mathbb{E}|F^\varepsilon(t) - f(x^\varepsilon(t))| = 0 \quad \text{for each} \quad t. \end{cases}$$

Using the truncation technique, for $x^{\varepsilon,N}(t)$, we need to look for the function $F^{\varepsilon,N}(\cdot) \in \mathscr{D}(\hat{\mathcal{L}}^{\varepsilon,N})$ and verify the corresponding conditions

$$\begin{cases} \sup\limits_{t,\varepsilon} \mathbb{E}|F^{\varepsilon,N}(t) - f(x^{\varepsilon,N}(t))| < \infty, \\ \lim\limits_{\varepsilon\to 0} \mathbb{E}|F^{\varepsilon,N}(t) - f(x^{\varepsilon,N}(t))| = 0 \quad \text{for each} \quad t. \end{cases} \tag{4.21}$$

Similarly, to prove (4.20) for the above $x^{\varepsilon,N}(t)$ and $f(\cdot)$, we need to verify the conditions

$$\begin{cases} \sup\limits_{t,\varepsilon} \mathbb{E}|\hat{\mathcal{L}}^{\varepsilon,N} F^{\varepsilon,N}(t) - L^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) f(x^{\varepsilon,N}(t))| < \infty, \\ \lim\limits_{\varepsilon\to 0} \mathbb{E}|\hat{\mathcal{L}}^{\varepsilon,N} F^{\varepsilon,N}(t) - L^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) f(x^{\varepsilon,N}(t))| = 0 \quad \text{for each} \quad t. \end{cases} \tag{4.22}$$

Proof of Theorem 4.5. We first prove the assertion of this theorem for $x^{\varepsilon,N}(t)$. For any $f(\cdot) \in C_0^3(\mathbb{R}^n;\mathbb{R})$, to use the perturbed test functional method, we need to prove (4.21) and (4.22). Let us define

$$\begin{aligned} V_2^{\varepsilon,N}(t,x,y) = \int_t^T \int_{v/\varepsilon^2}^{T/\varepsilon^2} &\{[\mathbb{E}^{\xi}_{t/\varepsilon^2} f_{xx}(x)\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi^\varepsilon(v)) \\ &-\mathbb{E}[f_{xx}(x)\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi^\varepsilon(v))\} du dv, \end{aligned} \tag{4.23}$$

$$\begin{aligned} V_3^{\varepsilon,N}(t,x,y) = \int_t^T \int_{v/\varepsilon^2}^{T/\varepsilon^2} &f_x(x)\{\mathbb{E}^{\xi}_{t/\varepsilon^2}[\nabla\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi^\varepsilon(v)) \\ &-\mathbb{E}[\nabla\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi^\varepsilon(v))\} du dv, \end{aligned} \tag{4.24}$$

$$V_4^{\varepsilon,N}(t,x,y) = \int_t^T \mathbb{E}^{\xi}_{t/\varepsilon^2} f_x(x)[\varphi^N(x,y,\xi^\varepsilon(u)) - \bar{\varphi}^N(x,y)] du, \tag{4.25}$$

and

$$f_i^{\varepsilon,N}(t) = V_i(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N}) \text{ for } i = 2,3,4.$$

Define

$$F^{\varepsilon,N}(t) = f(x^{\varepsilon,N}(t)) + \sum_{i=1}^{4} f_i^{\varepsilon,N}(t).$$

Making change of variable u/ε^2 to u gives

$$\begin{aligned} V_2^{\varepsilon,N}(t,x,y) = \varepsilon^2 \int_{t/\varepsilon^2}^{T/\varepsilon^2} \int_v^{T/\varepsilon^2} &\{[\mathbb{E}^{\xi}_{t/\varepsilon^2} f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v)) \\ &-\mathbb{E}[f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))\} du dv, \end{aligned} \tag{4.26}$$

$$\begin{aligned} V_3^{\varepsilon,N}(t,x,y) = \varepsilon^2 \int_{t/\varepsilon^2}^{T/\varepsilon^2} \int_v^{T/\varepsilon^2} &f_x(x)\{\mathbb{E}^{\xi}_{t/\varepsilon^2}[\nabla\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi(v)) \\ &-\mathbb{E}[\nabla\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi(v))\} du dv, \end{aligned} \tag{4.27}$$

$$V_4^{\varepsilon,N}(t,x,y)=\varepsilon^2\int_{t/\varepsilon^2}^{T/\varepsilon^2}\mathbb{E}^{\xi}_{t/\varepsilon^2}f_x(x)[\phi^N(x,y,\xi(u))-\bar{\phi}^N(x,y)]du. \tag{4.28}$$

According to Assumption (A1), for any $(x,y)\in S_N\times D([0,T];S_N)$, $\psi^N(x,y,\xi)$ is bounded for any $\xi\in\mathbb{R}^m$. Applying Lemma 2.3, together with the property of $\phi(\cdot)$ and the boundedness of $f_{xx}(\cdot)$, gives

$$\begin{aligned}&\Big|\int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_{v}^{T/\varepsilon^2}\{[\mathbb{E}^{\xi}_{t/\varepsilon^2}\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))-\mathbb{E}[\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))\}dudv\Big|\\&\le K\int_{t/\varepsilon^2}^{T/\varepsilon^2}\phi^{1/2}\Big(v-\frac{t}{\varepsilon^2}\Big)dv\int_{v}^{T/\varepsilon^2}\phi^{1/2}(u-v)du<\infty,\end{aligned}$$

which shows

$$\mathbb{E}\sup_{t\le T}|V_2(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})|=O(\varepsilon^2). \tag{4.29}$$

Similar techniques to (4.29) yield

$$\mathbb{E}\sup_{t\le T}|V_3(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})|=O(\varepsilon^2) \tag{4.30}$$

and

$$\mathbb{E}\sup_{t\le T}|V_4(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})|=O(\varepsilon^2). \tag{4.31}$$

(4.29)–(4.31), together with (4.10) give

$$\text{p-}\lim_{\varepsilon\to0}[F^{\varepsilon,N}(t)-f(x^{\varepsilon,N}(t))]=0. \tag{4.32}$$

Note that $f(\cdot)$ is bounded. Thus (4.21) is proved. Then let us consider $\mathcal{L}^{\varepsilon,N}F^{\varepsilon,N}(t)$. Similar to (4.11),

$$\begin{aligned}&\hat{\mathcal{L}}^{\varepsilon,N}F^{\varepsilon,N}(t)\\&\quad=\text{p-}\lim_{\delta\downarrow0}\frac{\mathbb{E}^{\xi^\varepsilon}_tF^{\varepsilon,N}(t+\delta)-F^{\varepsilon,N}(t)}{\delta}\\&\quad=\frac{df(x^{\varepsilon,N}(t))}{dt}+\sum_{i=1}^4\hat{\mathcal{L}}^{\varepsilon,N}f_i(t)\\&\quad=f_x'(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))+\varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))]\\&\qquad+\sum_{i=1}^4\mathcal{D}V_i^{\varepsilon,N}(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})\\&\qquad+\sum_{i=1}^4\nabla V_i^{\varepsilon,N}(t,x^{\varepsilon,N}(t),x_t^{\varepsilon,N})[\varphi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))+\varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))].\end{aligned} \tag{4.33}$$

We have that

$$
\begin{aligned}
\mathcal{D}V_2^{\varepsilon,N}(t,x,y) = &-\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_{xx}(x)\mathbb{E}^{\xi}_{t/\varepsilon^2}\psi^N(x,y,\xi(u))du\psi^N(x,y,\xi^\varepsilon(t)) \\
&+\int_{t/\varepsilon^2}^{T/\varepsilon^2} [\mathbb{E}f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi^\varepsilon(t))du, \\
\mathcal{D}V_3^{\varepsilon,N}(t,x,y) = &-\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_{x}(x)\mathbb{E}^{\xi}_{t/\varepsilon^2}[\nabla\psi^N(x,y,\xi(u))]du\psi^N(x,y,\xi^\varepsilon(t)) \\
&+\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_{x}(x)\mathbb{E}[\nabla\psi^N(x,y,\xi(u))]du\psi^N(x,y,\xi^\varepsilon(t)), \\
\mathcal{D}V_4^{\varepsilon,N}(t,x,y) = &-f_x(x)\varphi^N(x,y,\xi^\varepsilon(t)) + f_x(x)\bar{\varphi}^N(x,y).
\end{aligned}
$$

One can compute that

$$
\begin{aligned}
\nabla V_2^N(t,x,y) &= \varepsilon^2 \int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_{v}^{T/\varepsilon^2} \nabla\{[\mathbb{E}^{\xi}_{t/\varepsilon^2} f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v)) \\
&\qquad - \mathbb{E}[f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))\}dudv \\
&= \varepsilon^2 I_{21}^\varepsilon(t,x,y) + \varepsilon^2 I_{22}^\varepsilon(t,x,y),
\end{aligned} \tag{4.34}
$$

where

$$
\begin{aligned}
I_{21}^\varepsilon(t,x,y) = \int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_{v}^{T/\varepsilon^2} &\{[\mathbb{E}^{\xi}_{t/\varepsilon^2} f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v)) \\
&- \mathbb{E}[f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))\}_x dudv
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22}^\varepsilon(t,x,y) = \int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_{v}^{T/\varepsilon^2} &\partial\{[\mathbb{E}^{\xi}_{t/\varepsilon^2} f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v)) \\
&- \mathbb{E}[f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(v))\}_x dudv.
\end{aligned}
$$

According to Assumption (A1), $\psi^N(x,y,\xi(u)))$ is bounded. These, together with Lemma 2.3 and the properties of $\phi(\cdot)$ show that

$$
\begin{aligned}
\sup_{t\le T}|I_{21}^\varepsilon(t,x,y)| = \sup_{t\le T}\Big| \int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_{v}^{T/\varepsilon^2} &\{\mathbb{E}^{\xi}_{t/\varepsilon^2}[(f_{xx}(x)\psi^N(x,y,\xi(u)))'\psi^N(x,y,\xi(v))]_x \\
&- \mathbb{E}[(f_{xx}(x)\psi^N(x,y,\xi(u)))'\psi^N(x,y,\xi(v))]_x\}dudv\Big|
\end{aligned}
$$

$$= \sup_{t\le T}\int_{t/\varepsilon^2}^{T/\varepsilon^2}\phi^{1/2}\Big(v-\frac{t}{\varepsilon^2}\Big)dv\int_v^{T/\varepsilon^2}\phi^{1/2}(u-v)du$$
$$<\infty.$$

Similarly,

$$\sup_{t\le T}|I_{22}^\varepsilon(t,x,y)|<\infty.$$

These imply

$$\nabla V_2^{\varepsilon,N}(t,x,y)[\varphi^N(x,y,\xi^\varepsilon(t))+\varepsilon^{-1}\psi^N(x,y,\xi^\varepsilon(t))]=O(\varepsilon).$$

This, together with $\mathcal{D}V_2^{\varepsilon,N}(t,x,y)$, gives that

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N}f_2(t)=O(\varepsilon)+&\int_{t/\varepsilon^2}^{T/\varepsilon^2}[\mathbb{E}f_{xx}(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u))]'\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))du\\ -&\int_{t/\varepsilon^2}^{T/\varepsilon^2}f_{xx}(x^{\varepsilon,N}(t))\mathbb{E}^\xi_{t/\varepsilon^2}\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u))du\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t)).\end{aligned}\tag{4.35}$$

Let us compute $\nabla V_3^{\varepsilon,N}(t,x,y)$.

$$\begin{aligned}\nabla V_3^N(t,x,y)&=\varepsilon^2\int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_v^{T/\varepsilon^2}\nabla\{f_x(x)[\mathbb{E}^\xi_{t/\varepsilon^2}(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(v))\\&\qquad -\mathbb{E}(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(v))]\}dudv\\&=\varepsilon^2\int_{t/\varepsilon^2}^{T/\varepsilon^2}\int_v^{T/\varepsilon^2}\mathbb{E}^\xi_{t/\varepsilon^2}\nabla\{f_x(x)(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(v))\}\\&\qquad -\mathbb{E}\nabla\{f_x(x)(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(v))\}dudv.\end{aligned}\tag{4.36}$$

Define $h^{(2)}(x,y;u,v)=\nabla\{f_x(x)(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(v))$. Then

$$\begin{aligned}h^{(2)}(x,y;u,v)&=\nabla\{f_x(x)[\psi_x^N(x,y,\xi(u))+\partial\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi(v))\}\\&=f_{xx}(x)[\psi_x^N(x,y,\xi(u))+\partial\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi(v))\\&\quad+f_x(x)[\psi_{xx}^N(x,y,\xi(u))+\partial\psi_x^N(x,y,\xi(u))]\psi^N(x,y,\xi(v))\\&\quad+f_x(x)[\psi_x^N(x,y,\xi(u))+\partial\psi^N(x,y,\xi(u))]\psi_x^N(x,y,\xi(v))\\&\quad+f_x(x)[\partial\psi_x^N(x,y,\xi(u))+\partial^2\psi^N(x,y,\xi(u))]\psi^N(x,y,\xi(v))\\&\quad+f_x(x)[\psi_x^N(x,y,\xi(u))+\partial\psi^N(x,y,\xi(u))]\partial\psi^N(x,y,\xi(v)).\end{aligned}$$

Assumption (A1) shows that for any $(x, y) \in S_N \times D([0, T]; S_N)$ and any $\xi \in \mathbb{R}^m$, $\psi^N(x, y, \xi)$, $\psi_x^N(x, y, \xi)$, $\partial\psi^N(x, y, \xi)$, $\psi_{xx}^N(x, y, \xi)$, $\partial\psi_x^N(x, y, \xi)$ and $\partial^2\psi^N(x, y, \xi)$ are bounded. Applying Lemma 2.3 yields

$$\nabla V_3^N(t, x, y) = \varepsilon^2 K \int_{t/\varepsilon^2}^{T/\varepsilon^2} \phi^{1/2}\Big(v - \frac{t}{\varepsilon^2}\Big)dv \int_v^{T/\varepsilon^2} \phi^{1/2}(u - v)du = O(\varepsilon^2),$$

which implies

$$\nabla V_3^{\varepsilon,N}(t, x, y)[\varphi^N(x, y, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x, y, \xi^\varepsilon(t))] = O(\varepsilon).$$

This, together with $\mathcal{D}V_3^{\varepsilon,N}(t, x, y)$, leads to that

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N} f_3(t) = O(\varepsilon^2) + \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t))\mathbb{E}[\nabla\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]du\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)\\ - \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t))\mathbb{E}^\xi_{t/\varepsilon^2}[\nabla\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi(u))]du\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)).\end{aligned} \tag{4.37}$$

Let us compute $\nabla V_4^{\varepsilon,N}(t, x^{\varepsilon,N}(t), x_t^{\varepsilon,N})$. Using the same technique as the above estimate,

$$\begin{aligned}\nabla V_4^{\varepsilon,N}(t, x, y) &= \varepsilon^2 \int_{t/\varepsilon^2}^{T/\varepsilon^2} \nabla\mathbb{E}^\xi_{t/\varepsilon^2} f_x(x)[\varphi^N(x, y, \xi(u)) - \bar{\varphi}^N(x, y)]du\\ &= \varepsilon^2 \int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x)[\mathbb{E}^\xi_{t/\varepsilon^2}\nabla\varphi^N(x, y, \xi(u)) - \nabla\bar{\varphi}^N(x, y)]du\\ &= O(\varepsilon^2),\end{aligned} \tag{4.38}$$

which, together with $\mathcal{D}V_4^{\varepsilon,N}(t, x, y)$ implies that

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N} f_4(t) &= -f'_x(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) - \bar{\varphi}^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N})]\\ &\quad + O(\varepsilon^2)[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))]\\ &= -f'_x(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) - \bar{\varphi}^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N})] + O(\varepsilon).\end{aligned} \tag{4.39}$$

(4.12), (4.17), (4.35), (4.37), and (4.39) yield that

$$\begin{aligned}\hat{\mathcal{L}}^{\varepsilon,N} F^{\varepsilon,N}(t) &= f'_x(x^{\varepsilon,N}(t))[\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + \varepsilon^{-1}\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))]\\ &\quad + O(\varepsilon) - \varepsilon^{-1} f'_x(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))\\ &\quad - f'_x(x^{\varepsilon,N}(t))\varphi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t)) + f'_y(x^{\varepsilon,N}(t))\bar{\varphi}^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N})\\ &\quad + \int_{t/\varepsilon^2}^{T/\varepsilon^2} [\mathbb{E} f_{xx}(x^{\varepsilon,N}(t))\psi^N(y, x, \xi(u))]'\psi^N(x^{\varepsilon,N}(t), x_t^{\varepsilon,N}, \xi^\varepsilon(t))du\end{aligned}$$

$$
\begin{aligned}
&+\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t))\mathbb{E}(\nabla\psi^N(y,x,\xi(u)))\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))du\\
&=O(\varepsilon)+f_x'(x^{\varepsilon,N}(t))\bar{\varphi}^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N})\\
&+\int_{t/\varepsilon^2}^{T/\varepsilon^2} [\mathbb{E}f_{xx}(x^{\varepsilon,N}(t))\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u))]'\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))du\\
&+\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x^{\varepsilon,N}(t))\mathbb{E}(\nabla\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi(u)))\psi^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N},\xi^\varepsilon(t))du.
\end{aligned}
\tag{4.40}
$$

Let $f_{xxij}(x)$ be the ijth entry of $f_{xx}(x)$ and $S_{ij}^{0,N}(x,y)$ be the truncation of function of $S_{ij}^0(x,y)$, respectively. In view of Remark 4.1, as $\varepsilon\to 0$,

$$
\begin{aligned}
&\int_{t/\varepsilon^2}^{T/\varepsilon^2} [\mathbb{E}f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi^\varepsilon(t))du\\
&\to\int_0^\infty [\mathbb{E}f_{xx}(x)\psi^N(x,y,\xi(u))]'\psi^N(x,y,\xi(0))du=\frac{1}{2}\sum_{i,j} f_{xxij}(x)S_{ij}^{0,N}(x,y)
\end{aligned}
\tag{4.41}
$$

and

$$
\begin{aligned}
&\int_{t/\varepsilon^2}^{T/\varepsilon^2} f_x(x)\mathbb{E}(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi^\varepsilon(t))du\\
&\to\int_0^\infty f_x(x)\mathbb{E}(\nabla\psi^N(x,y,\xi(u)))\psi^N(x,y,\xi(0))du=f_x(x)\bar{\psi}^N(x,y).
\end{aligned}
\tag{4.42}
$$

These two limits, together with (4.40), yield that

$$
\begin{aligned}
\hat{\mathcal{L}}^{\varepsilon,N}F^{\varepsilon,N}(t)-[f_x'(x^{\varepsilon,N}(t))[\bar{\varphi}^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N})+\bar{\psi}^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N})]&\\
+\frac{1}{2}\sum_{i,j} f_{xxij}(x^{\varepsilon,N}(t))S_{ij}^{0,N}(x^{\varepsilon,N}(t),x_t^{\varepsilon,N})]\to 0&
\end{aligned}
\tag{4.43}
$$

as $\varepsilon\to 0$. Applying the generator L^N defined by (4.4) with truncation technique to the solution process $x^{\varepsilon,N}(t)$ in the stochastic truncated functional differential equation (3.4) yields

$$
L^N(x,y)f(x)=f_x'(x)[\bar{\varphi}^N(x,y)+\bar{\psi}^N(x,y)]+\frac{1}{2}\sum_{i,j} f_{xxij}(x)S_{ij}^{0,N}(x,y),
$$

which implies that

$$
\hat{\mathcal{L}}^{\varepsilon,N}F^{\varepsilon,N}(t)-L^N(x^{\varepsilon,N}(t),x_t^{\varepsilon,N})f(x^{\varepsilon,N}(t))\to 0,
\tag{4.44}
$$

as $\varepsilon \to 0$, which implies (4.20). This, together with (4.32) yields $x^{\varepsilon,N}(\cdot) \Rightarrow x^N(\cdot)$ as $\varepsilon \to 0$ by virtue of Lemma 4.6, where $x^N(\cdot)$ solves the martingale problem with operator L^N.

Moving from the truncated processes to that of un-truncated processes, the argument is similar to that of [14, p.46]. For any continuous deterministic initial value $x(0)$, let $\mathbb{P}(\cdot)$ and $\mathbb{P}^N(\cdot)$ denote the probabilities induced by $x(\cdot)$ and $x^N(\cdot)$, respectively, on the Borel sets of $D([0,T];\mathbb{R}^n)$. By (A3), the martingale problem has a unique solution for each $x(0)$, so $\mathbb{P}(\cdot)$ is unique. For each $T < \infty$, the uniqueness of $\mathbb{P}(\cdot)$ implies that $\mathbb{P}(\cdot)$ agrees with $\mathbb{P}^N(\cdot)$ on all Borel sets of the set of paths in $D([0,T];S_N)$ for each $t \le T$. However, $\mathbb{P}\{\sup_{t\le T}|x(t)| \le N\} \to 1$ as $N \to \infty$. This together with the weak convergence of $x^{\varepsilon,N}(\cdot)$ implies that $x^\varepsilon(\cdot) \Rightarrow x(\cdot)$. Moreover, the uniqueness implies that the limit does not depend on the chosen subsequences. The proof is thus completed. $\square$

5. Integro-differential equations under wideband noise perturbation

There are many important classes of differential equations that satisfy (1.1). As a special class, this section examines the following integro-differential equation with the wideband noise perturbation

$$\dot{x}^\varepsilon(t) = \zeta\Big(x^\varepsilon(t), \int_0^t \kappa(t-s)x^\varepsilon(s)ds, \xi^\varepsilon(t)\Big) + \varepsilon^{-1}\varsigma\Big(x^\varepsilon(t), \int_0^t \kappa(t-s)x^\varepsilon(s)ds, \xi^\varepsilon(t)\Big) \tag{5.1}$$

with the deterministic initial data $x(0) \in \mathbb{R}^n$, where $\zeta, \varsigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$. Define $y^\varepsilon(t) = \int_0^t \kappa(t-s)x^\varepsilon(s)ds$. It is clear that both $\zeta(\cdot)$ and $\varsigma(\cdot)$ are functions. Then Assumption (A1) may be simplified as

(A1') $\zeta(\cdot,\cdot,\cdot)$, $\varsigma(\cdot,\cdot,\cdot)$, $\varsigma_x(\cdot,\cdot,\cdot)$, $\varsigma_y(\cdot,\cdot,\cdot)$ are continuous for any $(x,y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, and $\varsigma_{xx}(\cdot,\cdot,\xi)$, $\varsigma_{xy}(\cdot,\cdot,\xi)$ and $\varsigma_{yy}(\cdot,\cdot,\xi)$ are bounded for any $\xi \in \mathbb{R}^m$ on $G \times G$, where $G \subset \mathbb{R}^n$ is a compact set.

Assumptions (A2) and (A3) can also be rewritten as

(A2') For any $x, y \in \mathbb{R}^n$ as parameters, assume that

$$\mathbb{E}\varsigma(x,y,\xi(t)) = 0, \tag{5.2a}$$

$$\mathbb{E}\zeta(x,y,\xi(t)) = \bar{\zeta}(x,y). \tag{5.2b}$$

For each $x, y \in \mathbb{R}^n$, denote

$$\frac{1}{2}S^0_{ij}(x,y) = \int_0^\infty \mathbb{E}\varsigma_i(x,y,\xi(t))\varsigma_j(x,y,\xi(0))dt, \tag{5.3a}$$

$$\bar{\varsigma}(x,y) = \int_0^\infty \mathbb{E}[\varsigma_x(x,y,\xi(t)) + \varsigma_y(x,y,\xi(t))]\varsigma(x,y,\xi(0))dt, \tag{5.3b}$$

$$S(x,y) = \frac{1}{2}[S^0(x,y) + (S^0(x,y))'],$$

and $\rho(x,y)$ is its square root, that is, $S(x,y) = \rho(x,y)\rho'(x,y)$.

(A3') The following equation

$$dx(t) = \Big[\bar{\zeta}\Big(x(t), \int_0^t \kappa(t-s)x(s)ds\Big) + \bar{\varsigma}\Big(x(t), \int_0^t \kappa(t-s)x(s)ds\Big)\Big]dt$$

$$+\rho\Big(x(t), \int_0^t \kappa(t-s)x(s)ds\Big)dB(t) \tag{5.4}$$

has a unique weak solution on $[0,T]$ for each deterministic initial value $x(0)$, where $B(t)$ is a standard Brownian motion.

Applying Theorem 4.5 leads to the following result.

Theorem 5.1. *Under Assumptions* (A1')–(A3')*, the solution* $\{x^\varepsilon(\cdot)\}$ *of* (5.1) *is tight in* $D([0,T];\mathbb{R}^n)$ *and the limit of any weakly convergent subsequence satisfies* (5.4) *with the same initial value* $x(0)$.

As an example, let us consider the Lotka-Volterra integro-differential equation (1.3) perturbed by wide-band noise of the form

$$\begin{aligned}\dot{x}^\varepsilon(t) = \mathrm{diag}(x_1^\varepsilon(t),\dots,x_n^\varepsilon(t))\Big[A(t) - D(t)x^\varepsilon(t) - \int_0^t \kappa_1(t-s)x^\varepsilon(s)ds\Big] \\ +\frac{1}{\varepsilon}\Sigma\Big(x^\varepsilon(t), \int_0^t \kappa_2(t-s)x^\varepsilon(s)ds\Big)\xi^\varepsilon(t),\end{aligned} \tag{5.5}$$

where $\mathrm{diag}(x_1,\dots,x_n)$ denotes the diagonal matrix with the indicated diagonal entries, $A(t)$ and $D(t)$ are continuous $n\times n$ matrix-valued functions, $\kappa_i(\cdot)$ for $i=1,2$ are appropriate kernel functions with compatible dimensions, $\Sigma(x,y)$ is an $n\times n$ matrix-valued function whose partial derivatives up to the second order with respect to x and y are continuous and $\xi^\varepsilon(t)$ is the wideband noise as mentioned before. Nevertheless, it can be unbounded.

Let us define

$$S = \mathbb{E}\int_0^\infty \xi(u)\xi'(0)du + \mathbb{E}\int_0^\infty \xi(0)\xi'(u)du \tag{5.6}$$

and

$$B^\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \xi^\varepsilon(u)du = \varepsilon\int_0^{t/\varepsilon^2} \xi(u)du.$$

Then we have the following lemma.

Lemma 5.2. $B^\varepsilon(\cdot)$ *converges weakly to a Brownian motion* $B(\cdot)$ *with covariance* St*, where* S *is given by* (5.6).

Lemma 5.2 is essentially a continuous-time version of Theorem 7.3.3 in [6, pp. 353]; see also Remark 7.3.4 in [6]. A proof can be carried out similar to the aforementioned theorem with modification to the continuous time case. We omit the verbatim argument.

With the above lemma at our hand, using the averaging argument as presented in this paper, we can show that $x^\varepsilon(\cdot)$ converges weakly to $x(\cdot)$ such that $x(\cdot)$ satisfies the following differential functional equation

$$
\begin{aligned}
dx(t) = {} & \operatorname{diag}(x_1(t), \ldots, x_n(t)) \Big[A(t) - D(t)x(t) - \int_0^t \kappa_1(t-s)x(s)ds \Big] dt \\
& + \Big[\Sigma_x \Big(x(t), \int_0^t \kappa_2(t-s)x(s)ds \Big) + \Sigma_y \Big(x(t), \int_0^t \kappa_2(t-s)x(s)ds \Big) \Big] \\
& \times \Big[\Sigma \Big(x(t), \int_0^t \kappa_2(t-s)x(s)ds \Big) dt \Big] S dt \\
& + \Sigma \Big(x(t), \int_0^t \kappa_2(t-s)x(s)ds \Big) S^{1/2} dB(t),
\end{aligned}
\tag{5.7}
$$

where $B(\cdot)$ is a standard n-dimensional Brownian motion and $S^{1/2}$ is the 'square root' matrix satisfying $S^{1/2}(S^{1/2})' = S$.

In lieu of the conditions above, we can deal with unbounded noise. In such a case, we may assume that $\Sigma(\cdot, \cdot)$ is bounded and continuous together with its patrial derivatives with respect to x and y up to the second order, and that $\xi(\cdot)$ satisfies for some $\Delta > 0$, $\mathbb{E}|\xi(t)|^{2+\Delta} < \infty$ and $\int_0^\infty [\phi(u)]^{\Delta/(1+\Delta)} du < \infty$, where instead of the ∞ norm, we use the $p = (2+\Delta)/(1+\Delta)$ norm. Then the result above still holds.

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