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A Generalization for Fourier Transforms of a Theorem due to Marcinkiewicz

Ferenc Weisz[1]

Department of Numerical Analysis, Eötvös L. University, H-1117 Budapest, Pázmány P. sétány 1/D, Hungary
E-mail: weisz@ludens.elte.hu

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It is shown that the maximal operator of the Marcinkiewicz means of a tempered distribution is bounded from $H_p(\mathbf{R}^2)$ to $L_p(\mathbf{R}^2)$ for all $p_0 < p \le \infty$ and, consequently, is of weak type $(1,1)$, where $p_0 < 1$. As a consequence we obtain a generalization for Fourier transforms of a summability result due to Marcinkiewicz and Zhizhiashvili, more exactly, the Marcinkiewicz means of a function $f \in L_1(\mathbf{R}^2)$ converge a.e. to the function in question. Moreover, we prove that the Marcinkiewicz means are uniformly bounded on the spaces $H_p(\mathbf{R}^2)$ and so they converge in the norm $(p_0 < p < \infty)$. Similar results for the Riesz transforms are also given.

Key Words: Hardy spaces; p-atom; atomic decomposition; interpolation; Marcinkiewicz means.

1. INTRODUCTION

The Hardy–Lorentz space $H_{p,q}(\mathbf{R}^2)$ of tempered distributions are introduced with the $L_{p,q}(\mathbf{R}^2)$ Lorentz norm of the nontangential maximal function. Of course, $H_p(\mathbf{R}^2) = H_{p,p}(\mathbf{R}^2)$ are the usual Hardy spaces $(0 < p \le \infty)$.

For multidimensional trigonometric–Fourier series Marcinkiewicz and Zygmund [7] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1, \dots, n_d) \to \infty$ provided that n is in a positive cone, i.e., provided that $2^{-\tau} \le n_k/n_j \le 2^\tau$ for every $k, j = 1, \dots, d$ and for some $\tau \ge 0$ $(n = (n_1, \dots, n_d) \in \mathbf{N}^d)$.

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Recently the author [13] obtained the same convergence result for the Cesàro means ς_n^α by proving the weak (L_1, L_1) inequality

$$\sup_{\rho>0} \rho\lambda(\varsigma_*^\alpha f > \rho) \le C\|f\|_1 \qquad \big(f \in L_1(\mathbf{T}^d)\big),$$

where $\varsigma_*^\alpha := \sup_{\substack{2^{-\tau} \le n_k/n_j \le 2^\tau \\ k,j=1,\dots,d}} |\sigma_n^\alpha|$, $\alpha = (\alpha_1, \dots, \alpha_d)$, and $0 < \alpha_k \le 1$. Moreover, the author [13] verified that ς_*^α is bounded from $H_{p,q}(\mathbf{T}^d)$ to $L_{p,q}(\mathbf{T}^d)$ if $p_0 < p \le \infty$ and $0 < q \le \infty$, where $p_0 < 1$. In the one-dimensional case these results for Fourier transforms and Riesz means are described in Weisz [14].

Marcinkiewicz [8] verified for two-dimensional Fourier series that the Marcinkiewicz means $\sigma_n^\alpha f$ of a function $f \in L \log L(\mathbf{T}^2)$ converge a.e. to f as $n \to \infty$ where $\alpha = 1$. Later Zhizhiashvili [17, 18] extended this result to all $f \in L_1(\mathbf{T}^2)$ and to all $0 < \alpha < \infty$.

In this paper we introduce the Marcinkiewicz means of Fourier transforms and generalize the results above. We will show that the maximal operator $\sigma_*^{\alpha,\gamma}$ of these Marcinkiewicz means is bounded from $H_{p,q}(\mathbf{R}^2)$ to $L_{p,q}(\mathbf{R}^2)$ whenever $p_0 < p \le \infty$, $0 < q \le \infty$, and $0 < \alpha < \infty$. Note that $p_0 < 1$ depends only on α. We introduce the Riesz transforms $\tilde{f}^{(i)} = R_i f$ $(i = 1, 2)$, the Marcinkiewicz means of the Riesz transforms $\tilde{\sigma}_T^{(i);\alpha,\gamma}$, and the corresponding maximal operators $\tilde{\sigma}_*^{(i);\alpha,\gamma}$. We obtain that the operator $\tilde{\sigma}_*^{(i);\alpha,\gamma}$ is also of type $(H_{p,q}(\mathbf{R}^2), L_{p,q}(\mathbf{R}^2))$ if $p_0 < p \le \infty$, $0 < q \le \infty$, and of weak type $(L_1(\mathbf{R}^2), L_1(\mathbf{R}^2))$.

A usual density argument implies then that the Marcinkiewicz means $\sigma_T^{\alpha,\gamma} f$ converge a.e. to f and the Marcinkiewicz means of the Riesz transforms $\tilde{\sigma}_T^{(i);\alpha,\gamma} f$ converge a.e. to $\tilde{f}^{(i)}$ as $T \to \infty$, provided that $f \in L_1(\mathbf{R}^2)$. Note that $\tilde{f}^{(i)}$ is not necessarily integrable whenever f is.

We will prove also that the operators $\sigma_T^{\alpha,\gamma}$ and $\tilde{\sigma}_T^{(i);\alpha,\gamma}$ $(T \in \mathbf{R}_+)$ are uniformly bounded from $H_{p,q}(\mathbf{R}^2)$ to $H_{p,q}(\mathbf{R}^2)$ if $p_0 < p \le \infty$, $0 < q \le \infty$. From this it follows that $\sigma_T^{\alpha,\gamma} f \to f$ and $\tilde{\sigma}_T^{(i);\alpha,\gamma} f \to \tilde{f}^{(i)}$ in the $H_{p,q}(\mathbf{R}^2)$ norm as $T \to \infty$.

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2. THE $H_p(\mathbf{R}^2)$ HARDY SPACES AND RIESZ TRANSFORMS

Let $\mathbf{R}$ denote the real numbers, $\mathbf{R}_+$ the positive real numbers, and λ the two-dimensional Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We write L_p instead of the real $L_p(\mathbf{R}^2, \lambda)$ space endowed with the norm $\|f\|_p := (\int_{\mathbf{R}^2} |f|^p \, d\lambda)^{1/p}$ $(0 < p \le \infty)$.

The distribution function of a two-dimensional Lebesgue-measurable function f is defined by

$$\lambda(|f| > \rho) := \lambda(\{(x, y) : |f(x, y)| > \rho\}) \qquad (\rho \geq 0).$$

The *weak* L_p space L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. For a measurable function f of the *nonincreasing rearrangement* is defined by

$$\check{f}(t) := \inf\{\rho : \lambda(|f| > \rho) \leq t\}.$$

The Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty$, $0 < q < \infty$,

$$\|f\|_{p,q} := \left(\int_0^\infty \check{f}(t)^q t^{q/p} \frac{dt}{t}\right)^{1/q},$$

while for $0 < p \leq \infty$

$$\|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \check{f}(t).$$

Let

$$L_{p,q} := L_{p,q}(\mathbf{R}^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can then show the equalities

$$L_{p,p} = L_p,\ L_{p,\infty} = L_p^* \qquad (0 < p \leq \infty)$$

(see e.g. Bennett and Sharpley [1] or Bergh and Löfström [2]).

We are going to introduce the H_p Hardy space. Let f be a tempered distribution on $C^\infty(\mathbf{R}^2)$. The *Fourier transform* of f is denoted by $\hat{f}$. In special case, if f is an integrable function then

$$\hat{f}(t, u) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x, y) e^{-\iota t x - \iota u y}\, dx\, dy \qquad (t, u \in \mathbf{R})$$

where $\iota = \sqrt{-1}$.

For a tempered distribution f and $t > 0$ define the harmonic function u by

$$u(x, y, t) := (f * P_t)(x, y)$$

where $*$ denotes the convolution and

$$P_t(x,y) := \frac{ct}{\left(t^2+x^2+y^2\right)^{3/2}} \qquad (x, y \in \mathbf{R})$$

is the Poisson kernel. Let $\Gamma := \{(x,y,t) : \sqrt{x^2+y^2} < t\}$, a cone whose vertex is the origin. We denote by $\Gamma(x,y)$ $(x, y \in \mathbf{R})$ the translate of Γ so that its vertex is (x,y). The nontangential maximal function is defined by

$$u^*(x,y) := \sup_{(x',y',t)\in\Gamma(x,y)} |u(x',y',t)|.$$

For $0 < p, q \leq \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{R}^2) =: H_{p,q}$ consists of all tempered distributions f for which $u^* \in L_{p,q}$, and we set

$$\|f\|_{H_{p,q}} := \|u^*\|_{p,q}.$$

Note that in the case $p = q$ the usual definition of Hardy spaces $H_{p,p} = H_p$ is obtained. Recall that $L_1 \subset H_{1,\infty}$ or, more exactly,

$$\|f\|_{H_{1,\infty}} = \sup_{\rho>0} \rho\lambda(u^* > \rho) \leq C\|f\|_1 \qquad (f \in L_1). \tag{1}$$

Moreover, $H_{p,q} \sim L_{p,q}$ for $1 < p < \infty$, $0 < q \leq \infty$, where $\sim$ denotes the equivalence of the norms and spaces (see Fefferman and Stein [5], Stein [10], and Fefferman *et al.* [4]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman *et al.*, Rivière and Sagher [4, 9], and also Weisz [12]).

THEOREM A. *If a sublinear (resp. (linear) operator V is bounded from H_{p_0} to L_{p_0} (resp. to H_{p_0}) and from L_{p_1} to L_{p_1} $(p_0 \leq 1 < p_1 \leq \infty)$ then it is also bounded from $H_{p,q}$ to $L_{p,q}$ (resp. to $H_{p,q}$) if $p_0 < p < p_1$ and $0 < q \leq \infty$.*

For a tempered distribution $f \in H_p$ $(0 < p < \infty)$ the *Riesz transforms* $\tilde{f}^{(i)} := R_i f$ $(i = 1, 2)$ are defined by

$$\left(\tilde{f}^{(1)}\right)^{\wedge}(t,u) := -\iota\frac{t}{\sqrt{t^2+u^2}}\hat{f}(t,u),$$

$$\left(\tilde{f}^{(2)}\right)^{\wedge}(t,u) := -\iota\frac{u}{\sqrt{t^2+u^2}}\hat{f}(t,u).$$

We use the notation $\tilde{f}^{(0)} := f$. As is well known, if f is an integrable function then the conjugate functions $\tilde{f}^{(i)}$ $(i = 1, 2)$ do exist almost everywhere, but they are not integrable, in general.

Fefferman and Stein [5], and Stein [10, 11] verified that if $f \in H_p$ $(0 < p < \infty)$ then the Riesz transforms are also in H_p and

$$\|\tilde{f}^{(i)}\|_{H_p} \le C_p \|f\|_{H_p} \qquad (i = 1, 2). \tag{2}$$

Furthermore, if $1/2 < p < \infty$ then the following equivalence holds:

$$\|f\|_{H_p} \sim \|f\|_p + \|\tilde{f}^{(1)}\|_p + \|\tilde{f}^{(2)}\|_p. \tag{3}$$

3. MARCINKIEWICZ SUMMABILITY OF TWO-DIMENSIONAL FOURIER TRANSFORMS

The definition of the Fourier transform can be extended to functions from $f \in L_p$ $(1 < p \le 2)$ in the usual way (see e.g. Butzer and Nessel [3]). Now suppose that $f \in L_p$ for some $1 \le p \le 2$. The *Dirichlet integral* $s_{t,u} f$ and the *Riesz transforms of the Dirichlet integral* $\tilde{s}_{t,u}^{(i)} f$ are introduced by

$$s_{t,u} f(x, y) := \frac{1}{2\pi} \int_{-t}^{t} \int_{-u}^{u} \hat{f}(v, w) e^{\iota x v + \iota y w} \, dv \, dw \qquad (t, u \in \mathbf{R}_+),$$

$$\tilde{s}_{t,u}^{(1)} f(x, y) := \frac{1}{2\pi} \int_{-t}^{t} \int_{-u}^{u} -\iota \frac{v}{\sqrt{v^2 + w^2}} \hat{f}(v, w) e^{\iota x v + \iota y w} \, dv \, dw \qquad (t, u \in \mathbf{R}_+),$$

and

$$\tilde{s}_{t,u}^{(2)} f(x, y) := \frac{1}{2\pi} \int_{-t}^{t} \int_{-u}^{u} -\iota \frac{w}{\sqrt{v^2 + w^2}} \hat{f}(v, w) e^{\iota x v + \iota y w} \, dv \, dw \qquad (t, u \in \mathbf{R}_+),$$

respectively. It is easy to see that

$$s_{t,u} f(x, y) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x - v, y - w) \frac{2}{\sqrt{2\pi}} \frac{\sin tv}{v} \frac{2}{\sqrt{2\pi}} \frac{\sin uw}{w} \, dv \, dw.$$

For $\alpha, \gamma, T \in \mathbf{R}_+$ the *Marcinkiewicz means* of a tempered distribution f are defined by

$$\begin{aligned}
\sigma_T^{\alpha,\gamma} f(x, y) &:= \frac{\alpha\gamma}{T} \int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha - 1} \left(\frac{t}{T}\right)^{\gamma - 1} s_{t,t} f(x, y) \, dt \\
&= \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x - v, y - w) K_T^{\alpha,\gamma}(v, w) \, dv \, dw,
\end{aligned}$$

where

$$\begin{aligned} K_T^{\alpha,\gamma}(v,w) &:= \frac{2}{\pi}\frac{\alpha\gamma}{T}\int_0^T \left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}\frac{\sin tv}{v}\frac{\sin tw}{w}\,dt \\ &= \frac{\alpha\gamma}{2\pi Tvw} \\ &\quad\times\int_0^T \left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}\left[\cos(v-w)-\cos t(v+w)\right]dt. \end{aligned}$$

The *Riesz transform of the Marcinkiewicz means* are introduced by

$$\begin{aligned} \tilde\sigma_T^{(i);\,\alpha,\gamma}f(x,y) &:= \frac{\alpha\gamma}{T}\int_0^T \left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}\tilde s_{t,t}^{(i)}f(x,y)\,dt \\ &= \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}}\tilde f^{(i)}(x-v,y-w)K_T^{\alpha,\gamma}(v,w)\,dv\,dw. \end{aligned}$$

We can extend the definition of the Marcinkiewicz means to tempered distributions as follows:

$$\sigma_T^{\alpha,\gamma}f := f * K_T^{\alpha,\gamma} \qquad (\alpha,\gamma,T\in\mathbf{R}_+).$$

One can show that $\sigma_T^{\alpha,\gamma}f$ is well defined for all tempered distributions $f\in H_p$ $(0<p\le\infty)$ and for all functions $f\in L_p$ $(1\le p\le\infty)$ (cf. Fefferman and Stein [5]). The extension of the Riesz transforms is

$$\tilde\sigma_T^{(i);\,\alpha,\gamma}f := \tilde f^{(i)} * K_T^{\alpha,\gamma} \qquad (\alpha,\gamma,T\in\mathbf{R}_+;\ i=1,2).$$

The *maximal operators* are defined by

$$\sigma_*^{\alpha,\gamma}f := \sup_{T\in\mathbf{R}_+}\left|\sigma_T^{\alpha,\gamma}f\right|, \qquad \tilde\sigma_*^{(i);\,\alpha,\gamma}f := \sup_{T\in\mathbf{R}_+}\left|\tilde\sigma_T^{(i);\,\alpha,\gamma}f\right|.$$

We write $\tilde\sigma_T^{(0);\,\alpha,\gamma}f = \sigma_T^{\alpha,\gamma}f$ and $\tilde\sigma_*^{(0);\,\alpha,\gamma} = \sigma_*^{\alpha,\gamma}$.

In this paper the constants C depend only on α,γ and the constants C_p (resp. $C_{p,q}$) depend only on p and α,γ (resp. p, q, and α,γ) and may denote different constants in different contexts.

4. ESTIMATIONS OF THE MARCINKIEWICZ KERNEL $K_T^{\alpha,\gamma}$

It is easy to see from the definition of $K_T^{\alpha,\gamma}$ that

$$\left|K_T^{\alpha,\gamma}(x,y)\right| \leq CT^2 \tag{4}$$

and

$$\left|K_T^{\alpha,\gamma}(x,y)\right| \leq Cx^{-1}y^{-1}. \tag{5}$$

By symmetry we can suppose in the whole paper that $0 < y < x$.

LEMMA 1. *Suppose that $0 < \alpha \leq 1$ and $\gamma \geq 2$, or $0 < \alpha \leq 1 = \gamma$, or $\alpha = 1 \leq \gamma$. Then*

$$\left|K_T^{\alpha,\gamma}(x,y)\right| \leq CT^{-\alpha}x^{-1}y^{-1}(x-y)^{-\alpha} \qquad (0<y<x).$$

Proof. Since $x - y \leq x + y$, it is enough to show that

$$T^{-1}\left|\int_0^T \left(1-\left(\frac{t}{T}\right)^\gamma\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}\cos tu\,dt\right| \leq C(Tu)^{-\alpha} \qquad (u>0).$$

The left-hand side is equal to

$$(Tu)^{-1}\left|\int_0^{Tu}\left(1-\left(\frac{x}{Tu}\right)^\gamma\right)^{\alpha-1}\left(\frac{x}{Tu}\right)^{\gamma-1}\cos x\,dx\right|$$
$$= A^{-\alpha\gamma}\left|\int_0^A (A^\gamma - x^\gamma)^{\alpha-1}x^{\gamma-1}\cos x\,dx\right|,$$

where $A = Tu$. Choose $n \in \mathbf{N}$ such that $2n\pi \leq A < 2(n+1)\pi$. Since

$$\left|\int_{2n\pi}^A (A^\gamma - x^\gamma)^{\alpha-1}x^{\gamma-1}\cos x\,dx\right| \leq CA^{\alpha(\gamma-1)}$$

by Lagrange's theorem, it is enough to verify that

$$\left|\int_0^{2n\pi} (A^\gamma - x^\gamma)^{\alpha-1}x^{\gamma-1}\cos x\,dx\right| \leq CA^{\alpha(\gamma-1)}. \tag{6}$$

We decompose the integral $\int_{2k\pi}^{(2k+2)\pi}$ as

$$\int_{2k\pi}^{(2k+2)\pi} = \int_{2k\pi}^{(2k+1/2)\pi} + \int_{(2k+1/2)\pi}^{(2k+1)\pi} + \int_{(2k+1)\pi}^{(2k+3/2)\pi} + \int_{(2k+3/2)\pi}^{(2k+2)\pi}.$$

Let us change the variables $x = y + 2k\pi$, $x = -y + (2k+1)\pi$, $x = y + (2k+1)\pi$, and $x = -y + (2k+2)\pi$ on the intervals $[2k\pi, (4k+1)\pi/$

2], $[(4k+1)\pi/2,(2k+1)\pi]$, $[(2k+1)\pi,(4k+3)\pi/2]$, and $[(4k+3)\pi/2,(2k+2)\pi]$, respectively. Then we obtain

$$\int_0^{2n\pi}(A^\gamma - x^\gamma)^{\alpha-1}x^{\gamma-1}\cos x\,dx = \sum_{k=0}^{n-1}\int_0^{\pi/2} g_k(y)\cos y\,dy \tag{7}$$

where

$$\begin{aligned} g_k(x) := {} & \big(A^\gamma - (x+2k\pi)^\gamma\big)^{\alpha-1}(x+2k\pi)^{\gamma-1} \\ & -\big(A^\gamma - (-x+(2k+1)\pi)^\gamma\big)^{\alpha-1}(-x+(2k+1)\pi)^{\gamma-1} \\ & -\big(A^\gamma - (x+(2k+1)\pi)^\gamma\big)^{\alpha-1}(x+(2k+1)\pi)^{\gamma-1} \\ & +\big(A^\gamma - (-x+(2k+2)\pi)^\gamma\big)^{\alpha-1}(-x+(2k+2)\pi)^{\gamma-1}. \end{aligned}$$

It is easy to check that $g_k'(x) < 0$, which means that g_k is decreasing. Since $g_k(\pi/2) = 0$, we conclude that $g_k(0) \ge 0$. Integrating in (7) we can establish that

$$\left|\int_0^{2n\pi}(A^\gamma - x^\gamma)^{\alpha-1}x^{\gamma-1}\cos x\,dx\right| \le C\big(|f_1(A)| + |f_2(A)|\big)$$

where

$$\begin{aligned} f_1(A) = \sum_{k=0}^{n-1}\Big[&\big(A^\gamma - (2k\pi)^\gamma\big)^\alpha - 2\big(A^\gamma - ((2k+\tfrac{1}{2})\pi)^\gamma\big)^\alpha \\ &+ \big(A^\gamma - ((2k+1)\pi)^\gamma\big)^\alpha\Big] \end{aligned}$$

and

$$\begin{aligned} f_2(A) = \sum_{k=0}^{n-1}\Big[&\big(A^\gamma - ((2k+1)\pi)^\gamma\big)^\alpha - 2\big(A^\gamma - ((2k+\tfrac{3}{2})\pi)^\gamma\big)^\alpha \\ &+ \big(A^\gamma - ((2k+2)\pi)^\gamma\big)^\alpha\Big]. \end{aligned}$$

We are going to verify that

$$|f_1(A)|, |f_2(A)| \le CA^{\alpha(\gamma-1)} \tag{8}$$

which will show (6).

The function $g_1(x) := (A^\gamma - x^\gamma)^\alpha$ $(0 \le x \le A)$ is concave, hence $f_1(A) < 0$ and $f_2(A) < 0$. We have

$$\begin{aligned} f_1'(A) = \sum_{k=0}^{n-1} \alpha\gamma\Big[&\big(A^\gamma - (2k\pi)^\gamma\big)^{\alpha-1} A^{\gamma-1} \\ &-2\Big(A^\gamma - \big((2k+\tfrac{1}{2})\pi\big)^\gamma\Big)^{\alpha-1} A^{\gamma-1} \\ &+\big(A^\gamma - ((2k+1)\pi)^\gamma\big)^{\alpha-1} A^{\gamma-1}\Big]. \end{aligned}$$

Since the function $g(x) := (A^\gamma - x^\gamma)^{\alpha-1}$ $(0 \le x \le A)$ is convex, the expressions in the square bracket are all positive. Hence $f_1'(A) > 0$ and f_1 is increasing. Therefore

$$\begin{aligned} f_1(A) &\ge f_1(2n\pi) \\ &= \pi^{\alpha\gamma} \sum_{k=0}^{n-1} \Big[\big((2n)^\gamma - (2k)^\gamma\big)^\alpha - 2\Big((2n)^\gamma - \big(2k+\tfrac{1}{2}\big)^\gamma\Big)^\alpha \\ &\qquad + \big((2n)^\gamma - (2k+1)^\gamma\big)^\alpha\Big]. \end{aligned} \tag{9}$$

If

$$h(x) := \big((2n)^\gamma - x^\gamma\big)^\alpha \qquad (0 \le x \le 2n)$$

then we get immediately that h' is negative and decreasing. By the Lagrange theorem there exists $2k < \xi < 2k+1$ such that

$$\big((2n)^\gamma - (2k)^\gamma\big)^\alpha - \Big((2n)^\gamma - \big(2k+\tfrac{1}{2}\big)^\gamma\Big)^\alpha = -h'(\xi) \ge -h'(2k)$$

and

$$\big((2n)^\gamma - (2k+1)^\gamma\big)^\alpha - \Big((2n)^\gamma - \big(2k+\tfrac{1}{2}\big)^\gamma\Big)^\alpha = h'(\xi) \ge h'(2k+1).$$

Consequently, by (9),

$$\begin{aligned} \frac{f_1(A)}{\pi^{\alpha\gamma}} &\ge \sum_{k=0}^{n-1} [-h'(2k) + h'(2k+1)] \\ &\ge h'(2n-1) \ge h(2n) - h(2n-1) = -h(2n-1). \end{aligned}$$

Since

$$(2n)^\gamma - (2n-1)^\gamma = \gamma\xi^{\gamma-1} \le \gamma(2n)^{\gamma-1} \qquad (2n-1 < \xi < 2n), \tag{10}$$

we can conclude that

$$\frac{f_1(A)}{\pi^{\alpha\gamma}} \geq -\gamma^{\alpha}(2n)^{(\gamma-1)\alpha} \geq -CA^{(\gamma-1)\alpha}.$$

Inequality (8) can be shown in the same way for the function $f_2(A)$. The proof of Lemma 1 is complete. ∎

LEMMA 2. *Suppose that* $0 < \alpha \leq 1$ *and* $\gamma \geq 2$, *or* $0 < \alpha \leq 1 = \gamma$, *or* $\alpha = 1 \leq \gamma$. *Then*

$$|K_T^{\alpha,\gamma}(x,y)| \leq CT^{1-\alpha}x^{-1}(x-y)^{-\alpha} \qquad (0 < y < x).$$

Proof. Since

$$\cos t(x-y) - \cos t(x+y) = 2ty \sin tu$$

for some $x - y < u < x + y$, we have

$$\begin{aligned} K_T^{\alpha,\gamma}(x,y) &= CT^{-1}x^{-1}\int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1} t \sin tu\,dt \\ &= Cx^{-1}u^{-1}(A)^{-\alpha\gamma}\int_0^A (A^\gamma - x^\gamma)^{\alpha-1}x^\gamma \sin x\,dx. \end{aligned}$$

Similar to the proof of Lemma 1, it is enough to verify that

$$\left|\int_0^{2n\pi}(A^\gamma - x^\gamma)^{\alpha-1}x^\gamma \sin x\,dx\right| \leq CA^{\alpha(\gamma-1)+1} \tag{11}$$

where $A = Tu$ and $2n\pi \leq A < 2(n+1)\pi$.

Changing the variables $x = y + 2k\pi$ and $x = y + (2k+1)\pi$ on the intervals $[2k\pi, (2k+1)\pi]$ and $[(2k+1)\pi, (2k+2)\pi]$, respectively, we obtain

$$\begin{aligned} &\int_0^{2n\pi}(A^\gamma - x^\gamma)^{\alpha-1}x^\gamma \sin x\,dx \\ &\quad = \sum_{k=0}^{n-1}\left(\int_{2k\pi}^{(2k+1)\pi} + \int_{(2k+1)\pi}^{(2k+2)\pi}\right)(A^\gamma - x^\gamma)^{\alpha-1}x^\gamma \sin x\,dx \\ &\quad = \sum_{k=0}^{n-1}\int_0^{\pi} g_k(y)\sin y\,dx, \end{aligned}$$

where

$$g_k(x) := \left(A^\gamma - (x + 2k\pi)^\gamma\right)^{\alpha-1}(x+2k\pi)^\gamma - \left(A^\gamma - (x+(2k+1)\pi)^\gamma\right)^{\alpha-1}(x+(2k+1)\pi)^\gamma.$$

We can show that $g_k < 0$ and $g_k' < 0$. Since

$$f(A) := \sum_{k=0}^{n-2} g_k(\pi) \le \sum_{k=0}^{n-2} g_k(x) \qquad (x \in [0,\pi]),$$

we get that

$$\left| \int_0^{2(n-1)\pi} (A^\gamma - x^\gamma)^{\alpha-1} x^\gamma \sin x\, dx \right| \le |f(A)|. \tag{12}$$

It is easy to see that $f' > 0$ and so we have

$$f(A) \ge f(2n\pi) = \pi^{\alpha\gamma+1} \sum_{k=0}^{n-1} \Big[\left((2n)^\gamma - (2k+1)^\gamma\right)^{\alpha-1}(2k+1)^\gamma - \left((2n)^\gamma - (2k+2)^\gamma\right)^{\alpha-1}(2k+2)^\gamma\Big].$$

If

$$h(x) := \left((2n)^\gamma - x^\gamma\right)^{\alpha-1} x^\gamma \qquad (0 \le x < 2n)$$

then, by an easy computation, h' is positive and increasing. Thus

$$h(2k+1) - h(2k+2) = -h'(\xi) \ge -h'(2k+2) \qquad (2k+1 < \xi < 2k+2).$$

Consequently, by (10),

$$\begin{aligned}
\frac{f(A)}{\pi^{\alpha\gamma+1}} &\ge \sum_{k=0}^{n-2} -h'(2k+2) \ge \frac{1}{2}\int_2^{2n-2} -h'\, d\lambda - h'(2n-2) \\
&\ge -\frac{1}{2}h(2n-2) + h(2n-2) - h(2n-1) \\
&\ge \left((2n)^\gamma - (2n-1)^\gamma\right)^{\alpha-1}(2n-1)^\gamma \\
&\ge -CA^{(\gamma-1)(\alpha-1)}A^\gamma \ge -CA^{\alpha(\gamma-1)+1}.
\end{aligned} \tag{13}$$

Similarly, we can verify that

$$g_{n-1} \ge -CA^{\alpha(\gamma-1)+1},$$

hence

$$\left| \int_{2(n-1)\pi}^{2n\pi} (A^\gamma - x^\gamma)^{\alpha-1} x^\gamma \sin x \, dx \right| \le C A^{\alpha(\gamma-1)+1},$$

which, together with (12) and (13), shows (11). ∎

If $T \ge y^{-1}$ then Lemma 1 implies

$$\left| K_T^{\alpha,\gamma}(x,y) \right| \le C x^{-1} y^{\alpha-1} (x-y)^{-\alpha} \le C y^{\alpha-1} (x-y)^{-1-\alpha}. \quad (14)$$

If $T < y^{-1}$ then we get the same inequality from Lemma 2.

PROPOSITION 1. *Suppose that* $0 < \alpha \le 1$ *and* $\gamma \ge 2$, *or* $0 < \alpha \le 1 = \gamma$, *or* $\alpha = 1 \le \gamma$. *Then* $\int_{\mathbf{R}^2} |K_T^{\alpha,\gamma}| \, d\lambda \le C$ $(T \in \mathbf{R}_+)$.

Proof. It is enough to integrate the kernel function over the set $\{(x,y): 0 < y < x\}$. Let us decompose this set into the union $A \cup B \cup C \cup D \cup E$ where

$$\begin{aligned}
A &:= \{(x,y): 0 < x \le 2/T, 0 < y < x\}, \\
B &:= \{(x,y): x > 2/T, 0 < y \le 1/T\}, \\
C &:= \{(x,y): x > 2/T, 1/T < y \le x/2\}, \\
D &:= \{(x,y): x > 2/T, x/2 < y \le x - 1/T\}, \\
E &:= \{(x,y): x > 2/T, x - 1/T < y < x\}.
\end{aligned}$$

Equation (4) implies $\int_A |K_T^{\alpha,\gamma}| \, d\lambda \le C$. By (14),

$$\int_B \left| K_T^{\alpha,\gamma}(x,y) \right| d\lambda \le C \int_{2/T}^{\infty} \int_0^{1/T} (x - 1/T)^{-1-\alpha} y^{\alpha-1} \, dx \, dy \le C.$$

Since $x - y > x/2$ and $x - y > y$ on the set C, we get from Lemma 1 that

$$\left| K_T^{\alpha,\gamma}(x,y) \right| \le C T^{-\alpha} x^{-1-\alpha/2} y^{-1-\alpha/2}. \quad (15)$$

Thus

$$\int_C \left| K_T^{\alpha,\gamma}(x,y) \right| d\lambda \le C T^{-\alpha} \int_{2/T}^{\infty} \int_{1/T}^{x/2} x^{-1-\alpha/2} y^{-1-\alpha/2} \, dx \, dy \le C.$$

$y > x/2$ and $y > x - y$ on D, hence

$$\begin{aligned}
\int_D \left| K_T^{\alpha,\gamma}(x,y) \right| d\lambda &\le C T^{-\alpha} \int_{2/T}^{\infty} \int_{x/2}^{x-1/T} x^{-1-\alpha/2} (x-y)^{-1-\alpha/2} \, dx \, dy \\
&\le C.
\end{aligned}$$

Finally, by (5),

$$\int_E |K_T^{\alpha,\gamma}(x,y)|\,d\lambda \le C\int_{2/T}^{\infty}\int_{x-1/T}^{x} x^{-2}\,dx\,dy \le C,$$

which completes the proof of the proposition. ∎

5. THE BOUNDEDNESS OF THE MAXIMAL MARCINKIEWICZ OPERATOR ON H_p

For $0 < p < \infty$ a bounded measurable function a is a *p-atom* if there exists a square $Q \subset \mathbf{R}^2$ such that

(i) $\int_Q a(x,y)x^{\mu}y^{\nu}\,dx\,dy = 0$ for all $\mu, \nu \in \mathbf{N}$ with $\sqrt{\mu^2+\nu^2} \le [2(1/p-1)]$, where $[2(1/p-1)]$ denotes the integer part of $2(1/p-1)$,

(ii) $\|a\|_\infty \le |Q|^{-1/p}$,

(iii) $\operatorname{supp} a \subset Q$.

The basic result of the atomic decomposition is stated as follows (see Latter [6], Wilson [16], and also Weisz [12]).

THEOREM B. *A tempered distribution f is in H_p $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a_k = f \tag{16}$$

in the sense of distributions, and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, the equivalence of norms

$$\|f\|_{H_p} \sim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (16), *holds.*

If I is an interval then let $8I$ be the interval with the same center as I and with length $8|I|$. For a square $Q = I_1 \times I_2$ with $|I_1| = |I_2|$ let $8Q = 8I_1 \times 8I_2$.

An operator V which maps the set of distributions into the collection of measurable functions will be called *p-quasi-local* if there exists a constant $C_p > 0$ such that

$$\int_{\mathbf{R}^2 \setminus 8Q} |Va|^p \, d\lambda \le C_p$$

for ever p-atom a where Q is the support of the atom. With the help of the atomic decomposition the following result was proved by the author [13].

THEOREM C. *Suppose that the operator V is sublinear and p-quasi-local for some $0 < p \le 1$. If V is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \le \infty$ then*

$$\|Vf\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Now we can formulate our main result.

THEOREM 1. *If $0 < \alpha \le 1$ and $\gamma \ge 2$, or $0 < \alpha \le 1 = \gamma$, or $\alpha = 1 \le \gamma$, then*

$$\|\sigma_*^{\alpha,\gamma} f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q}) \tag{17}$$

for all $2(\alpha+3)/3(\alpha+2) < p < \infty$ and $0 < q \le \infty$. In particular, if $f \in L_1$ then

$$\lambda(\sigma_*^{\alpha,\gamma} f > \rho) \le \frac{C}{\rho} \|f\|_1 \qquad (\rho > 0). \tag{18}$$

Proof. By Theorems A and C the proof of Theorem 1 will be complete if we show that the operator $\sigma_*^{\alpha,\gamma}$ is p-quasi-local for each $2(\alpha+3)/3(\alpha+2) < p < 1$ and is bounded from L_∞ to L_∞.

The boundedness follows from Proposition 1. Let a be an arbitrary p-atom with support $Q = I \times J$ and $2^{K-1} < |I| = |J| \le 2^K$ $(K \in \mathbf{Z})$. We can suppose that the center of Q is zero. In this case

$$[-2^{K-2}, 2^{K-2}] \subset I, J \subset [-2^{K-1}, 2^{K-1}].$$

To prove the quasi-locality of the operator $\sigma_*^{\alpha,\gamma}$ we have to integrate $|\sigma_*^{\alpha,\gamma} a|^p$ over $\mathbf{R}^2 \setminus 8Q$. It is enough to integrate over the set $A \cup B \cup C \cup$

D where

$$A := \left\{(x, y) \in \mathbf{R}^2 : x \in \left[i2^K, (i+1)2^K\right), y \in \left[j2^K, (j+1)2^K\right), \right.$$
$$\left. 4 \le i < \infty, j = 0\right\}$$

$$B := \left\{(x, y) \in \mathbf{R}^2 : x \in \left[i2^K, (i+1)2^K\right), y \in \left[j2^K, (j+1)2^K\right), \right.$$
$$\left. 4 \le i < \infty, 1 \le j \le i/2 - 2\right\}$$

$$C := \left\{(x, y) \in \mathbf{R}^2 : x \in \left[i2^K, (i+1)2^K\right), y \in \left[j2^K, (j+1)2^K\right), \right.$$
$$\left. 4 \le i < \infty, i/2 - 2 \le j < i - 1\right\}$$

$$D := \left\{(x, y) \in \mathbf{R}^2 : x \in \left[i2^K, (i+1)2^K\right), y \in \left[j2^K, (j+1)2^K\right), \right.$$
$$\left. 4 \le i < \infty, i - 1 \le j \le i\right\}.$$

First we integrate over A. Obviously,

$$\int_A |\sigma_*^{\alpha,\gamma} a(x, y)|^p \, dx \, dy \le \sum_{i=4}^{\infty} \int_{i2^K}^{(i+1)2^K} \int_0^{2^K} |\sigma_*^{\alpha,\gamma} a(x, y)|^p \, dx \, dy. \quad (19)$$

By the definition of the atom,

$$|\sigma_T^{\alpha,\gamma} a(x, y)| = \frac{1}{2\pi} \left| \int_I \int_J a(t, u) K_T^{\alpha,\gamma}(x - t, y - u) \, dt \, du \right|$$
$$\le C_p 2^{-2K/p} \int_I \int_J |K_T^{\alpha,\gamma}(x - t, y - u)| \, dt \, du$$
$$\le C_p 2^{-2K/p} \int_{x - 2^{K-1}}^{x + 2^{K-1}} \int_{y - 2^{K-1}}^{y + 2^{K-1}} |K_T^{\alpha,\gamma}(v, w)| \, dv \, dw. \quad (20)$$

Applying (14) we get

$$\int_{x - 2^{K-1}}^{x + 2^{K-1}} \int_{y - 2^{K-1}}^{y + 2^{K-1}} |K_T^{\alpha,\gamma}(v, w)| \, dv \, dw$$
$$\le C \int_{x - 2^{K-1}}^{x + 2^{K-1}} \int_{y - 2^{K-1}}^{y + 2^{K-1}} (v - |w|)^{-1-\alpha} |w|^{\alpha - 1} \, dv \, dw$$
$$\le C \int_{x - 2^{K-1}}^{x + 2^{K-1}} (v - 3 \cdot 2^{K-1})^{-1-\alpha} (3 \cdot 2^{K-1})^{\alpha} \, dv \, dw$$
$$\le C 2^K \left((i-2) 2^K\right)^{-1-\alpha} \left(2^K\right)^{\alpha} = C i^{-1-\alpha}.$$

Hence

$$\int_A |\sigma_*^{\alpha,\gamma} a(x,y)|^p \, dx\, dy \le C_p 2^{2K} 2^{-2K} \sum_{i=4}^{\infty} i^{-(1+\alpha)p}$$

which is finite if $p > 1/(\alpha + 1)$.

Next we integrate over B. Similar to (19), we get that

$$\begin{aligned}
&\int_B |\sigma_*^{\alpha,\gamma} a(x,y)|^p \, dx\, dy \\
&\quad \le \sum_{i=4}^{\infty} \sum_{j=1}^{i/2-2} \int_{i2^K}^{(i+1)2^K} \int_{j2^K}^{(j+1)2^K} \sup_{T \ge r_{i,j}} |\sigma_T^{\alpha,\gamma} a(x,y)|^p \, dx\, dy \\
&\quad \le \sum_{i=4}^{\infty} \sum_{j=1}^{i/2-2} \int_{i2^K}^{(i+1)2^K} \int_{j2^K}^{(j+1)2^K} \sup_{T < r_{i,j}} |\sigma_T^{\alpha,\gamma} a(x,y)|^p \, dx\, dy \\
&\quad = (B_1) + (B_2)
\end{aligned}$$

where $r_{i,j} := [2^{-K}/(ij)^{\delta/2}]$ with $\delta > 0$ chosen later. For (B_1) we have by (15) that

$$\begin{aligned}
&\int_{x-2^{K-1}}^{x+2^{K-1}} \int_{y-2^{K-1}}^{y+2^{K-1}} |K_T^{\alpha,\gamma}(v,w)| \, dv\, dw \\
&\quad \le C T^{-\alpha} \int_{x-2^{K-1}}^{x+2^{K-1}} \int_{y-2^{K-1}}^{y+2^{K-1}} v^{-1-\alpha/2} w^{-1-\alpha/2} \, dv\, dw \\
&\quad \le C 2^{K\alpha} (ij)^{\alpha\delta/2} 2^{2K} (i2^K)^{-1-\alpha/2} (j2^K)^{-1-\alpha/2} = C (ij)^{\alpha\delta/2-\alpha/2-1}.
\end{aligned}$$

Taking into account (20) we can see that

$$(B_1) \le C_p \sum_{i=4}^{\infty} \sum_{j=1}^{i/2-2} (ij)^{(\alpha\delta/2-\alpha/2-1)p}$$

and this is finite if

$$\delta < \frac{(2+\alpha)p - 2}{\alpha p}. \tag{21}$$

It is easy to see that

$$\sigma_T^{\alpha,\gamma} a(x,y) = \frac{\alpha\gamma}{2\pi T}\int_0^T \left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}$$
$$\times \int_{-t}^{t}\int_{-t}^{t} \hat{a}(v,w)e^{\iota xv+\iota yw}\,dv\,dw\,dt$$
$$= \int_{-T}^{T}\int_{-T}^{T} \hat{a}(v,w)e^{\iota xv+\iota yw}\left(1-\left(\frac{\max(|v|,|w|)}{T}\right)^{\gamma}\right)^{\alpha} dv\,dw.$$

Since

$$|\hat{a}(v,w)| \le C(|v|+|w|)|I|^{3-2/p}$$

(see Weisz [13]), we conclude

$$|\sigma_T^{\alpha,\gamma} a| \le \int_{-T}^{T}\int_{-T}^{T} |\hat{a}(v,w)|\,dv\,dw \le CT^3|I|^{3-2/p}.$$

Thus

$$\sup_{T<r_{i,j}} |\sigma_T^{\alpha,\gamma} a(x,y)| \le C(ij)^{-3\delta/2}2^{-2K/p} \tag{22}$$

and so

$$(B_2) \le C_p \sum_{i=4}^{\infty}\sum_{j=1}^{i/2-2} 2^{2K}2^{-2K}(ij)^{-3\delta p/2}$$

which converges if $\delta > 2/3p$. This together with (21) implies

$$p > \frac{2(\alpha+3)}{3(2+\alpha)}. \tag{23}$$

We define (C_1) and (C_2) in the same way as (B_1) and (B_2), with the only difference being that we take the sum $\sum_{j=i/2-2}^{i-2}$. By Lemma 1,

$$\int_{x-2^{K-1}}^{x+2^{K-1}}\int_{y-2^{K-1}}^{y+2^{K-1}} |K_T^{\alpha,\gamma}(v,w)|\,dv\,dw$$
$$\le CT^{-\alpha}\int_{x-2^{K-1}}^{x+2^{K-1}}\int_{y-2^{K-1}}^{y+2^{K-1}} v^{-1}w^{-1}(v-w)^{-\alpha}\,dv\,dw$$
$$\le C2^{K\alpha}(ij)^{\alpha\delta/2}(i2^K)^{-1}(j2^K)^{-1}((i-j)2^K)^{-\alpha} = Ci^{\alpha\delta-2}(i-j)^{-\alpha}$$

and, consequently,

$$(C_1) \le C_p \sum_{i=4}^{\infty} \sum_{j=i/2-2}^{i-2} i^{\alpha\delta p - 2p}(i-j)^{-\alpha p} \le C_p \sum_{i=4}^{\infty} i^{\alpha\delta p - 2p - \alpha p + 1},$$

because $p < 1/\alpha$. Thus $\delta < ((2+\alpha)p - 2)/\alpha p$. (C_2) can be estimated in the same way as (B_2), hence (23) holds.

Now we integrate over D. Observe that

$$\int_D |\sigma_*^{\alpha,\gamma} a(x,y)|^p \, dx\, dy$$
$$\le \sum_{i=4}^{\infty} \sum_{j=i-1}^{i} \int_{i2^K}^{(i+1)2^K} \int_{j2^K}^{(j+1)2^K} |\sigma_*^{\alpha,\gamma} a(x,y)|^p \, dx\, dy.$$

By (5),

$$\int_{x-2^{K-1}}^{x+2^{K-1}} \int_{y-2^{K-1}}^{y+2^{K-1}} |K_T^{\alpha,\gamma}(v,w)| \, dv\, dw \le C \int_{x-2^{K-1}}^{x+2^{K-1}} \int_{y-2^{K-1}}^{y+2^{K-1}} v^{-1} w^{-1} \, dv\, dw$$
$$\le C 2^{2K} (i2^K)^{-1} (j2^K)^{-1} = C(ij)^{-1}.$$

Hence

$$\int_D |\sigma_*^{\alpha,\gamma} a(x,y)|^p \, dx\, dy \le C_p \sum_{i=4}^{\infty} \sum_{j=i-1}^{i} (ij)^{-p} \le C_p \sum_{i=4}^{\infty} i^{-2p}$$

and this implies that $p > 1/2$.

Thus we have proved that $\sigma_*^{\alpha,\gamma}$ is p-quasi-local for each $2(\alpha+3)/3(\alpha+2) < p < 1$. Hence (17) for $p = q$ follows from Theorem C. Applying Theorem A, we obtain the general case of (17). Let us point out this result for $p = 1$ and $q = \infty$. If $f \in L_1$ then (1) implies

$$\|\sigma_*^{\alpha,\gamma} f\|_{1,\infty} = \sup_{\rho > 0} \rho \lambda(\sigma_*^{\alpha,\gamma} f > \rho) \le C \|f\|_{H_{1,\infty}} \le C \|f\|_1$$

which shows (18). The proof of the theorem is complete. ∎

We can state the same for the maximal operator of the Riesz transforms of the Marcinkiewicz means.

THEOREM 2. *Suppose that* $0 < \alpha \le 1$ *and* $\gamma \ge 2$, *or* $0 < \alpha \le 1 = \gamma$, *or* $\alpha = 1 \le \gamma$. *For* $i = 1, 2$, *we have*

$$\left\|\tilde{\sigma}_*^{(i);\alpha,\gamma} f\right\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $2(\alpha+3)/3(\alpha+2) < p < \infty$ *and* $0 < q \le \infty$. *In particular, if* $f \in L_1$ *then*

$$\lambda\big(\tilde{\sigma}_*^{(i);\,\alpha,\gamma} f > \rho\big) \le \frac{C}{\rho}\|f\|_1 \qquad (\rho > 0).$$

Proof. Using (2), Theorem 1 for $p = q$, and the fact that $\tilde{\sigma}_*^{(i);\,\alpha,\gamma} f = \sigma_*^{\alpha,\gamma}\tilde{f}^{(i)}$ we obtain

$$\big\|\tilde{\sigma}_*^{(i);\,\alpha,\gamma} f\big\|_p = \big\|\sigma_*^{\alpha,\gamma}\tilde{f}^{(i)}\big\|_p \le C_p\big\|\tilde{f}^{(i)}\big\|_{H_p} \le C_p\|f\|_{H_p} \qquad \big(f \in H_p\big).$$

Now Theorem 2 follows from Theorem A in the usual way. ∎

Since the set of those functions $f \in L_1$ whose Fourier transform has a compact support is dense in L_1 (see Wiener [15]), the weak-type inequalities of Theorems 1 and 2 and the usual density argument (see Marcinkiewicz and Zygmund [7]) imply

COROLLARY 1. *Suppose that* $0 < \alpha \le 1$ *and* $\gamma \ge 2$, *or* $0 < \alpha \le 1 = \gamma$, *or* $\alpha = 1 \le \gamma$. *If* $f \in L_1$ *then* $\sigma_T^{\alpha,\gamma} f \to f$ *a.e. and* $\tilde{\sigma}_T^{(i);\,\alpha,\gamma} f \to \tilde{f}^{(i)}$ *a.e.* $(i = 1, 2)$ *as* $T \to \infty$.

Note that $\tilde{f}^{(i)}$ $(i = 1, 2)$ is not necessarily integrable whenever f is. The first convergence result for Fourier series and for $\gamma = 1$ is due to Zhizhiashvili [17, 18].

THEOREM 3. *Suppose that* $0 < \alpha \le 1$ *and* $\gamma \ge 2$, *or* $0 < \alpha \le 1 = \gamma$, *or* $\alpha = 1 \le \gamma$. *For* $i = 0, 1, 2$ *we have*

$$\big\|\tilde{\sigma}_T^{(i);\,\alpha,\gamma} f\big\|_{H_{p,q}} \le C_{p,q}\|f\|_{H_{p,q}} \qquad \big(f \in H_{p,q}\big)$$

uniformly in T, *whenever* $2(\alpha+3)/3(\alpha+2) < p < \infty$ *and* $0 < q \le \infty$. *In this case* $\tilde{\sigma}_T^{(i);\,\alpha,\gamma} f \to \tilde{f}^{(i)}$ *in the* $H_{p,q}$ *norm as* $T \to \infty$.

Proof. Since $(\sigma_T^{\alpha,\gamma} f)^{\sim(i)} = \tilde{\sigma}_T^{(i),\,\alpha,\gamma} f$, we have by Theorems 1 and 2 that

$$\big\|\big(\sigma_T^{\alpha,\gamma} f\big)^{\sim(i)}\big\|_p \le C_p\|f\|_{H_p} \qquad \big(f \in H_p,\, i = 0, 1, 2\big)$$

for all $T > 0$. Inequality (3) implies that

$$\|\sigma_T^{\alpha,\gamma} f\|_{H_p} \le C_p\|f\|_{H_p} \qquad \big(f \in H_p,\, T > 0\big).$$

Hence, by (2),

$$\big\|\tilde{\sigma}_T^{(i);\,\alpha,\gamma} f\big\|_{H_p} \le C_p\|f\|_{H_p} \qquad \big(f \in H_p,\, T > 0,\, i = 1, 2\big)$$

and the theorem follows by interpolation from Theorem A. ∎

It is an open question whether Theorems 1–3 are true for $p \le 2(\alpha + 3)/3(\alpha + 2)$.

We can prove similarly to the proof of Lemma 4 in [14] that also in this case

$$\sigma_T^{1+h,\gamma} f(x,y) = \frac{h(h+1)\gamma}{T} \int_0^T \left(1 - \left(\frac{s}{T}\right)^{\gamma}\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \sigma_s^{1,\gamma} f(x,y)\, ds$$

where $h > 0$. From this it follows that $\sigma_*^{\alpha,\gamma} f \le C \sigma_*^{1,\gamma} f$ whenever $\alpha > 1$. This shows that Theorems 1, 2, and 3 hold also for $\alpha > 1$.

COROLLARY 2. *If $\alpha, \gamma \ge 1$ then all inequalities of Theorems* 1, 2, *and* 3 *and the convergence results of Corollary* 1 *hold for every $8/9 < p < \infty$ and $0 < q \le \infty$.*

Remark. If we define the Marcinkiewicz means by

$$\sigma_T^{\alpha,\gamma} f(x,y) = \frac{\alpha\gamma}{T} \int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1} \left(\frac{t}{T}\right)^{\gamma-1} s_{\mu t, \nu t} f(x,y)\, dt,$$

where $\mu, \nu > 0$, then the result above can be proved in the same way.

REFERENCES

1. C. Bennett and R. Sharpley, "Interpolation of Operators," Pure and Applied Mathematics, Vol. 129, Academic Press, New York, 1988.
2. J. Bergh and L. Löfström, "Interpolation Spaces, an Introduction," Springer, Berlin, 1976.
3. P. L. Butzer and R. J. Nessel, "Fourier Analysis and Approximation," Birkhäuser Verlag, Basel, 1971.
4. C. Fefferman, N. M. Rivière, and Y. Sagher, Interpolation between H^p spaces: The real method, *Trans. Amer. Math. Soc.* **191** (1974), 75–81.
5. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–194.
6. R. H. Latter, A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms, *Studia Math.* **62** (1978), 92–101.
7. J. Marcinkiewicz and A. Zygmund, On the summability of double Fourier series, *Fund. Math.* **32** (1939), 122–132.
8. J. Marcinkiewicz, Sur une méthode remarquable de sommation des séries doubles de Fourier, *Ann. Scuola Norm. Super. Pisa* **8** (1939), 149–160.
9. N. M. Rivière and Y. Sagher, Interpolation between L^∞ and H^1: The real method, *J. Funct. Anal.* **14** (1973), 401–409.
10. E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
11. E. M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals," Princeton Univ. Press, Princeton, NJ, 1993.

12. F. Weisz, "Martingale Hardy Spaces and their Applications in Fourier Analysis," Lecture Notes in Mathematics, Vol. 1568, Springer, Berlin, 1994.
13. F. Weisz, Cesàro summability of multi-dimensional trigonometric-Fourier series, *J. Math. Anal. Appl.* **204** (1996), 419–431.
14. F. Weisz, Riesz means of Fourier transforms and Fourier series on Hardy spaces, *Studia Math.* **131** (1998), 253–270.
15. N. Wiener, "The Fourier Integral and Certain of Its Applications," Dover, New York, 1959.
16. J. M. Wilson, A simple proof of the atomic decomposition for $H^p(\mathbf{R})$, $0 < p \leq 1$, *Studia Math.* **74** (1982), 25–33.
17. L. Zhizhiashvili, A generalization of a theorem of Marcinkiewicz, *Math. USSR Izv.* **2** (1968), 1065–1075.
18. L. Zhizhiashvili, "Trigonometric Fourier Series and Their Conjugates," Kluwer Academic, Dordrecht, 1996.