

# Linear orderings and the sets of periods for star maps

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## Abstract

We show that Baldwin's characterization of the set of periods of continuous self maps of the  $n$ -star can be expressed in terms of a finite number of linear orderings.

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## 1. Introduction

In an interesting paper [4], which extends Sharkovskii's theorem to the  $n$ -star, Baldwin has shown that the set of periods of a continuous map from an  $n$ -star into itself can be expressed as a union of “tails” of a finite set of partial orderings of the natural numbers. On the other hand, in [1] it was shown that for the class of continuous maps of the 3-star into itself which leave the branching point fixed, the set of periods can be expressed as “tails” of three *linear* orderings (one of which was Sharkovskii's ordering and the other two were called red and green orderings). In [2] it was noted that these three orderings can be thought of as certain orderings associated to the fractions  $1/2$  and  $1/3$ . This suggests that this is

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in fact the general situation. To be more precise, this suggests that the set of periods of a continuous map from an  $n$ -star into itself can be expressed as the union of “tails” of linear orderings associated to all fractions in the interval  $(0, 1)$  with denominator smaller than or equal to  $n$ , defined in certain subsets of the natural numbers. This result was meant to be proved in the first part of the paper [3] by Alsedà and Moreno. However, despite of the fact that the strategy used in proving that result was correct, the orderings associated to fractions considered in [3] were not well defined (they were not antisymmetric). The aim of this paper is to give a correct definition of the orderings associated to fractions whilst proving the correct version of Alsedà and Moreno’s result [3, Theorem 3.1]. This gives a constructive proof of Theorem 1.6 of [4] which, in particular, proves Conjecture 13.4 of [1].

As it was already noted in [3], the fact that it is possible to characterize the sets of periods of continuous self maps of the  $n$ -star in terms of linear orderings associated to fractions suggests that the sets of periods of such maps may arise in some way from “rotation intervals” (see [2] where an example of such a situation was given). However, this relation is still far from being understood.

The paper is organized as follows. In Section 2 we give the basic definitions and preliminary results. In particular, we recall Baldwin’s theorem on the set of periods of star maps. In Section 3 we discuss a general approach to the problem of constructing finitely many linear orderings such that any “tail” of a Baldwin ordering can be expressed as a finite union of “tails” of those linear orderings. As we shall see this approach is not completely satisfactory and in Section 4 we will adopt a constructive strategy. This section, for clarity, is divided into four subsections. In the first two we define and study the orderings associated to fractions in the coprime and non-coprime cases, respectively. In the third subsection of Section 4 we state and prove the main results of the paper. The last subsection is devoted to the study of the structure of the orderings associated to fractions and to giving a finite algorithm for their construction. As an application, an example of one of these orderings is given.

## 2. Basic definitions and preliminary results

We define an  $n$ -star as the subspace of the complex numbers consisting on all  $z \in \mathbb{C}$  such that  $z^n \in [0, 1]$ . The  $n$ -star will be denoted by  $\mathbb{X}_n$  and the class of all continuous maps from  $\mathbb{X}_n$  to itself will be denoted by  $\mathcal{X}_n$ . Each map from  $\mathcal{X}_n$  will be called an  $n$ -star map. The class of all  $n$ -star maps  $f$  such that  $f(0) = 0$  will be denoted by  $\mathcal{X}_n^\circ$ . We note that the 1-star and the 2-star are homeomorphic to a closed interval of the real line. Thus, in what follows, when talking about  $\mathbb{X}_n$  or  $\mathcal{X}_n$  we shall always assume that  $n \geq 2$ .

As usual, if  $f \in \mathcal{X}_n$  we shall write  $f^k$  to denote  $f \circ f \circ \dots \circ f$  ( $k$  times). A point  $x \in \mathbb{X}_n$  such that  $f^k(x) = x$  but  $f^j(x) \neq x$  for  $j = 1, 2, \dots, k - 1$  will be called a *periodic point of  $f$  of period  $k$* . The set of periods of all periodic points of a map  $f \in \mathcal{X}_n$  will be denoted by  $\text{Per}(f)$ . As is always the case when we consider sets of periods, the set  $\mathbb{N}$  of natural numbers does not contain 0.

In the next subsection we summarize the characterization of the set of periods for maps from  $\mathcal{X}_n$ .

### 2.1. Baldwin's theorem on the set of periods of maps from $\mathcal{X}_n$

Baldwin's characterization of the set of periods for  $n$ -star maps is given in terms of Baldwin's partial orderings (see [4]). To define them we first recall the Sharkovskii's ordering  $_{\text{Sh}} \geq$ , which is defined on  $\mathbb{N} \cup \{2^\infty\}$  as follows:

$$\begin{aligned} 3 &_{\text{Sh}} > 5 &_{\text{Sh}} > 7 &_{\text{Sh}} > \cdots &_{\text{Sh}} > 2 \cdot 3 &_{\text{Sh}} > 2 \cdot 5 &_{\text{Sh}} > 2 \cdot 7 &_{\text{Sh}} > \cdots &_{\text{Sh}} > 4 \cdot 3 \\ &_{\text{Sh}} > 4 \cdot 5 &_{\text{Sh}} > 4 \cdot 7 &_{\text{Sh}} > \cdots &_{\text{Sh}} > \cdots &_{\text{Sh}} > 2^n \cdot 3 &_{\text{Sh}} > 2^n \cdot 5 &_{\text{Sh}} > 2^n \cdot 7 &_{\text{Sh}} > \cdots \\ &_{\text{Sh}} > 2^\infty &_{\text{Sh}} > \cdots &_{\text{Sh}} > 2^n &_{\text{Sh}} > \cdots &_{\text{Sh}} > 16 &_{\text{Sh}} > 8 &_{\text{Sh}} > 4 &_{\text{Sh}} > 2 &_{\text{Sh}} > 1. \end{aligned}$$

Now we define the *Baldwin partial orderings*  $_t \geq$  for all positive integers  $t \geq 2$ . We denote by  $\mathbb{N}_t$  the set  $\{t, t+1, t+2, t+3, \dots\} \cup \{1, t \cdot 2^\infty\}$  and by  $\mathbb{N}_t^\vee$  the set  $\{mt : m \in \mathbb{N}\} \cup \{1, t \cdot 2^\infty\}$ . Then the ordering  $_t \geq$  (we will also use the symbol  $_t >$  in the natural way) is defined in  $\mathbb{N}_t$  as follows. For  $k, m \in \mathbb{N}_t$  we write  $m_t \geq k$  if one of the following cases holds:

- (i)  $k = 1$  or  $k = m$ ,
- (ii)  $k, m \in \mathbb{N}_t^\vee \setminus \{1\}$  and  $m/t_{\text{Sh}} > k/t$ ,
- (iii)  $k \in \mathbb{N}_t^\vee$  and  $m \notin \mathbb{N}_t^\vee$ ,
- (iv)  $k, m \notin \mathbb{N}_t^\vee$  and  $k = im + jt$  with  $i, j \in \mathbb{N}$ ,

where in case (ii) we use the following arithmetic rule for the symbol  $t \cdot 2^\infty$ :  $t \cdot 2^\infty / t = 2^\infty$ . We note that thanks to the inclusion of the symbol  $t \cdot 2^\infty$ , each subset  $A$  of  $\mathbb{N}_t$  has a least upper bound  $m$  with respect to the ordering  $_t \geq$  and if  $m \neq t \cdot 2^\infty$  then  $m \in A$ . We also note that the ordering  $_2 \geq$  on  $\mathbb{N}_2 = \mathbb{N} \cup \{2 \cdot 2^\infty\}$  coincides with Sharkovskii's ordering on  $\mathbb{N} \cup \{2^\infty\}$  (by identifying the symbol  $2 \cdot 2^\infty$  with  $2^\infty$ ).

Let  $\star \geq$  be an ordering on some set  $\mathbb{N}_\star$ . A non-empty set  $S \subset \mathbb{N}_\star \cap \mathbb{N}$  will be called a *tail of the ordering*  $\star \geq$  if for each  $m \in S$  we have  $\{k \in \mathbb{N} : m_\star \geq k\} \subset S$ . Clearly, the union of tails of an ordering  $\star \geq$  is also a tail of  $\star \geq$ . There exists a particular type of tails of orderings that plays a special role in this theory. It is the set of all elements which are smaller than or equal to a given element of  $\mathbb{N}_\star$ : Given  $m \in \mathbb{N}_\star$  we will denote by  $S_\star(m)$  the set  $\{k \in \mathbb{N} : m_\star \geq k\}$ , which is clearly a tail of  $\star \geq$ . With this notation we have that  $S \subset \mathbb{N}_\star \cap \mathbb{N}$  is a tail of the ordering  $\star \geq$  if and only if for each  $m \in S$  it follows that  $S_\star(m) \subset S$ .

**Remark 2.1.** Assume that  $\star \geq$  is a linear ordering on  $\mathbb{N}_\star$  such that each subset  $A$  of  $\mathbb{N}_\star$  has a least upper bound  $m$  with respect to the  $\star \geq$  ordering and if  $m \in \mathbb{N}$  then  $m \in A$ . Then,  $S$  is a tail of  $\star \geq$  if and only if there exists  $m \in \mathbb{N}_\star$  such that  $S = S_\star(m)$ . The “if part” of this statement is obvious. To prove the “only if part” take  $m$  equal to the least upper bound of  $S$  and observe that  $S_\star(m) \subset S \subset S_\star(m)$ .

The following result is due to Baldwin [4] and characterizes the set of periods of  $n$ -star maps.

**Theorem 2.1.** *Let  $f \in \mathcal{X}_n$ . Then  $\text{Per}(f)$  is a finite union of tails of the orderings  ${}_t \succcurlyeq$  with  $2 \leq t \leq n$ . Conversely, given a set  $A$  which can be expressed as a finite union of tails of the orderings  ${}_t \succcurlyeq$  with  $2 \leq t \leq n$ , there exists a map  $f \in \mathcal{X}_n^\circ$  such that  $\text{Per}(f) = A$ .*

As it has been said in the introduction, the aim of this paper is to prove that the set of periods of an  $n$ -star map (which, as the preceding theorem shows, is a finite union of tails of the orderings  ${}_t \succcurlyeq$  with  $2 \leq t \leq n$ ) can be obtained as the union of tails of the linear orderings associated to all fractions in the interval  $(0, 1)$  with denominator smaller than or equal to  $n$ , defined in certain subsets of the natural numbers. In fact we will prove that any tail of the  ${}_t \succcurlyeq$  ordering can be expressed as a union of tails of the linear orderings associated to the fractions of the form  $s/t$  with  $s \in \{1, 2, \dots, t-1\}$ .

### 3. General approach

As we mentioned, we want to find  $t-1$  linear orderings such that any tail of the ordering  ${}_t \succcurlyeq$  can be expressed as the union of tails of those orderings. In this section we prove the existence of such orderings in a general framework. However, as we shall see, this approach has the serious drawback that it does not give us any information on the obtained linear orderings from the point of view of the dynamics. The following result is the key tool of this paper.

**Proposition 3.1.** *Let  $\star \succcurlyeq$  be an ordering on a set  $\mathbb{N}_\star$ , let  $\{\mathbb{N}_\star^1, \mathbb{N}_\star^2, \dots, \mathbb{N}_\star^{t-1}\}$  be a cover of  $\mathbb{N}_\star$  and, for each  $s \in \{1, 2, \dots, t-1\}$ , let  $\star_s \succcurlyeq$  be an ordering on  $\mathbb{N}_{\star_s}$  such that  $\mathbb{N}_\star^s \subset \mathbb{N}_{\star_s} \subset \mathbb{N}_\star$  and*

- (a) *if  $m \in \mathbb{N}_{\star_s}$ ,  $k \in \mathbb{N}_\star$  and  $m \star_s k$  then  $k \in \mathbb{N}_{\star_s}$  and  $m \star_s k$ ,*
- (b) *if  $m \in \mathbb{N}_\star^s$ ,  $k \in \mathbb{N}_{\star_s}$  and  $m \star_s k$  then  $m \star k$ .*

*Then any tail of  $\star_s \succcurlyeq$  is also a tail of  $\star \succcurlyeq$ . Conversely, if  $A \subset \mathbb{N}_\star$  is a tail of  $\star \succcurlyeq$  then there exists  $R \subset \{1, 2, \dots, t-1\}$  such that  $A = \bigcup_{s \in R} A_s$ , for some tail  $A_s$  of  $\star_s \succcurlyeq$ .*

**Proof.** Let  $A \subset \mathbb{N}_{\star_s}$  be a tail of  $\star_s \succcurlyeq$ , let  $m \in A$  and let  $k \in \mathbb{N}$  be such that  $m \star k$ . From (a) it follows that  $m \star_s k$  and, hence,  $k \in A$ . Consequently,  $A$  is also a tail of  $\star \succcurlyeq$ .

Now assume that  $A \subset \mathbb{N}_\star$  is a tail of  $\star \succcurlyeq$ . For each  $s \in \{1, 2, \dots, t-1\}$  we define  $A_s$  as the set of all  $k \in \mathbb{N} \cap \mathbb{N}_\star$  such that there is  $m_k \in A \cap \mathbb{N}_\star^s$  with  $m_k \star k$ . From (a) it follows that  $k \in \mathbb{N}_{\star_s}$  and  $m_k \star_s k$ . Consequently,  $A_s \subset \mathbb{N}_{\star_s}$ . Also, observe that if  $m \in A$  then  $m \in A \cap \mathbb{N}_\star^s$  for some  $s$  and  $m \star_s m$ . Hence,  $m \in A_s$ , and so  $A \subset \bigcup_{s=1}^{t-1} A_s$ . Assume now that  $k \in A_s$ . Since  $m_k$  is in  $A$ , and  $A$  is a tail of  $\star \succcurlyeq$ , we get  $k \in A$ . Therefore

$$A = \bigcup_{s=1}^{t-1} A_s = \bigcup_{s \in R} A_s,$$

where  $R = \{s \in \{1, 2, \dots, t-1\} : A_s \neq \emptyset\}$ .

To complete the proof we have to show that  $A_s$  is a tail of  $\star_s \geq$  for each  $s \in R$ . Let  $k \in A_s$  and  $u \in \mathbb{N} \cap \mathbb{N}_{\star_s}$  be such that  $k \star_s \geq u$ . We have  $m_k \star_s \geq k \star_s \geq u$ . Thus, from (b) it follows that  $m_k \star_s \geq u$ , which implies  $u \in A_s$ .  $\square$

Now we will use the above proposition to prove the existence of the  $t - 1$  linear orderings we are looking for. Since  $\geq_t$  is linear on the set  $\mathbb{N}_t^\vee$ , it is enough to define those linear orderings on the set  $\mathbb{N}_t \setminus \mathbb{N}_t^\vee$  and then attach  $\mathbb{N}_t^\vee$  with the ordering  $\geq_t$  at the end of each of them. Let us look closer at  $\geq_t$  restricted to  $\mathbb{N}_t \setminus \mathbb{N}_t^\vee$ . This ordering is defined by the condition (iv) from the preceding section (or  $k = m$ ). Assume that  $k - m$  is divisible by  $t$ . If  $m_t > k$  then  $k = im + jt$  with  $i, j \in \mathbb{N}$ , so  $m < k$ . On the other hand, if  $m < k$  then  $k = m + jt$  for some  $j \in \mathbb{N}$ , so  $m_t > k$ . Thus, for  $s = 1, 2, \dots, t - 1$ , the ordering  $\geq_t$  restricted to the set

$$\mathbb{N}_t^s = \{m \in \mathbb{N}_t \setminus \mathbb{N}_t^\vee : m \equiv s \pmod{t}\} = \{s + jt : j \in \mathbb{N}\}$$

is linear.

Hence, our aim is to find linear orderings  $\star_s \geq$  on some sets  $\mathbb{N}_{\star_s} \subset \mathbb{N}_{\star} = \mathbb{N}_t \setminus \mathbb{N}_t^\vee$  such that  $\mathbb{N}_{\star_s} \supset \mathbb{N}_{\star}^s = \mathbb{N}_t^s$  and the assumptions of Proposition 3.1 are satisfied.

**Lemma 3.1.** *Let  $\star \geq$  be an ordering on a set  $\mathbb{N}_{\star}$  and let  $\{\mathbb{N}_{\star}^1, \mathbb{N}_{\star}^2, \dots, \mathbb{N}_{\star}^{t-1}\}$  be a cover of  $\mathbb{N}_{\star}$  such that  $\star \geq$  restricted to each  $\mathbb{N}_{\star}^s$  is a linear ordering. Then there exist linear orderings  $\star_s \geq$  defined on*

$$\mathbb{N}_{\star_s} := \{k \in \mathbb{N}_{\star} : m \star k \text{ for some } m \in \mathbb{N}_{\star}^s\} \supset \mathbb{N}_{\star}^s \quad (1)$$

such that (a) and (b) of Proposition 3.1 are satisfied.

**Proof.** Any ordering  $\star_s \geq$  defined on  $\mathbb{N}_{\star_s}$  will be identified (accordingly to the usual definition of an ordering) with the set

$$\mathcal{C}_{\star_s} = \{(m, k) \in \mathbb{N}_{\star_s} \times \mathbb{N}_{\star_s} : m \star_s k\}.$$

With this notation and in view of (1), conditions (a) and (b) of Proposition 3.1 can be restated as follows:

- (A)  $\{(m, k) \in \mathcal{C}_{\star} : m \in \mathbb{N}_{\star_s}\} \subset \mathcal{C}_{\star_s}$ ,
- (B)  $\{(m, k) \in \mathcal{C}_{\star_s} : m \in \mathbb{N}_{\star}^s\} \subset \mathcal{C}_{\star}$ .

When we say that some ordering satisfies (A) or (B), we mean that it does so when it replaces  $\star_s \geq$  in these conditions.

Denote by  $\star_s \geq$  the ordering  $\star \geq$  restricted to  $\mathbb{N}_{\star_s}$ . Clearly,  $\star_s \geq$  satisfies (A) and (B). Moreover, any ordering on  $\mathbb{N}_{\star_s}$  containing  $\mathcal{C}_{\star_s}$  also satisfies (A).

Now we consider the family  $\mathcal{F}_s$  of all orderings (non-necessarily linear) defined on  $\mathbb{N}_{\star_s}$  which contain  $\mathcal{C}_{\star_s}$  and satisfy (B), ordered by inclusion. Observe that  $\mathcal{F}_s$  is non-empty since  $\star_s \geq$  is an element of  $\mathcal{F}_s$ . Moreover, the union of an ascending sequence of elements of  $\mathcal{F}_s$  is also an element of  $\mathcal{F}_s$ . Hence, by the Zorn's Lemma, there is a maximal (with respect to the inclusion) element  $\star_s \geq$  of  $\mathcal{F}_s$ . Now we have to prove that  $\star_s \geq$  is linear. To

do it we assume that there exist  $u, v \in \mathbb{N}_{\star_s}$  which are not comparable by  $\star_s \geq$  and we will arrive to a contradiction.

We enlarge the ordering  $\star_s \geq$  to a new ordering  $\star_s^u \geq$  in  $\mathbb{N}_{\star_s}$  by adding to  $\mathcal{C}_{\star_s}$  the pairs  $(m, k)$  (that is, we add the relations  $m \star_s^u \geq k$ ) such that  $m \star_s \geq u$  and  $v \star_s \geq k$ . It is easy to see that  $\star_s^u \geq$  is still an ordering and contains  $\mathcal{C}_{\star_s}$ .

Since the ordering  $\star_s \geq$  is maximal, the ordering  $\star_s^u \geq$  cannot satisfy (B). That is, there exist  $m \in \mathbb{N}_{\star_s}^s$  and  $k \in \mathbb{N}_{\star_s}$  such that  $m \star_s^u \geq k$  and  $v \star_s \geq k$  but not  $m \star_s \geq k$ . Thus, since  $\star_s \geq$  satisfies (B), we cannot have  $m \star_s \geq k$ , so we cannot have  $m \star_s \geq v$ . By reversing the roles of  $u$  and  $v$  we get  $m' \in \mathbb{N}_{\star_s}^s$  such that  $m' \star_s \geq v$ , but not  $m' \star_s^u \geq u$ . However, the ordering  $\star_s \geq$  is linear, so either  $m \star_s \geq m' \star_s \geq v$  or  $m' \star_s \geq m \star_s \geq u$ . In both cases we get a contradiction. This completes the proof.  $\square$

In our concrete situation  $\mathbb{N}_{\star}$  is  $\mathbb{N}_t \setminus \mathbb{N}_t^\vee$ ,  $\mathbb{N}_{\star}^s$  is  $\{s + jt : j \in \mathbb{N}\}$  and  $\star \geq$  is  $_t \geq$ . Thus,

$$\mathbb{N}_{\star_s} = \{k \in \mathbb{N}_t \setminus \mathbb{N}_t^\vee : s + jt \geq k \text{ for some } j \in \mathbb{N}\}.$$

If  $s + jt \neq k$  then the condition  $s + jt \geq k$  is equivalent to  $k = i(s + jt) + j't$  with  $i, j' \in \mathbb{N}$ , that is to  $k = is + j''t$  with  $i, j'' \in \mathbb{N}$  and  $j'' > i$ . If we rename  $\mathbb{N}_{\star_s}$  as  $\mathbb{N}_{t,s}^*$  then we can write

$$\mathbb{N}_{t,s}^* = \{s + t\} \cup \{is + tj : i, j \in \mathbb{N}, j > i\}.$$

We rename also  $\star_s \geq$  as  $\star_{t,s} \geq$ . We can now extend  $\star_{t,s} \geq$  from  $\mathbb{N}_{t,s}^*$  to  $\mathbb{N}_{t,s}^* \cup \mathbb{N}_t^\vee$  by attaching  $\mathbb{N}_t^\vee$  with the ordering  $_t \geq$  at the end. Then as a consequence of Proposition 3.1, Lemma 3.1 and Remark 2.1 we get the following theorem.

**Theorem 3.1.** *Given  $t \geq 2$  there exist linear orderings  $\star_{t,s} \geq$  on  $\mathbb{N}_{t,s}^* \cup \mathbb{N}_t^\vee$  ( $s = 1, 2, \dots, t - 1$ ) such that any set  $A \subset \mathbb{N}_t$  is a tail of  $_t \geq$  if and only if there exists a non-empty subset  $R$  of  $\{1, 2, \dots, t - 1\}$  and tails  $A_s$  of  $\star_{t,s} \geq$  for  $s \in R$  such that  $A = \bigcup_{s \in R} A_s$ .*

This theorem, together with Theorem 2.1, asserts that it is possible to state the characterization of the set of periods of  $n$ -star maps in terms of finitely many linear orderings. This proves Conjecture 13.4 of [1].

It may seem that Theorem 3.1 solves our problem completely. However, observe that Lemma 3.1 does not give us any information on the orderings  $\star_{t,s} \geq$  related to the dynamics. Consequently, it cannot serve as the base for the investigations of a possible deeper nature of those orderings. It is like trying to develop number theory by considering  $\mathbb{N}$  not with the natural ordering, but with a random one.

Thus, while using Proposition 3.1, we will apply a more constructive approach. We will define the linear orderings we are looking for, by the order of capturing of denominators of fractions in the interval  $(0, 1)$  as we move from some rational number of the form  $p/q$  to the left (the left will suffice because moving from  $p/q$  to the right gives the same result as moving from  $(q - p)/q$  to the left). The formal definition of these orderings is given in the next section.

#### 4. Orderings associated to fractions

When defining the linear ordering associated to a fraction  $p/q$  and proving the main results of the paper, we will consider separately the case when  $p$  and  $q$  are coprime and the case when  $p$  and  $q$  have common factors, since the ideas involved are of completely different nature. We will consider these two cases in next two subsections. In Section 4.3 we will state and prove the main results of the paper and, finally, in Section 4.4 we will describe how to compute these orderings and we will give an example of the algorithm.

##### 4.1. Coprime case

Fix  $p, q \in \mathbb{N}$  with  $p < q$  and  $p$  and  $q$  coprime. We define the *ordering associated to the fraction  $p/q$*  on  $\mathbb{N}_{p,q} = \mathbb{N}_q$ , which we denote by  ${}_{p,q}\geq$  (as usual, the symbol  ${}_{p,q}>$  will be also used in the natural way), as follows. For each  $n \in \mathbb{N}$  we will denote the largest integer  $l$  such that  $l/n < p/q$  by  $l_{p,q}(n)$  (that is,  $l_{p,q}(n) = \lceil np/q \rceil - 1$ ), where  $\lceil \cdot \rceil$  denotes the ceiling function. Assume that  $k, m \in \mathbb{N}_{p,q}$ . Then we write  $m {}_{p,q}\geq k$  if one of the following cases holds:

- (i)  $k = 1$  or  $k = m$ ,
- (ii)  $k, m \in \mathbb{N}_q^\vee \setminus \{1\}$  and  $m/q_{\text{sh}} > k/q$ ,
- (iii)  $k \in \mathbb{N}_q^\vee$  and  $m \notin \mathbb{N}_q^\vee$ ,
- (iv)  $k, m \notin \mathbb{N}_q^\vee$  and  $l_{p,q}(m)/m < l_{p,q}(k)/k$ ,
- (v)  $k, m \notin \mathbb{N}_q^\vee$ ,  $l_{p,q}(m)/m = l_{p,q}(k)/k$  and  $m > k$

(where in (ii) we use the arithmetic rule  $q \cdot 2^\infty / q = 2^\infty$ ). Clearly,  ${}_{p,q}\geq$  is a well-defined linear ordering. Moreover, observe that conditions (i)–(iii) above coincide with conditions (i)–(iii) of the definition of the ordering  ${}_q\geq$ . Thus, when  $k \in \mathbb{N}_q^\vee$ , the condition  $m {}_{p,q}\geq k$  is equivalent to  $m {}_q\geq k$ . Note also that, by means of the inclusion of the symbol  $q \cdot 2^\infty$ , each subset  $A$  of  $\mathbb{N}_{p,q}$  has a least upper bound  $m$  with respect to the  ${}_{p,q}\geq$  ordering, and if  $m \neq q \cdot 2^\infty$  then  $m = \max_{{}_{p,q}\geq} A$ .

Next, we will show that the orderings  ${}_{p,q}\geq$  satisfy statements (a) and (b) of Proposition 3.1 with  $\star\geq$ ,  $\mathbb{N}_\star$ ,  $\star_s\geq$  and  $\mathbb{N}_{\star_s}$  replaced by  ${}_q\geq$ ,  $\mathbb{N}_q$ ,  ${}_{p,q}\geq$  and  $\mathbb{N}_{p,q}$ , respectively. Furthermore, we replace  $\mathbb{N}_\star^s$  by the set  $\mathbb{N}_q^{\vee p}$  defined as follows:

$$\mathbb{N}_q^{\vee p} = \{m \in \mathbb{N}_q \setminus \mathbb{N}_q^\vee : mp \equiv 1 \pmod{q}\} \cup \mathbb{N}_q^\vee.$$

Although in this case  $\mathbb{N}_{p,q} = \mathbb{N}_q$ , we will distinguish between these two sets in order to have the same statements as in the other case.

**Proposition 4.1.** *Let  $p$  and  $q$  be coprime positive integers such that  $p < q$ . Then the following statements hold:*

- (a) if  $m \in \mathbb{N}_{p,q}$ ,  $k \in \mathbb{N}_q$  and  $m {}_q\geq k$  then  $k \in \mathbb{N}_{p,q}$  and  $m {}_{p,q}\geq k$ ,
- (b) if  $m \in \mathbb{N}_q^{\vee p}$ ,  $k \in \mathbb{N}_{p,q}$  and  $m {}_{p,q}\geq k$  then  $m {}_q\geq k$ .

To prove Proposition 4.1 we will introduce more notation and prove two easy technical lemmas. Let  $p, q \in \mathbb{N}$  be such that  $0 < p < q$  and let  $k \in \mathbb{N}$ . Then there is a unique  $\gamma_{p,q}(k) \in \{1, 2, \dots, q\}$  such that  $\gamma_{p,q}(k) \equiv kp \pmod{q}$ . Note that if  $k \equiv m \pmod{q}$  then  $\gamma_{p,q}(k) = \gamma_{p,q}(m)$ .

**Lemma 4.1.** *Let  $p, q \in \mathbb{N}$  be such that  $0 < p < q$  and let  $k, m \in \mathbb{N}$ . Then  $l_{p,q}(m)/m < l_{p,q}(k)/k$  is equivalent to  $m/\gamma_{p,q}(m) < k/\gamma_{p,q}(k)$ .*

**Proof.** Since  $kp - \gamma_{p,q}(k)$  is divisible by  $q$  and

$$\frac{kp - \gamma_{p,q}(k)}{q} < \frac{kp}{q} \leq \frac{kp - \gamma_{p,q}(k)}{q} + 1,$$

we see that  $l_{p,q}(k) = (kp - \gamma_{p,q}(k))/q$ . Therefore,  $l_{p,q}(m)/m < l_{p,q}(k)/k$  is equivalent to  $(mp - \gamma_{p,q}(m))/m < (kp - \gamma_{p,q}(k))/k$ , which in turn is equivalent to  $m/\gamma_{p,q}(m) < k/\gamma_{p,q}(k)$ .  $\square$

**Lemma 4.2.** *Let  $p, q \in \mathbb{N}$  be such that  $0 < p < q$  and let  $m, k \in \mathbb{N}$  be such that  $k = im + jq$  for some  $i, j \in \mathbb{N}$ . Then  $m/\gamma_{p,q}(m) < k/\gamma_{p,q}(k)$ .*

**Proof.** Clearly,  $\gamma_{p,q}(k) = \gamma_{p,q}(im) \leq i\gamma_{p,q}(m)$ . Thus,

$$m\gamma_{p,q}(k) \leq im\gamma_{p,q}(m) < k\gamma_{p,q}(m). \quad \square$$

**Proof of Proposition 4.1.** If  $k \in \mathbb{N}_q^\vee$  then  $m_q \geq k$  is equivalent to  $m_{p,q} \geq k$ . Hence, (a) and (b) follow in this case. In the rest of the proof we will assume that  $k \notin \mathbb{N}_q^\vee$  and  $m \neq k$ .

To prove (a) note that from the definition of the ordering  ${}_q \geq$  it follows that  $m \notin \mathbb{N}_q^\vee$  and  $k = im + jq$  with  $i, j \in \mathbb{N}$ . Hence, we get  $m_{p,q} \geq k$  from Lemmas 4.2 and 4.1.

Now we prove (b). As above,  $m \notin \mathbb{N}_q^\vee$ , and hence  $mp \equiv 1 \pmod{q}$ . Then  $\gamma_{p,q}(m) = 1$ . Moreover, the number  $kp - m\gamma_{p,q}(k)$  is divisible by  $q$  and hence  $k - m\gamma_{p,q}(k)$  is also divisible by  $q$ . Since  $m_{p,q} \geq k$ , by Lemma 4.1, we have either  $k > m\gamma_{p,q}(k)$  or  $k = m\gamma_{p,q}(k)$  and  $k < m$ . However, if  $k = m\gamma_{p,q}(k)$  then  $k \geq m$ , so we must have  $k > m\gamma_{p,q}(k)$ . Then  $j = (k - m\gamma_{p,q}(k))/q$  is a natural number and  $k = \gamma_{p,q}(k)m + jq$ . This implies  $m_q \geq k$ .  $\square$

#### 4.2. Non-coprime case

We are going to define the ordering associated to the fraction  $np/nq$ , where  $p$  and  $q$  are coprime positive integers such that  $p < q$  and  $n > 1$  is an integer. To this end we set

$$\mathbb{N}_{np,nq} := n \cdot (\mathbb{N}_q \setminus \{1\}) \cup \{1\} \quad \text{and} \quad \mathbb{N}_{nq}^{\vee np} := n \cdot (\mathbb{N}_q^{\vee p} \setminus \{1\}) \cup \{1\}.$$

That is,  $m \in \mathbb{N}_{np,nq} \setminus \{1\}$  if and only if  $m/n \in \mathbb{N}_q \setminus \{1\}$  and  $m \in \mathbb{N}_{nq}^{\vee np} \setminus \{1\}$  if and only if  $m/n \in \mathbb{N}_q^{\vee p} \setminus \{1\}$ . Moreover,  $m \in \mathbb{N}_{nq}^{\vee p} \setminus \{1\}$  if and only if  $m/n \in \mathbb{N}_q^{\vee} \setminus \{1\}$ . Since  $\mathbb{N}_q^{\vee p} \subset \mathbb{N}_{p,q}$ , we have also  $\mathbb{N}_{nq}^{\vee np} \subset \mathbb{N}_{np,nq}$ .

Now, we define the ordering associated to the fraction  $np/nq$ , denoted by  ${}_{np,nq} \geq$ , as follows. For each  $k, m \in \mathbb{N}_{np,nq}$  we write  $m_{np,nq} \geq k$  if and only if either  $k = 1$  or



$1 \notin \{k, m\}$  and  $m/n_{p,q} > k/n$ . Clearly,  $_{np,nq} \geq$  is a well-defined linear ordering, each subset  $A$  of  $\mathbb{N}_{np,nq}$  has a least upper bound  $m$  with respect to  $_{np,nq} \geq$ , and if  $m \neq nq \cdot 2^\infty$  then  $m = \max_{_{np,nq} \geq} A$ . Moreover, one can see that  $m_{np,nq} \geq k$  is equivalent to  $m_{nq} \geq k$  whenever  $k \in \mathbb{N}_{nq}^\vee$ . In what follows we will also use the symbol  $_{np,nq} >$  in the natural way.

The analogue of Proposition 4.1 is now:

**Proposition 4.2.** *Let  $p$  and  $q$  be coprime positive integers such that  $p < q$  and let  $n > 1$  be an integer. Then, the following statements hold:*

- (a) *if  $m \in \mathbb{N}_{np,nq}$ ,  $k \in \mathbb{N}_{nq}$  and  $m_{nq} \geq k$  then  $k \in \mathbb{N}_{np,nq}$  and  $m_{np,nq} \geq k$ ,*
- (b) *if  $m \in \mathbb{N}_{np,nq}^\vee$ ,  $k \in \mathbb{N}_{np,nq}$  and  $m_{np,nq} \geq k$  then  $m_{nq} \geq k$ .*

**Proof.** As in the proof of Proposition 4.1, the proposition follows if  $k \in \mathbb{N}_{nq}^\vee$ . So, in the rest of the proof we will assume that  $k \notin \mathbb{N}_{nq}^\vee$  and  $m \neq k$ . In both cases this implies that  $m \in \mathbb{N}_{np,nq} \setminus \mathbb{N}_{nq}^\vee$ . Hence,  $m$  is a multiple of  $n$  and  $m/n \in \mathbb{N}_{p,q} \setminus \mathbb{N}_q^\vee = \mathbb{N}_q \setminus \mathbb{N}_q^\vee$ .

Under the assumptions of (a) it follows that  $k = im + jnq$  with  $i, j \in \mathbb{N}$  and hence  $k$  is a multiple of  $n$ . Thus,  $k/n \in \mathbb{N}_q \setminus \mathbb{N}_q^\vee$  and  $k \in \mathbb{N}_{np,nq}$ . On the other hand,  $k/n = im/n + jq$  which implies  $m/n_q > k/n$ . Consequently, in view of Proposition 4.1(a),  $m/n_{p,q} \geq k/n$ , which gives us  $m_{np,nq} \geq k$ . This completes the proof of (a).

Now we prove (b). In this case we have  $k/n \in \mathbb{N}_q \setminus \mathbb{N}_q^\vee$  and  $m/n \in \mathbb{N}_q^{\vee p}$ . On the other hand, since  $m_{np,nq} > k$ , we have  $m/n_{p,q} > k/n$ . Thus,  $m/n_q > k/n$  by Proposition 4.1(b). This is equivalent to the existence of  $i, j \in \mathbb{N}$  such that  $k/n = im/n + jq$ , so  $k = im + jnq$ , and hence  $m_{nq} > k$ . This completes the proof of (b).  $\square$

#### 4.3. Main results

The following theorem is the main result of the paper. It relates the orderings  $_{p,q} \geq$  with the Baldwin's partial orderings  $_t \geq$ . It is the analogue of Theorem 3.1 for the orderings  $_{p,q} \geq$ .

**Theorem 4.1.** *Let  $q \geq 2$  and let  $p \in \{1, 2, \dots, q-1\}$ . Any tail of the ordering  $_{p,q} \geq$  is also a tail of  $_q \geq$ . Conversely, for each tail  $A$  of the ordering  $_q \geq$  there exist  $\alpha_p \in \mathbb{N}_{p,q} \cap (A \cup \{q \cdot 2^\infty\})$  for  $p = 1, 2, \dots, q-1$  such that  $A = \bigcup_{p=1}^{q-1} S_{p,q}(\alpha_p)$ .*

**Proof.** Since for each  $p \in \{1, 2, \dots, q-1\}$  every subset  $A$  of  $\mathbb{N}_{p,q}$  has a least upper bound  $m$  with respect to the ordering  $_{p,q} \geq$ , and if  $m \neq q \cdot 2^\infty$  then  $m \in A$ , by Remark 2.1 it follows that each tail of  $_{p,q} \geq$  is of the form  $S_{p,q}(m)$  for some  $m \in \mathbb{N}_{p,q}$ . Now, the theorem follows from Propositions 3.1, 4.1 and 4.2 (together with the fact that 1 belongs to each tail of each ordering considered), provided that we show that the sets  $\mathbb{N}_q^{\vee p}$ ,  $p = 1, 2, \dots, q-1$ , cover  $\mathbb{N}_q$ .

To prove this property, note first that  $\mathbb{N}_q^\vee \subset \mathbb{N}_q^{\vee 1}$ . Now, consider  $m \in \mathbb{N}_q \setminus \mathbb{N}_q^\vee$ . If  $m$  and  $q$  are coprime, there is  $p \in \{1, 2, \dots, q-1\}$  such that  $mp \equiv 1 \pmod{q}$  and then  $m \in \mathbb{N}_q^{\vee p}$ . If the greatest common divisor  $n$  of  $m$  and  $q$  is larger than 1 then similarly,  $m/n \in \mathbb{N}_{q/n}^{\vee p'}$  for some  $p' \in \{1, 2, \dots, q/n-1\}$ , and then  $m \in \mathbb{N}_q^{\vee p}$  for  $p = p'n$ . This completes the proof.  $\square$

The following corollary to Theorem 4.1 characterizes the possible sets of periods of maps from  $\mathcal{X}_n$  in terms of the orderings  $_{p,q} \geq$ . It generalizes to the  $n$ -star the main theorem of [1].

**Corollary 4.1.** *Let  $f \in \mathcal{X}_n$ . Then there exist  $\alpha_l^t \in \mathbb{N}_{l,t}$  for  $t = 2, 3, \dots, n$  and  $l = 1, 2, \dots, t-1$  such that  $\text{Per}(f) = \bigcup_{t=2}^n \bigcup_{l=1}^{t-1} S_{l,t}(\alpha_l^t)$ . Conversely, given  $\alpha_l^t \in \mathbb{N}_{l,t}$  for  $t = 2, 3, \dots, n$  and  $l = 1, 2, \dots, t-1$ , there exists a map  $f \in \mathcal{X}_n$  such that  $\text{Per}(f) = \bigcup_{t=2}^n \bigcup_{l=1}^{t-1} S_{l,t}(\alpha_l^t)$ .*

**Proof.** By Theorem 2.1 we know that  $\text{Per}(f) = \bigcup_{t=2}^n A_t$  where each  $A_t$  is a tail of the ordering  $_{t} \geq$ . Therefore, in view of the second statement of Theorem 4.1, for each  $t \in \{2, 3, \dots, n\}$  and  $l \in \{1, 2, \dots, t-1\}$  there exist  $\alpha_l^t \in \mathbb{N}_{l,t} \cap (A_t \cup \{t \cdot 2^\infty\})$  such that  $A_t = \bigcup_{l=1}^{t-1} S_{l,t}(\alpha_l^t)$ . This proves the first statement. Now we prove the second one. Since each  $S_{l,t}(\alpha_l^t)$  is a tail of  $_{l,t} \geq$ , in view of Theorem 4.1, it follows that it is a tail of  $_{t} \geq$ . Therefore, for each  $t \in \{2, 3, \dots, n\}$ ,  $\bigcup_{l=1}^{t-1} S_{l,t}(\alpha_l^t)$  is a tail of  $_{t} \geq$  because the union of tails of an ordering is a tail of the same ordering. Then the corollary follows from Theorem 2.1.  $\square$

#### 4.4. Computation of the orderings $_{p,q} \geq$

This subsection is devoted to the study of the structure of the orderings  $_{p,q} \geq$  in order to give a simple algorithm to construct them. As an application we will produce an example of an ordering  $_{p,q} \geq$ .

As we mentioned before, the orderings  $_{p,q} \geq$  have two parts. The second part consists of all elements of  $\mathbb{N}_q^\vee$  ordered as in the orderings  $_{q} \geq$ , and this part can be derived immediately from the Sharkovskii's ordering. The first part consists of all elements of  $\mathbb{N}_{p,q} \setminus \mathbb{N}_q^\vee$ , and these elements are larger than the elements of  $\mathbb{N}_q^\vee$  in the ordering  $_{p,q} \geq$ . Therefore, we only need to compute the orderings  $_{p,q} \geq$  on  $\mathbb{N}_{p,q} \setminus \mathbb{N}_q^\vee$ . Moreover, we only need to compute them in the case  $(p, q) = 1$ . Therefore, in the rest of this subsection we assume that  $p$  and  $q$  are coprime positive integers such that  $p < q$ . Set  $\mathbb{N}_q^\bullet = \mathbb{N} \setminus q \cdot \mathbb{N} \supset \mathbb{N}_{p,q} \setminus \mathbb{N}_q^\vee$ . We extend the ordering  $_{p,q} \geq$  to  $\mathbb{N}_q^\bullet$  according to the rules (iv) and (v) of the definition of the ordering  $_{p,q} \geq$  in the coprime case. To display the ordering  $_{p,q} \geq$  on  $\mathbb{N}_q^\bullet$ , we will compute the sequence

$$n_1 \text{ }_{p,q} > n_2 \text{ }_{p,q} > n_3 \text{ }_{p,q} > \dots, \quad (2)$$

given by  $_{p,q} \geq$  on  $\mathbb{N}_q^\bullet$ .

The following lemma will allow us to produce recursively the sequence (2). For  $k \in \mathbb{N}_q^\bullet$  we denote by  $b_{p,q}(k)$  and  $c_{p,q}(k)$  the quotient and the remainder of  $k$  divided by  $\gamma_{p,q}(k)q$ , respectively. That is,  $k = b_{p,q}(k)\gamma_{p,q}(k)q + c_{p,q}(k)$  where  $b_{p,q}(k) \geq 0$  and  $0 < c_{p,q}(k) < \gamma_{p,q}(k)q$ .

**Lemma 4.3.** *Let  $p, q \in \mathbb{N}$  be such that  $0 < p < q$  with  $p$  and  $q$  coprime and let  $k, m \in \mathbb{N}$ . If either  $b_{p,q}(k) < b_{p,q}(m)$  or  $b_{p,q}(k) = b_{p,q}(m)$  and  $c_{p,q}(k) \text{ }_{p,q} > c_{p,q}(m)$  then  $k \text{ }_{p,q} > m$ .*

**Proof.** From the definition of  $b_{p,q}(k)$  and  $c_{p,q}(k)$  it follows that

$$\frac{m}{\gamma_{p,q}(m)} - \frac{k}{\gamma_{p,q}(k)} = q(b_{p,q}(m) - b_{p,q}(k)) + \frac{c_{p,q}(m)}{\gamma_{p,q}(m)} - \frac{c_{p,q}(k)}{\gamma_{p,q}(k)}. \quad (3)$$

Hence, by Lemma 4.1, if the right-hand side of (3) is positive then  $k_{p,q} > m$ .

If  $b_{p,q}(k) < b_{p,q}(m)$  then  $q(b_{p,q}(m) - b_{p,q}(k)) \geq q$ . The lemma follows in this case because both  $c_{p,q}(m)/\gamma_{p,q}(m)$  and  $c_{p,q}(k)/\gamma_{p,q}(k)$  are strictly between 0 and  $q$  and, hence, the right-hand side of (3) is positive. If  $b_{p,q}(k) = b_{p,q}(m)$  and

$$\frac{c_{p,q}(k)}{\gamma_{p,q}(c_{p,q}(k))} < \frac{c_{p,q}(m)}{\gamma_{p,q}(c_{p,q}(m))},$$

then, since  $\gamma_{p,q}(c_{p,q}(k)) = \gamma_{p,q}(k)$  and  $\gamma_{p,q}(c_{p,q}(m)) = \gamma_{p,q}(m)$ , the right-hand side of (3) is also positive.

In the remaining case we have  $b_{p,q}(k) = b_{p,q}(m)$ ,  $c_{p,q}(k)/\gamma_{p,q}(c_{p,q}(k)) = c_{p,q}(m)/\gamma_{p,q}(c_{p,q}(m))$  and  $c_{p,q}(k) > c_{p,q}(m)$ . Hence, since  $\gamma_{p,q}(c_{p,q}(k)) = \gamma_{p,q}(k)$  and  $\gamma_{p,q}(c_{p,q}(m)) = \gamma_{p,q}(m)$ , it follows that  $\gamma_{p,q}(k) > \gamma_{p,q}(m)$ . From (3) we get  $k/\gamma_{p,q}(k) = m/\gamma_{p,q}(m)$ , and thus  $k > m$ . Hence,  $k_{p,q} > m$  because, in view of Lemma 4.1,  $k/\gamma_{p,q}(k) = m/\gamma_{p,q}(m)$  is equivalent to  $l_{p,q}(m)/m = l_{p,q}(k)/k$ .  $\square$

Since  $k \equiv i \pmod{q}$  implies  $\gamma_{p,q}(k) = \gamma_{p,q}(i)$ , for each  $k \in \mathbb{N}_q^\Delta$  the number  $c_{p,q}(k)$  belongs to the set

$$\mathcal{B} = \{i + jq : i \in \{1, 2, \dots, q-1\} \text{ and } j \in \{0, 1, \dots, \gamma_{p,q}(i)-1\}\}.$$

Since  $\{\gamma_{p,q}(i) : 0 < i < q\} = \{1, 2, \dots, q-1\}$ , the cardinality of  $\mathcal{B}$  is  $q(q-1)/2$ . On the other hand,  $\gamma_{p,q}(n_i + q\gamma_{p,q}(n_i)) = \gamma_{p,q}(n_i)$ . Consequently,  $b_{p,q}(n_i + q\gamma_{p,q}(n_i)) = b_{p,q}(n_i) + 1$  and  $c_{p,q}(n_i + q\gamma_{p,q}(n_i)) = c_{p,q}(n_i)$ . Therefore, from Lemma 4.3 it follows that

- (i)  $n_{i+jq(q-1)/2} = n_i + jq\gamma_{p,q}(n_i)$  for each  $i \in \{1, 2, \dots, q(q-1)/2\}$  and  $j \in \mathbb{N} \cup \{0\}$ ,
- (ii)  $\{n_1, n_2, \dots, n_{q(q-1)/2}\} = \mathcal{B}$ .

Thus, to construct the sequence (2), it is enough to order the elements of the set  $\mathcal{B}$  according to  $_{p,q} \geq$  (using the definition of  $_{p,q} \geq$ ) and then use (i) to construct the whole sequence.

**Remark 4.1.** From the above algorithm one can easily see that the ordering  $_{1,2} \geq$  is just the Sharkovskii's ordering, and the ordering  $_{1,3} \geq$  is the green ordering considered in [1] to study the set of periods of maps from  $\mathcal{X}_3^\circ$ . Moreover,  $_{2,3} \geq$  (restricted to  $\mathbb{N}_{2,3} \setminus \{4\}$ ) is the red ordering defined in [1].

As an example we compute the ordering  $_{2,5} \geq$  by using the above algorithm.

**Example 4.1.** Here we construct the ordering  $_{2,5} \geq$  on  $N_{2,5} = \mathbb{N}_5$ . The sequence  $n_{1,2,5} > n_{2,2,5} > \dots > n_{q(q-1)/2,2,5}$  of the leading elements of  $\mathbb{N}_5^\Delta$  is

$$2_{2,5} > 1_{2,5} > 4_{2,5} > 7_{2,5} > 12_{2,5} > 9_{2,5} > 6_{2,5} > 3_{2,5} > 17_{2,5} > 14$$

and the corresponding sequence  $5 \cdot \gamma_{2,5}(\cdot)$  is

$$20, 10, 15, 20, 20, 15, 10, 5, 20, 15.$$

Hence, the ordering  ${}_{2,5} \geq$  on  $\mathbb{N}_5^\Delta$  is

$$\begin{aligned} 2_{2,5} &> 1_{2,5} > 4_{2,5} > 7_{2,5} > 12_{2,5} > 9_{2,5} > 6_{2,5} > 3_{2,5} > 17_{2,5} > 14_{2,5} > 22_{2,5} \\ 2_{2,5} &> 11_{2,5} > 19_{2,5} > 27_{2,5} > 32_{2,5} > 24_{2,5} > 16_{2,5} > 8_{2,5} > 37_{2,5} > 29_{2,5} > \cdots \\ 2_{2,5} &> 2 + i \cdot 20_{2,5} > 1 + i \cdot 10_{2,5} > 4 + i \cdot 15_{2,5} > 7 + i \cdot 20_{2,5} > 12 + i \cdot 20_{2,5} \\ 2_{2,5} &> 9 + i \cdot 15_{2,5} > 6 + i \cdot 10_{2,5} > 3 + i \cdot 5_{2,5} > 17 + i \cdot 20_{2,5} \\ 2_{2,5} &> 14 + i \cdot 15_{2,5} > \cdots \end{aligned}$$

and the  ${}_{2,5} \geq$  ordering on  $\mathbb{N}_{2,5}$  is

$$\begin{aligned} 7_{2,5} &> 12_{2,5} > 9_{2,5} > 6_{2,5} > 17_{2,5} > 14_{2,5} > 22_{2,5} > 11_{2,5} > 19_{2,5} > 27_{2,5} > 32_{2,5} \\ 2_{2,5} &> 24_{2,5} > 16_{2,5} > 8_{2,5} > 37_{2,5} > 29_{2,5} > \cdots_{2,5} > 5 \cdot 3_{2,5} > 5 \cdot 5_{2,5} > 5 \cdot 7_{2,5} \\ 2_{2,5} &> \cdots_{2,5} > 5 \cdot 3 \cdot 2_{2,5} > 5 \cdot 5 \cdot 2_{2,5} > 5 \cdot 7 \cdot 2_{2,5} > \cdots_{2,5} > 5 \cdot 3 \cdot 4_{2,5} > 5 \cdot 5 \cdot 4_{2,5} \\ 2_{2,5} &> 5 \cdot 7 \cdot 4_{2,5} > \cdots_{2,5} > \cdots_{2,5} > 5 \cdot 3 \cdot 2^n_{2,5} > 5 \cdot 5 \cdot 2^n_{2,5} > 5 \cdot 7 \cdot 2^n_{2,5} > \cdots_{2,5} \\ 2_{2,5} &> 5 \cdot 2^\infty_{2,5} > \cdots_{2,5} > 5 \cdot 2^n_{2,5} > \cdots_{2,5} > 5 \cdot 16_{2,5} > 5 \cdot 8_{2,5} > 5 \cdot 4_{2,5} > 5 \cdot 2_{2,5} \\ 2_{2,5} &> 5 \cdot 1_{2,5} > 1. \end{aligned}$$

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