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Normal families and value distribution in connection with composite functions

E.F. Clifford *

Department of Mathematical Sciences, Loughborough University, Leicestershire, LE11 3TU, UK

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Abstract

We prove a value distribution result which has several interesting corollaries. Let $k \in \mathbb{N}$, let $\alpha \in \mathbb{C}$ and let f be a transcendental entire function with order less than $1/2$. Then for every nonconstant entire function g, we have that $(f \circ g)^{(k)} - \alpha$ has infinitely many zeros. This result also holds when $k = 1$, for every transcendental entire function g. We also prove the following result for normal families. Let $k \in \mathbb{N}$, let f be a transcendental entire function with $\rho(f) < 1/k$, and let $a_0, \ldots, a_{k-1}, a$ be analytic functions in a domain Ω. Then the family of analytic functions g such that

$$(f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z),$$

in Ω, is a normal family.

Keywords: Normal families; Value distribution; Composite functions; Nevanlinna theory

* Fax: +44 (0) 1509 223 969.
E-mail address: e.clifford@lboro.ac.uk.

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1. Introduction

In [1], Hinchliffe proves the following result which provides a criterion for normal families in connection with composite functions.

Theorem 1.1 (Hinchliffe [1]). *Let f be a transcendental meromorphic function in the plane, and let Ω be a domain in $\mathbb{C}$. If $\mathbb{C}^* \setminus f(\mathbb{C}) = \emptyset$, $\{\infty\}$ or $\{\alpha, \beta\}$, where α and β are two distinct values in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, then the family*

$$\mathcal{G} = \{g\colon\ g \text{ is analytic in } \Omega,\ \ f \circ g \text{ has no fixpoints in } \Omega\}$$

is a normal family in Ω.

We note that this criterion is that $(f \circ g)(z) \neq z$ in Ω, or that $(f \circ g)^{(0)}(z) - a(z)$ has no zeros in Ω, where $a(z) \equiv z$, for $g \in \mathcal{G}$. Theorem 1.1 then motivates the idea of a criterion for normal families in connection with composite functions involving $(f \circ g)^{(k)}(z) \neq 0$ for $k \in \mathbb{N}$. This idea is reinforced by the following theorem and corollary by Langley and Zheng, where n.e. is used as an abbreviation for *"nearly everywhere,"* that is, to denote the phrase *"outside a set of finite measure."*

Theorem 1.2 (Langley and Zheng [2]). *Let $k \in \mathbb{N}$. Suppose that f and g are transcendental entire functions of finite order. Suppose also that*

$$\bar{N}\big(r, 1/(f \circ g)^{(k)}\big) = O\big(T(r, g)\big) \quad (n.e.). \tag{1}$$

Then

$$T(r, f) \neq o(r^{1/k}) \quad \text{as } r \to \infty.$$

Corollary 1.3. *Let $k \in \mathbb{N}$. Suppose that f is a transcendental entire function such that $\rho(f) < 1/k$. Suppose that g is an entire function of finite order such that*

$$(f \circ g)^{(k)}(z) \neq 0$$

on $\mathbb{C}$. Then g is a polynomial.

We note that the example $f(z) = e^z$ shows that Corollary 1.3 cannot be strengthened to $\rho(f) \leqslant 1/k$.

And so, given a transcendental function f with $\rho(f) < 1/k$ for some $k \in \mathbb{N}$, the Bloch Principle (see [3]), Theorem 1.1 and Corollary 1.3 motivate the question whether the family $\mathcal{G}$ of analytic functions g in a domain Ω, such that $(f \circ g)^{(k)}(z) \neq 0$ in Ω, or more generally, $(f \circ g)^{(k)}(z) \neq Q(z)$ for some analytic function Q, is a normal family. This is true, and is a special case of the following result.

Theorem 1.4. *Let $k \in \mathbb{N}$. Let f be a transcendental entire function with $\rho(f) < 1/k$. Let $a_0, \ldots, a_{k-1}, a$ be analytic functions in a domain Ω. Then*

$$\mathcal{G} = \left\{ g\colon\ g \text{ is analytic in } \Omega,\ (f\circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f\circ g)^{(j)}(z) \neq a(z) \text{ in } \Omega \right\}$$

is a normal family in Ω.

In the proof of Theorem 1.4, we use the following theorem, which is an interesting value distribution result in its own right.

Theorem 1.5. *Let* $k \in \mathbb{N}$. *Let* f *be a transcendental entire function with* $\rho(f) < 1/2$. *Let* g *and* Q *be polynomials, with* g *nonconstant. Then*

$$(f\circ g)^{(k)} - Q$$

has infinitely many zeros.

We note that in Theorem 1.5, we must have that $\rho(f) < 1/2$, since we apply a theorem of $\cos \pi\rho$ type. However, if $Q \equiv 0$, we can prove Theorem 1.5 for $\rho(f) < 1$, for the extended case where g is a nonconstant entire function. We state the result as follows.

Theorem 1.6. *Let* f *be a transcendental entire function with* $\rho(f) < 1$. *Let* g *be a nonconstant entire function. Then* $(f\circ g)'$ *has infinitely many zeros.*

From Theorems 1.5 and 1.6, we prove the following corollary which strengthens Corollary 1.3 and which is used in the proof of Theorem 1.4.

Corollary 1.7. *Let* $k \in \mathbb{N}$. *Suppose that* f *is a transcendental entire function such that* $\rho(f) < 1/k$. *Suppose that* g *is an entire function of finite order such that*

$$(f\circ g)^{(k)}(z) \neq 0$$

on $\mathbb{C}$. *Then* g *is constant.*

Finally, we note that Theorems 1.5 and 1.6 have the following corollaries.

Corollary 1.8. *Let* k *be an integer,* $k \geqslant 2$. *Let* f *be a transcendental entire function with* $\rho(f) < 1/k$. *Let* $\alpha \in \mathbb{C}$. *Then for every nonconstant entire function* g,

$$(f\circ g)^{(k)} - \alpha$$

has infinitely many zeros.

Again, although the $k = 1$ case is omitted in Corollary 1.8, we can prove the $k = 1$ case when g is a transcendental entire function. We state the result as follows.

Corollary 1.9. *Let* f *be a transcendental entire function with* $\rho(f) < 1$. *Let* $\alpha \in \mathbb{C}$. *Then for every transcendental entire function* g,

$$(f\circ g)' - \alpha$$

has infinitely many zeros.

Since the proof of Theorem 1.4 depends on Theorems 1.5, 1.6 and Corollary 1.7, we prove these results in Sections 2, 3 and 4, respectively. We then prove Theorem 1.4 in Section 5. Finally, we prove Corollaries 1.8 and 1.9 in Section 6.

2. Proof of Theorem 1.5

The following lemma is a version of Taylor's theorem and is easily proved by induction.

Lemma 2.1. *If f is an entire function and $a \in \mathbb{C}$, then for $k \in \mathbb{N}$ we have*

$$f(z) = f(a) + (z-a)f'(a) + \cdots + \frac{(z-a)^{k-1}}{(k-1)!} f^{(k-1)}(a) + \int_a^z \frac{(z-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,dt.$$

We also need the following lemma. We include the proof here for completeness.

Lemma 2.2. *Let $k \in \mathbb{N}$. Let P_1 and P_2 be polynomials of degree m and n respectively, with $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we can choose a straight line Γ from 0 to ∞ such that*

$$I = \left| \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} P_1(t) e^{P_2(t)}\,dt \right| \leqslant c|z|^{k+m},$$

as $z \to \infty$ along Γ, for some positive constant c.

Proof. The behaviour of P_2 is dominated by the leading term $b_n t^n$. Setting $t = re^{i\theta}$, we have that $|e^{b_n t^n}| = e^{(\alpha\cos(n\theta)+\beta\sin(n\theta))r^n}$ for some $\alpha, \beta \in \mathbb{R}$, not both 0. Then choose θ such that $\alpha\cos(n\theta) + \beta\sin(n\theta) = -d < 0$, and let Γ be the straight line $z = re^{i\theta}$, for $0 \leqslant r < \infty$. Then for t on Γ between 0 and z we have that $|e^{P_2(t)}| = e^{-dr^n + O(r^{n-1})} \to 0$ as $r \to \infty$, for fixed θ as above. Thus $|e^{P_2(t)}| \leqslant c_0$ for some positive constant c_0, and since P_1 has degree m, we have that for t on Γ between 0 and z, $|P_1(t)| \leqslant c_1|t|^m$ for some positive constant c_1, as $t \to \infty$. The result follows. □

Finally, we need a theorem of $\cos\pi\rho$ type, as follows. We refer the reader to [4] for further reading.

Theorem 2.3 [4]. *Let f be a nonconstant entire function with $\rho(f) = \rho < 1/2$. For $r > 0$, define $A(r)$ and $B(r)$ as follows*

$$A(r) = \inf\bigl\{\log\bigl|f(z)\bigr|:\ |z| = r\bigr\}, \qquad B(r) = \sup\bigl\{\log\bigl|f(z)\bigr|:\ |z| = r\bigr\}.$$

If $\rho < \alpha < 1/2$, then

$$\underline{\log\mathrm{dens}}\bigl\{r:\ A(r) > (\cos\pi\alpha)B(r)\bigr\} \geqslant 1 - \rho/\alpha,$$

where if E is a subset of $(1,+\infty)$ the lower logarithmic density of E is defined by

$$\underline{\log\operatorname{dens}}(E)=\liminf_{r\to\infty}\left(\int_1^r \chi(t)\,dt/t\right)\Big/\log r,$$

where $\chi(t)$ is the characteristic function of E.

We now prove Theorem 1.5.

Proof of Theorem 1.5. We use a proof by contradiction. Suppose that $(f\circ g)^{(k)}-Q$ has m zeros in $\mathbb{C}$, for some $m\in\mathbb{N}\cup\{0\}$. Then we can write

$$(f\circ g)^{(k)}(z)-Q(z)=P_1(z)e^{P_2(z)}$$

for some polynomials P_1 and P_2 of degree m and n respectively, with m as above and $n\in\mathbb{N}$. Then, by Lemma 2.1, we have for $a=0$ that

$$\begin{aligned}(f\circ g)(z)&=(f\circ g)(0)+z(f\circ g)'(0)+\cdots+\frac{z^{k-1}}{(k-1)!}(f\circ g)^{(k-1)}(0)\\&\quad+\int_0^z\frac{(z-t)^{k-1}}{(k-1)!}(f\circ g)^k(t)\,dt\\&=Q_{k-1}(z)+\int_0^z\frac{(z-t)^{k-1}}{(k-1)!}\left(Q(t)+P_1(t)e^{P_2(t)}\right)dt,\end{aligned}$$

where Q_{k-1} is a polynomial of degree at most $k-1$. Then

$$\begin{aligned}\left|(f\circ g)(z)\right|&\leqslant\left|Q_{k-1}(z)\right|+\left|\int_0^z\frac{(z-t)^{k-1}}{(k-1)!}Q(t)\,dt\right|\\&\quad+\left|\int_0^z\frac{(z-t)^{k-1}}{(k-1)!}P_1(t)e^{P_2(t)}\,dt\right|.\end{aligned}\tag{2}$$

For the remainder of this proof, we use c_j to denote positive constants.

Since Q_{k-1} is a polynomial of degree at most $k-1$, and since Q is a polynomial of degree q say, $q\geqslant 0$, we have that $|Q_{k-1}(z)|\leqslant c_1|z|^{k-1}$ as $z\to\infty$ and that $|Q(t)|\leqslant c_2|t|^q$ as $t\to\infty$ on any straight line Γ between 0 and z. Then, as $t\to\infty$, we have that integrating along any straight line Γ between 0 and z gives

$$\left|\int_0^z\frac{(z-t)^{k-1}}{(k-1)!}Q(t)\,dt\right|\leqslant\frac{c_2|z|^{k-1}}{(k-1)!}\int_0^z O\left(|t|^q\right)dt\leqslant c_3|z|^{k+q}$$

since $|z-t|\leqslant|z|$.

In particular, by Lemma 2.2, we can choose a straight line path Γ from 0 to ∞ such that

$$\left| \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} P_1(t) e^{P_2(t)}\, dt \right| \leqslant c_4 |z|^{k+m}$$

as $z \to \infty$ along Γ. Then we have that $|(f \circ g)(z)| \leqslant c_5 |z|^{k+q+m}$, which gives

$$\log\bigl|(f \circ g)(z)\bigr| \leqslant c_6 \log|z| \tag{3}$$

as $z \to \infty$ along Γ.

Since $\rho = \rho(f) < 1/2$, we can apply Theorem 2.3 to f. For $r > 0$, define $A(r)$ and $B(r)$ as in Theorem 2.3. Then for $\rho < \alpha < 1/2$ we have

$$\underline{\log\mathrm{dens}}\bigl\{r\colon\ A(r) > (\cos\pi\alpha)B(r)\bigr\} \geqslant 1 - \rho/\alpha. \tag{4}$$

Next, since g is a polynomial and is nonconstant, we have that $|g(z)| \geqslant c_7|z|$ as $z \to \infty$, then by (3), we have that

$$\log\bigl|\bigl(f(g(z))\bigr| = \log\bigl|(f \circ g)(z)\bigr| \leqslant c_6 \log|z| \leqslant c_8 \log\bigl|g(z)\bigr| \tag{5}$$

as $z \to \infty$ along Γ. Now choose R large such that $R \in \{r\colon\ A(r) > (\cos\pi\alpha)B(r)\}$. Choose w such that $|w| = R$ and $w = g(z)$ for some z on Γ. Then by (5), we have that

$$(\cos\pi\alpha)B(R) < A(R) \leqslant \log\bigl|f(w)\bigr| \leqslant c_8 \log R.$$

This is a contradiction since f is a transcendental function, which implies that $B(R)/\log R \to +\infty$ as $R \to \infty$. $\square$

3. Proof of Theorem 1.6

We need the following lemma.

Lemma 3.1. *If f is a transcendental entire function with $\rho(f) < 1$, then f' has infinitely many zeros.*

Proof of Theorem 1.6. Since g is a nonconstant entire function, we have by Picard's theorem that g omits at most one value in $\mathbb{C}$. Since f is a transcendental entire function with $\rho(f) < 1$, we have by Lemma 3.1 that f' has infinitely many zeros. Then since g omits at most one of these zeros, we have that $f'(g(z))$ has infinitely many zeros. Therefore, since $(f \circ g)'(z) = f'(g(z)).g'(z)$, we have that $(f \circ g)'$ has infinitely many zeros. $\square$

4. Proof of Corollary 1.7

By Corollary 1.3, we have that g is a polynomial. However, by Theorem 1.6 for $k = 1$ and by Theorem 1.5 for $k \geqslant 2$, if g is a nonconstant polynomial then $(f \circ g)^{(k)}$ has infinitely many zeros. Therefore g is constant.

5. Proof of Theorem 1.4

First we need the following result, which is a version of Hurwitz' theorem (see [3]).

Lemma 5.1. *Let (f_n) be a sequence of analytic functions on a domain Ω, which converge spherically uniformly on compact subsets to a function f. Let (s_n) be a sequence of analytic functions tending to* 0 *on some disc $B(\alpha,\delta)=\{z\colon\ |z-\alpha|<\delta\}\subseteq\Omega$, for some $\delta>0$. If $f\not\equiv 0$ and $f(\alpha)=0$, then for large n, we have $f_n(z)=s_n(z)$ for some z near α.*

Next, we note that Lemma 5.1 has the following corollary, which we will use in the proof of Theorem 1.4. We provide a proof for completeness.

Corollary 5.2. *Let $k\in\mathbb{N}$. Let Ω be the open unit disc $B(0,1)$. Let a be an analytic function on Ω. Let (f_n) be a sequence of analytic functions on Ω, such that $f_n(z)\neq a(z)$ on Ω. Let (z_n) be a sequence of points tending to $z_0\in\Omega$, and let (ρ_n) be a positive sequence tending to* 0*. Suppose g is an entire function such that*

$$\lim_{n\to\infty}\rho_n^k f_n(z_n+\rho_n z)=g(z)$$

locally uniformly on $\mathbb{C}$. Then either $g\equiv 0$ on $\mathbb{C}$, or $g(z)\neq 0$ on $\mathbb{C}$.

Proof. Suppose there exists $\alpha\in\mathbb{C}$ such that $g(\alpha)=0$. If $g\equiv 0$, then we are done. Otherwise, we note that $z_n+\rho_n z\in\Omega$ for n large, and that $(\rho_n^k f_n)$ is a sequence of analytic functions which converge to g locally uniformly on $\mathbb{C}$. We note also that since a is analytic, and therefore bounded near z_0, and since (ρ_n^k) is a sequence tending to 0, then $(\rho_n^k a(z_n+\rho_n z))$ is a sequence of functions tending to 0, for n large, on $B(\alpha,\delta)$, for some $\delta>0$. Then by Lemma 5.1, we obtain $\rho_n^k f_n(z_n+\rho_n z)=\rho_n^k a(z_n+\rho_n z)$ for n large, for some z near α. Since (ρ_n^k) is a positive sequence, we therefore have that $f_n(z_n+\rho_n z)=a(z_n+\rho_n z)$, which is a contradiction since $z_n+\rho_n z\in\Omega$ for n large. □

We need the following lemma, which is called the *Zalcman lemma* (see [5]).

Theorem 5.3. *A family $\mathcal{G}$ of analytic functions in the open unit disc $B(0,1)$ is not normal at the origin, if and only if there exist a sequence of functions $g_n\in\mathcal{G}$, a sequence of points $z_n\to 0$, a positive sequence $\rho_n\to 0$ and a nonconstant entire function g on $\mathbb{C}$ such that*

$$\lim_{n\to\infty}g_n(z_n+\rho_n z)=g(z)$$

locally uniformly on $\mathbb{C}$, with respect to the spherical metric, such that the spherical derivative of g is bounded, $g^\sharp(z)\leqslant g^\sharp(0)=1$.

Finally, we need the following lemma which is an immediate consequence of the definition of the order of a meromorphic function, using the Ahlfors–Shimizu form of the Nevanlinna characteristic.

Lemma 5.4. *Let f be a meromorphic function with bounded spherical derivative. Then the order of f is at most* 2.

We now prove Theorem 1.4.

Proof of Theorem 1.4. Since normality is a local property, we can assume, without loss of generality, that Ω is a disc and $a_0, \ldots, a_{k-1}, a$ are bounded on Ω. Using a linear change of variables $h(z) = g(\alpha + \beta z)$, and the fact that $(f \circ h)^{(j)}(z) = \beta^j (f \circ g)^{(j)}(\alpha + \beta z)$, for a suitable choice of $\alpha, \beta \in \mathbb{C}$, we may assume that Ω is $B(0,1)$. Suppose that $\mathcal{G}$ is not normal on Ω. Then $\mathcal{G}$ must be not normal at least one point in Ω, and without loss of generality we can suppose that $\mathcal{G}$ is not normal at 0.

Since $\mathcal{G}$ is a family of analytic functions, we can apply Lemma 5.3. Then there exist points (z_n) tending to 0, a sequence (g_n) in $\mathcal{G}$, a positive sequence (ρ_n) tending to 0 and a nonconstant entire function g such that

$$h_n(z) = g_n(z_n + \rho_n z) \to g(z) \tag{6}$$

locally uniformly on $\mathbb{C}$, with respect to the spherical metric, with $g^\sharp(z) \leqslant 1$. Then since g has bounded spherical derivative, we have by Lemma 5.4 that g is a function of finite order.

Next, since f is an entire function, we have that

$$(f \circ h_n)(z) \to (f \circ g)(z),$$

locally uniformly on $\mathbb{C}$. Then by the Weierstrass theorem (see [3]), for $j \in \mathbb{N}$,

$$(f \circ h_n)^{(j)}(z) = \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \to (f \circ g)^{(j)}(z), \tag{7}$$

locally uniformly on $\mathbb{C}$. However, since each $g_n \in \mathcal{G}$, we have that for $z_n + \rho_n z \in \Omega$,

$$\begin{aligned} F_n(z) &= (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} a_j(z_n + \rho_n z)(f \circ g_n)^{(j)}(z_n + \rho_n z) \\ &\neq a(z_n + \rho_n z). \end{aligned}$$

Then we have that

$$\begin{aligned} \rho_n^k F_n(z) &= \rho_n^k (f \circ g_n)^{(k)}(z_n + \rho_n z) \\ &\quad + \sum_{j=0}^{k-1} \rho_n^{k-j} a_j(z_n + \rho_n z) \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \neq \rho_n^k a(z_n + \rho_n z). \end{aligned}$$

Next, since $\rho_n^{k-j} \to 0$ for $j = 0, \ldots, k-1$, and since the a_j are assumed bounded on Ω, we have by (7) that

$$\lim_{n \to \infty} \rho_n^k F_n(z) = (f \circ g)^{(k)}(z)$$

locally uniformly on $\mathbb{C}$. However, we can write $F_n(z) = G_n(z_n + \rho_n z)$ where $G_n(z) \neq a(z)$ on Ω, and so we have by Corollary 5.2, that either $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$, or $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$.

Case 1. $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$.

Then integrating this equation $k-1$ times, we have that $(f\circ g)'(z)=P_{k-2}(z)$, where P_{k-2} is a polynomial of degree at most $k-2$. Since P_{k-2} has at most $k-2$ zeros, counting multiplicities, $(f\circ g)'(z)=f'(g(z)).g'(z)$ has also. However, f is a transcendental entire function and $\rho(f)<1$, and so by Lemma 3.1, we have that f' has infinitely many zeros on $\mathbb{C}$. Then since g is a nonconstant entire function, we must have that g omits infinitely many zeros of f' on $\mathbb{C}$, which is a contradiction by Picard's theorem.

Case 2. $(f\circ g)^{(k)}(z)\neq 0$ on $\mathbb{C}$.

Suppose first that $k=1$. Then by Theorem 1.6, we have that $(f\circ g)'$ has infinitely many zeros on $\mathbb{C}$ and so we have a contradiction.

Suppose second that $k\geqslant 2$. Then since f is a transcendental entire function with $\rho(f)<1/k$ and g is a nonconstant entire function of finite order, we have by Corollary 1.7 that $(f\circ g)^{(k)}$ has at least one zero in $\mathbb{C}$. Therefore we have a contradiction.

Therefore $\mathcal{G}$ is a normal family.

6. Proof of Corollaries 1.8 and 1.9

Proof of Corollary 1.8. Suppose that $(f\circ g)^{(k)}-\alpha$ has finitely many zeros.

Suppose first that g is a function of finite order. If g is a polynomial, then we have a contradiction by Theorem 1.5. If g is a transcendental function, then since $N(r,1/((f\circ g)^{(k)}-\alpha))=O(\log r)=o(T(r,g))$ and since $T(r,f)=o(r^{1/k})$, then we have a contradiction by Theorem 1.2.

Suppose second that g is a function of infinite order. Then, by Lemma 5.4, we have that g has unbounded spherical derivative, that is, we can choose a sequence of points (α_n) tending to ∞, such that $g^\sharp(\alpha_n)\to\infty$ as $n\to\infty$. Then the family of functions $\{g_n(z)=g(\alpha_n+z)\colon n\in\mathbb{N}\}$ is not a normal family on the open unit disc $B(0,1)$. Then by Theorem 1.4 we have a contradiction. □

Proof of Corollary 1.9. Suppose that $(f\circ g)'-\alpha$ has finitely many zeros. If g is a function of finite order, then by the argument in Corollary 1.8, we have that g is a polynomial. This is a contradiction since g is a transcendental function. If g is a function of infinite order, then by the argument in Corollary 1.8, we have a contradiction by Theorem 1.4. □

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