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An analytic approximate method for solving stochastic integrodifferential equations ☆

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Abstract

In this paper we compare the solution of a general stochastic integrodifferential equation of the Ito type, with the solutions of a sequence of appropriate equations of the same type, whose coefficients are Taylor series of the coefficients of the original equation. The approximate solutions are defined on a partition of the time-interval. The rate of the closeness between the original and approximate solutions is measured in the sense of the L^p-norm, so that it decreases if the degrees of these Taylor series increase, analogously to real analysis. The convergence with probability one is also proved.

Keywords: Stochastic integrodifferential equation; Taylor approximation; Approximate solution; L^p-convergence; Convergence with probability one

1. Introduction and preliminary results

In many natural and life sciences, engineering and economics too, there are physical phenomena depending on a Gaussian white noise excitation, and, because of that, governed by certain probability laws and evolving in time. Because a Gaussian white noise is an abstraction, not a real process, mathematically described as a formal derivative of a

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Brownian motion process, the behavior of such phenomena are mathematically modeled by stochastic processes which are solutions of stochastic differential or, in more complicated case, stochastic integrodifferential equations of the Ito type [4,7]. Certainly, the class of explicitly solvable stochastic equations is very small, and, from the theoretical point of view and much more from various applications, it is important to find an analytic or numerical approximate solution of such an equation. The researcher's interest is focused on exploring the bifurcational behavior and stability, for example, of the solution of the initial equation, by comparing it with the corresponding approximate one, as well as on conditions under which these solutions are close in some sense. One type of an analytic approximation, based on the Taylor expansion, and because of that suitable for numerical approximations, will be the object of the present paper.

Further, we suppose that all random variables and stochastic processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Likewise, we usually restrict ourselves on the time interval $[0, 1]$ instead of $[t_0, T]$.

In this paper we study a very general stochastic integrodifferential equation of the Ito type,

$$\begin{aligned} dx_t &= \left[a_1(t, x_t) + \int_0^t a_2(t, s, x_s)\,ds + \int_0^t a_3(t, s, x_s)\,dw_s\right]dt \\ &\quad + \left[b_1(t, x_t) + \int_0^t b_2(t, s, x_s)\,ds + \int_0^t b_3(t, s, x_s)\,dw_s\right]dw_t, \quad t \in [0, 1], \\ x(0) &= x_0 \quad \text{a.s.}, \end{aligned} \tag{1}$$

introduced and studied earlier in detail in paper [4] by Berger and Mizel. Here $w = (w_t, t \in R)$ is an R^d-valued normalized Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with a natural filtration $\{\mathcal{F}_t, t \geqslant 0\}$ of non-decreasing sub-σ-algebras of $\mathcal{F}$ $(\mathcal{F}_t = \sigma\{x_s, 0 \leqslant s \leqslant t\})$, x_0 is an R^k-random variable independent of w, the coefficients of this equation are real functions

$$\begin{aligned} &a_1 : [0, 1] \times R^k \to R^k, && b_1 : [0, 1] \times R^k \to R^k \times R^d, \\ &a_2 : J \times R^k \to R^k, && b_2 : J \times R^k \to R^k \times R^d, \\ &a_3 : J \times R^k \to R^k \times R^d, && b_3 : J \times R^k \to R^k \times R^k \times R^d, \end{aligned}$$

where $J = \{(t, s)\colon\ 0 \leqslant s \leqslant t \leqslant 1\}$, which are Borel measurable on their domains, and $x = (x_t, t \in [0, 1])$ is an R^k-valued unknown stochastic process. The process x is a solution of Eq. (1) if it is adapted to $\{\mathcal{F}_t, t \geqslant 0\}$, all Lebesgue and Ito integrals in the integral form of Eq. (1) are well defined and Eq. (1) holds a.s. for each $t \in [0, 1]$. The basic existence and uniqueness theorem, proved in paper [4] and based on the classical theory of stochastic differential equations (see [5,12,13,15], for example), requires that the functions $a_i, b_i,\ i = 1, 2, 3$, satisfy the global Lipschitz condition and the usual linear growth condition on the last argument, i.e., there exists a constant $L > 0$, so that, for all $(t, s) \in J$, $x, y \in R^k$,

$$\big|a_2(t, s, x) - a_2(t, s, y)\big| < L|x - y|, \qquad \big|a_2(t, s, x)\big| \leqslant L\big(1 + |x|\big), \tag{2}$$

and similarly for the other functions ($|\cdot|$ are appropriate Euclidean matrix norms). If $E|x_0|^2 < \infty$, then there exists a unique a.s. continuous solution $x = (x_t, t \in [0,1])$ of Eq. (1), satisfying $E\{\sup_{t\in[0,1]} |x_t|^2\} < \infty$. Moreover, following the procedures in [10,13], it can be proved that if $E|x_0|^p < \infty$ for any number $p > 0$, then $E\{\sup_{t\in[0,1]} |x_t|^p\} < \infty$.

The main purpose of the present paper is to compare in the L^p-norm, $p \geqslant 2$, and with probability one, the solution of Eq. (1) with the solutions of a sequence of appropriate equations of the same type, whose coefficients are Taylor expansions of the coefficients of Eq. (1). The basis of such approximation is that the approximate solutions are defined on an arbitrary partition Γ: $0 = t_0 < t_1 < \cdots < t_n = 1$ of the time interval $[0,1]$, where $\delta_n = \max_{0\leqslant k\leqslant n-1}(t_{k+1} - t_k)$. In fact, the basic ideas of the present paper go back to papers [1,2] by Atalla, and earlier to [9] by Kanagawa under some more restrictive conditions. In papers [1,9], the solution of the stochastic differential equation of the Ito type $dx_t = a(t,x_t)\,dt + b(t,x_t)\,dw_t,\ x_0 = \eta,\ t \in [0,1]$ was approximated by the solutions x^n, $n \in N$, of the equations $dx_t^n = a(t_k, x_{t_k}^n)\,dt + b(t_k, x_{t_k}^n)\,dw_t,\ x_0^n = \eta,\ t \in [t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$, by successively connecting the solutions $(x_t^n, t \in [t_k, t_{k+1}])$ at the points t_k of the partition Γ. It was shown that the rate of such approximation, in the sense of the L^p-norm, $p \geqslant 2$, was $O(\delta_n^{p/2})$ when $n \to \infty$ and $\delta_n \to 0$. This result has earlier been obtained in [5] for $p = 2$ and generalized in paper [6] by Ilić and Janković to a stochastic integrodifferential equation, more general than Eq. (1).

In paper [2] Atalla improved his own result by using the sequence of stochastic differential equations determined on a partition Γ, whose drift and diffusion coefficients were Taylor approximations of $a(t,x)$ and $b(t,x)$ up to the first derivatives in argument x, i.e., by linear stochastic differential equations of the Ito type $dx_t^n = [a(t_k, x_{t_k}^n) + a_x'(t_k, x_{t_k}^n)(x_t^n - x_{t_k}^n)]\,dt + [b(t_k, x_{t_k}^n) + b_x'(t_k, x_{t_k}^n)(x_t^n - x_{t_k}^n)]\,dw_t$, $x_0^n = \eta$, $t \in [t_k, t_{k+1})$, $0 \leqslant k \leqslant n-1$. The rate of this approximation, in the sense of the L^p-norm, was $O(\delta_n^p)$ when $n \to \infty$ and $\delta_n \to 0$.

By applying the previously considered concept, in paper [8] by Janković and Ilić the solution of the equation $dx_t = a(t,x_t)\,dt + b(t,x_t)\,dw_t$, $x_0 = \eta$, $t \in [0,1]$ is approximated on a partition Γ by the solutions of the equations whose drift and diffusion coefficients are Taylor series of the functions $a(t,x)$ and $b(t,x)$ with respect to the argument x, up to arbitrary fixed derivatives m_1 and m_2, respectively. The rate of such approximation, in the sense of the L^p-norm, is found as $O(\delta_n^{(m+1)p/2})$ when $n \to \infty$ and $\delta_n \to 0$, where $m = \min\{m_1, m_2\}$.

Having in mind that it is almost impossible to solve explicitly Eq. (1), and that Taylor series, as polynomials, could be a very useful tool to approximate its solution analytically or numerically (see [10,11], for example), the main results of the present paper are devoted to a construction of its approximate solution by applying partially the reasons from paper [8].

2. Main results

Let

$$0 = t_0^n < t_1^n < \cdots < t_{q_n}^n = 1, \qquad \delta_n = \max_{0\leqslant k\leqslant q_n-1}\left(t_{k+1}^n - t_k^n\right) \tag{3}$$

be an arbitrary partition of the interval [0, 1]. Together with Eq. (1) in its equivalent integral form

$$x_t = x_0 + \int_0^t \left[a_1(s, x_s) + \int_0^s a_2(s, u, x_u)\, du + \int_0^s a_3(s, u, x_u)\, dw_u \right] ds$$
$$+ \int_0^t \left[b_1(s, x_s) + \int_0^s b_2(s, u, x_u)\, du + \int_0^s b_3(s, u, x_u)\, dw_u \right] dw_s, \quad t \in [0, 1], \tag{4}$$

we consider the sequence of the equations, defined on the partition (3),

$$x_t^n = x_{t_k^n}^n + \int_{t_k^n}^t \left[A_{1k}\big(s, x_s^n\big) + \int_{t_k^n}^s A_{2k}\big(s, u, x_u^n\big)\, du + \int_{t_k^n}^s A_{3k}\big(s, u, x_u^n\big)\, dw_u \right] ds$$
$$+ \int_{t_k^n}^t \left[B_{1k}\big(s, x_s^n\big) + \int_{t_k^n}^s B_{2k}\big(s, u, x_u^n\big)\, du + \int_{t_k^n}^s B_{3k}\big(s, u, x_u^n\big)\, dw_u \right] dw_s,$$
$$t \in \big[t_k^n, t_{k+1}^n\big],\ 0 \leqslant k \leqslant q_n - 1,\ x_{t_0^n}^n = \eta, \tag{5}$$

where

$$A_{1k}\big(s, x_s^n\big) = \sum_{i=0}^{m_1} \frac{a_{1x}^{(i)}(s, x_{t_k^n}^n)}{i!} \big(x_s^n - x_{t_k^n}^n\big)^i,$$

$$A_{jk}\big(s, u, x_u^n\big) = \sum_{i=0}^{m_j} \frac{a_{jx}^{(i)}(s, u, x_{t_k^n}^n)}{i!} \big(x_u^n - x_{t_k^n}^n\big)^i, \quad j = 1, 2,$$

$$B_{1k}\big(s, x_s^n\big) = \sum_{i=0}^{m_1} \frac{b_{1x}^{(i)}(s, x_{t_k^n}^n)}{i!} \big(x_s^n - x_{t_k^n}^n\big)^i,$$

$$B_{jk}\big(s, u, x_u^n\big) = \sum_{i=0}^{n_j} \frac{b_{jx}^{(i)}(s, u, x_{t_k^n}^n)}{i!} \big(x_u^n - x_{t_k^n}^n\big)^i, \quad j = 1, 2,$$

and $a_{jx}^{(i)}, b_{jx}^{(i)}$ are the ith derivatives of the functions a_j, b_j, with respect to x.

The approximate solution $x^n = (x_t^n, t \in [0, 1])$ is constructed as an a.s. continuous process, by successively connecting the processes $(x_t^n, t \in [t_k^n, t_{k+1}^n])$ at the points t_k^n, $1 \leqslant k \leqslant q_n - 1$, of the partition (3).

Let us introduce the following assumptions:

($\mathcal{A}_1$) The functions a_j and b_j, $j = 1, 2, 3$, have Taylor approximations in the argument x, up to the m_jth and n_jth derivatives, respectively;

$(\mathcal{A}_2)$ The functions $a_{jx}^{(m_j+1)}(\cdot)$ and $b_{jx}^{(n_j+1)}(\cdot)$ are uniformly bounded, i.e., there exist positive constants L_j and $\bar{L}_j$, such that

$$\sup_{[0,1]\times R^k}\left|a_{1x}^{(m_1+1)}(t,x)\right|\leqslant L_1,\qquad \sup_{J\times R^k}\left|a_{jx}^{(m_j+1)}(t,s,x)\right|\leqslant L_j,\quad j=2,3,$$

$$\sup_{[0,1]\times R^k}\left|b_{1x}^{(n_1+1)}(t,x)\right|\leqslant \bar{L}_1,\qquad \sup_{J\times R^k}\left|b_{jx}^{(n_j+1)}(t,s,x)\right|\leqslant \bar{L}_j,\quad j=2,3;$$

$(\mathcal{A}_3)$ Without special emphasizing of the conditions for the coefficients of Eqs. (4) and (5), we suppose that there exist unique, a.s. continuous solutions x and x^n of these equations respectively satisfying $E\{\sup_{[0,1]}|x_t|^p\}<\infty$ and $E\{\sup_{[0,1]}|x_t^n|^{(M+1)^2 p}\}\leqslant Q<\infty$, where $M=\max\{m_j,n_j,\ j=1,2,3\}$ and $Q>0$ is a constant independent on n and δ_n. Likewise, all Lebesgue and Ito integrals employed further are also well defined.

In the proofs of the next assertions, we apply many times, without special emphasizing, the elementary inequality $(\sum_{i=1}^m q_i)^s\leqslant m^{s-1}\sum_{i=1}^m q_i^s$, $q_i>0$, $s\in N$, the usual Ito stochastic integral isometry, Hölder inequality to Lebesgue integrals and Burkholder–Davis–Gundy inequality to Ito integrals [5,10,13,15]: For any $l>0$, there exists a constant $c_l>0$, such that

$$E\left\{\sup_{s\in[t_0,t]}\left|\int_{t_0}^{s} f_u\,dw_u\right|^l\right\}\leqslant c_l E\left(\int_{t_0}^{t}|f_u|^2\,du\right)^{l/2}$$

for any measurable $\mathcal{F}_t$-adapted process $(f_t, t\in[0,T])$, satisfying $\int_{t_0}^T|f_t|^2\,dt<\infty$ a.s. Clearly, the last inequality can be also used if the left-hand side is minorized by omitting the supremum.

In order to prove the closeness, in the sense of the L^p-norm, between the solutions x and x^n of Eqs. (4) and (5), respectively, we must first prove the following assertion:

Proposition 1. *Let x^n be the solution of Eq.* (5)*, the conditions* (2) *and the assumptions* $(\mathcal{A}_1)$*,* $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *be satisfied. Then, for* $1\leqslant r\leqslant(M+1)p$*,*

$$E\left|x_t^n-x_{t_k^n}^n\right|^r\leqslant D_r\,\delta_n^{r/2},\quad t\in\left[t_k^n,t_{k+1}^n\right],\ 0\leqslant k\leqslant q_n-1,$$

where D_r is a generic constant independent of n and δ_n.

Proof. By applying the previously cited elementary inequality to Eq. (5), and after that Hölder and Burkholder–Davis–Gundy inequalities to Lebesgue and Ito integrals, respectively, we obtain, for all $t\in[t_k^n,t_{k+1}^n]$, $0\leqslant k\leqslant q_n-1$,

$$E\left|x_t^n-x_{t_k^n}^n\right|^r$$
$$\leqslant 2^{r-1}\left\{E\left|\int_{t_k^n}^{t}\left[A_{1k}\left(s,x_s^n\right)+\int_{t_k^n}^{s}A_{2k}\left(s,u,x_u^n\right)du+\int_{t_k^n}^{s}A_{3k}\left(s,u,x_u^n\right)dw_u\right]ds\right|^r\right.$$

$$
\begin{aligned}
&+ E\left|\int_{t_k^n}^{t}\left[B_{1k}\big(s,x_s^n\big)+\int_{t_k^n}^{s}B_{2k}\big(s,u,x_u^n\big)\,du+\int_{t_k^n}^{s}B_{3k}\big(s,u,x_u^n\big)\,dw_u\right]dw_s\right|^r\Bigg\}\\
&\leqslant 2^{r-1}\Bigg\{\big(t-t_k^n\big)^{r-1}\int_{t_k^n}^{t}E\left|A_{1k}\big(s,x_s^n\big)+\cdots+\int_{t_k^n}^{s}A_{3k}\big(s,u,x_u^n\big)\,dw_u\right|^r ds\\
&\quad + c_r\big(t-t_k^n\big)^{r/2-1}\int_{t_k^n}^{t}E\left|B_{1k}\big(s,x_s^n\big)+\cdots+\int_{t_k^n}^{s}B_{3k}\big(s,u,x_u^n\big)\,dw_u\right|^r ds\Bigg\}\\
&\equiv 2^{r-1}\big[\big(t-t_k^n\big)^{r-1}J_1(t)+c_r\big(t-t_k^n\big)^{r/2-1}J_2(t)\big],
\end{aligned}
\tag{6}
$$

where $J_1(t)$ and $J_2(t)$ are the corresponding integrals, which must be estimated.

First,

$$
\begin{aligned}
J_1(t)\leqslant 3^{r-1}\int_{t_k^n}^{t}\Bigg[&E\big|a_1\big(s,x_s^n\big)-\big[a_1\big(s,x_s^n\big)-A_{1k}\big(s,x_s^n\big)\big]\big|^r\\
&+\big(s-t_k^n\big)^{r-1}\int_{t_k^n}^{s}E\big|a_2\big(s,u,x_u^n\big)-\big[a_2\big(s,u,x_u^n\big)-A_{2k}\big(s,u,x_u^n\big)\big]\big|^r du\\
&+c_r\big(s-t_k^n\big)^{r/2-1}\int_{t_k^n}^{s}E\big|a_3\big(s,u,x_u^n\big)\\
&-\big[a_3\big(s,u,x_u^n\big)-A_{3k}\big(s,u,x_u^n\big)\big]\big|^r du\Bigg]ds.
\end{aligned}
$$

By using the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we find that, for any $\theta\in(0,1)$,

$$
\begin{aligned}
&E\big|a_1\big(s,x_s^n\big)-A_{1k}\big(s,x_s^n\big)\big|^r\\
&\quad\leqslant E\left\{\frac{|a_{1x}^{(m_1+1)}(s,x_{t_k^n}^n+\theta(x_s^n-x_{t_k^n}^n))|^r}{[(m_1+1)!]^r}\big|x_s^n-x_{t_k^n}^n\big|^{(m_1+1)r}\right\}\\
&\quad\leqslant\frac{L_1}{(m_1+1)!}E\big|x_s^n-x_{t_k^n}^n\big|^{(m_1+1)r},
\end{aligned}
\tag{7}
$$

and analogously for a_2 and a_3. Further, we apply the linear growth condition (2), after that the previous estimations (7) and the assumption $(\mathcal{A}_3)$ from which it follows that $E|x_s^n|^r\leqslant 1+Q=S$ for every $1\leqslant r\leqslant(M+1)p$ and $s\in[0,1]$. Thus, we get

$$
J_1(t)\leqslant 6^{r-1}\Bigg\{\int_{t_k^n}^{t}\left[L^r\big(1+E\big|x_s^n\big|^r\big)+\frac{L_1^r}{[(m_1+1)!]^r}E\big|x_s^n-x_{t_k^n}^n\big|^{(m_1+1)r}\right]ds
$$

$$
\begin{aligned}
&+\int_{t_k^n}^{t}\left(s-t_k^n\right)^{r-1}\int_{t_k^n}^{s}\biggl[L^r\left(1+E\left|x_u^n\right|^r\right)\\
&+\frac{L_2^r}{[(m_2+1)!]^r}E\left|x_u^n-x_{t_k^n}^n\right|^{(m_2+1)r}\biggr]du\,ds\\
&+c_r\int_{t_k^n}^{t}\left(s-t_k^n\right)^{r/2-1}\int_{t_k^n}^{s}\biggl[L^r\left(1+E\left|x_u^n\right|^r\right)\\
&+\frac{L_3^r}{[(m_3+1)!]^r}E\left|x_u^n-x_{t_k^n}^n\right|^{(m_2+1)r}\biggr]du\,ds\biggr\}\\
\leqslant{}& 6^{r-1}\biggl\{\biggl[L^r(1+S)+\frac{L_1^r2^{(m_1+1)r}S}{[(m_1+1)!]^r}\biggr]\left(t-t_k^n\right)\\
&+\biggl[L^r(1+S)+\frac{L_2^r2^{(m_2+1)r}S}{[(m_2+1)!]^r}\biggr]\frac{(t-t_k^n)^{r+1}}{r+1}\\
&+c_r\biggl[L^r(1+S)+\frac{L_3^r2^{(m_3+1)r}S}{[(m_3+1)!]^r}\biggr]\frac{(t-t_k^n)^{r/2+1}}{r/2+1}\biggr\}.
\end{aligned}
$$

Because $0\leqslant t-t_k^n\leqslant 1$, it follows that

$$J_1(t)\leqslant C_1\left(t-t_k^n\right), \tag{8}$$

where C_1 is a generic constant independent of n and δ_n.

Analogously, by repeating completely the previous procedure, we find that

$$J_2(t)\leqslant C_2\left(t-t_k^n\right), \tag{9}$$

where C_2 is also a generic constant independent of n and δ_n. The estimations (8) and (9) together with (6) imply that

$$E\left|x_t^n-x_{t_k^n}^n\right|^r\leqslant 2^{r-1}\left[C_1\left(t-t_k^n\right)^r+C_2\left(t-t_k^n\right)^{r/2}\right]\leqslant D_r\left(t-t_k^n\right)^{r/2}\leqslant D_r\delta_n^{r/2},$$

where D_r is a constant. Thus, the proof is completed. □

The following assertion enables us to estimate the closeness between the solutions x and x^n, in the pth moment sense, uniformly on the time interval [0, 1].

Proposition 2. *Let x and x^n be the solutions of Eqs.* (4) *and* (5)*, respectively, the conditions* (2) *and the assumptions* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *be satisfied. Then, for $p\geqslant 2$,*

$$\sup_{t\in[0,1]}E\left|x_t-x_t^n\right|^p\leqslant H\delta_n^{(m+1)p/2}, \tag{10}$$

where $m=\min\{m_j,n_j,\ j=1,2,3\}$ and H is a generic constant independent of n and δ_n.

Proof. Let $p>2$ and $t\in[t_k^n,t_{k+1}^n]$. By subtracting Eqs. (4) and (5) and after that by applying the Ito formula [12,13,15] to the function $f(x)=|x|^p$, we obtain

$$
\begin{aligned}
\left|x_t - x_t^n\right|^p = {}& \left|x_{t_k^n} - x_{t_k^n}^n\right|^p \\
& + p \int_{t_k^n}^{t} \Bigg[a_1(s, x_s) - A_{1k}\left(s, x_s^n\right) + \int_{t_k^n}^{s} \left[a_2(s,u,x_u) - A_{2k}\left(s,u,x_u^n\right)\right] du \\
& + \int_{t_k^n}^{s} \left[a_3(s,u,x_u) - A_{3k}\left(s,u,x_u^n\right)\right] dw_u \Bigg] \left|x_s - x_s^n\right|^{p-1} ds \\
& + \frac{p(p-1)}{2} \int_{t_k^n}^{t} \Bigg[b_1(s, x_s) - B_{1k}\left(s, x_s^n\right) \\
& + \int_{t_k^n}^{s} \left[b_2(s,u,x_u) - B_{2k}\left(s,u,x_u^n\right)\right] du \\
& + \int_{t_k^n}^{s} \left[b_3(s,u,x_u) - B_{3k}\left(s,u,x_u^n\right)\right] dw_u \Bigg]^2 \left|x_s - x_s^n\right|^{p-2} ds \\
& + p \int_{t_k^n}^{t} \Bigg[b_1(s, x_s) - B_{1k}\left(s, x_s^n\right) + \int_{t_k^n}^{s} \left[b_2(s,u,x_u) - B_{2k}\left(s,u,x_u^n\right)\right] du \\
& + \int_{t_k^n}^{s} \left[b_3(s,u,x_u) - B_{3k}\left(s,u,x_u^n\right)\right] dw_u \Bigg] \left|x_s - x_s^n\right|^{p-1} dw_s.
\end{aligned}
$$

Let us denote that $\Delta_t = E|x_t - x_t^n|^p$. Having in mind that the expectation of the Ito integral is equal to zero, it follows that

$$
\begin{aligned}
E\left|x_t - x_t^n\right|^p \leqslant {}& E\left|x_{t_k^n} - x_{t_k^n}^n\right|^p + pE \int_{t_k^n}^{t} \Bigg[\left|a_1(s, x_s) - A_{1k}\left(s, x_s^n\right)\right| \\
& + \left| \int_{t_k^n}^{s} \left[a_2(s,u,x_u) - A_{2k}\left(s,u,x_u^n\right)\right] du \right| \\
& + \left| \int_{t_k^n}^{s} \left[a_3(s,u,x_u) - A_{3k}\left(s,u,x_u^n\right)\right] dw_u \right| \Bigg] \left|x_s - x_s^n\right|^{p-1} ds \\
& + \frac{3p(p-1)}{2} E \int_{t_k^n}^{t} \Bigg[\left|b_1(s, x_s) - B_{1k}\left(s, x_s^n\right)\right|^2
\end{aligned}
$$

$$+\left|\int_{t_k^n}^{s}\left[b_2(s,u,x_u)-B_{2k}\big(s,u,x_u^n\big)\right]du\right|^2$$

$$+\left|\int_{t_k^n}^{s}\left[b_3(s,u,x_u)-B_{3k}\big(s,u,x_u^n\big)\right]dw_u\right|^2\Bigg]\big|x_s-x_s^n\big|^{p-2}\,ds$$

$$\equiv \Delta_{t_k}+p\sum_{j=1}^{3}I_j(t)+\frac{3p(p-1)}{2}\sum_{j=4}^{6}I_j(t), \tag{11}$$

where $I_j(t)$, $j=1,\dots,6$, are the corresponding integrals. To estimate $I_1(t)$, we use the Lipschitz condition (2) and the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, so that we obtain

$$\begin{aligned}
I_1(t) &= E\int_{t_k^n}^{t}\big|a_1(s,x_s)-A_{1k}\big(s,x_s^n\big)\big|\big|x_s-x_s^n\big|^{p-1}\,ds\\
&\leqslant E\int_{t_k^n}^{t}\Big[\big|a_1(s,x_s)-a_1\big(s,x_s^n\big)\big|+\big|a_1\big(s,x_s^n\big)-A_{1k}\big(s,x_s^n\big)\big|\Big]\big|x_s-x_s^n\big|^{p-1}\,ds\\
&\leqslant L\int_{t_k^n}^{t}\Delta_s\,ds+\frac{L_1}{(m_1+1)!}\int_{t_k}^{t}E\big|x_s^n-x_{t_k^n}^n\big|^{m_1+1}\big|x_s-x_s^n\big|^{p-1}\,ds.
\end{aligned}$$

If we apply Hölder inequality for $\mu=p$, $\nu=p/(p-1)$, $1/\mu+1/\nu=1$, after that Young inequality (for every $a,b>0$ and $\mu>1$, $1/\mu+1/\nu=1$, it follows that $ab\leqslant a^{\mu}/\mu+b^{\nu}/\nu$) and Proposition 1, we find that

$$\begin{aligned}
I_1(t) &\leqslant L\int_{t_k^n}^{t}\Delta_s\,ds+\frac{L_1}{(m_1+1)!}\int_{t_k^n}^{t}\Big(E\big|x_s^n-x_{t_k^n}^n\big|^{(m_1+1)p}\Big)^{1/p}(\Delta_s)^{(p-1)/p}\,ds\\
&\leqslant\left[L+\frac{L_1\,(p-1)}{(m_1+1)!p}\right]\int_{t_k^n}^{t}\Delta_s\,ds+\frac{L_1\,D_{(m_1+1)p}}{(m_1+1)!p}\delta_n^{(m_1+1)p/2}\big(t-t_k^n\big)\\
&\equiv \alpha_1\int_{t_k}^{t}\Delta_s\,ds+\beta_1\,\delta_n^{(m_1+1)p/2}\big(t-t_k^n\big),
\end{aligned}\tag{12}$$

where α_1 and β_1 are generic constants independent of n and δ_n.

To estimate $I_2(t)$, we apply the previous procedure and the integration by parts to double integrals. Hence, we deduce that

$$I_2(t)=E\int_{t_k^n}^{t}\left|\int_{t_k^n}^{s}\left[a_2(s,x_s)-A_{2k}\big(s,x_s^n\big)\right]du\right|\big|x_s-x_s^n\big|^{p-1}\,ds$$

$$
\begin{aligned}
&\leqslant \int_{t_k^n}^{t} \left(E \left| \int_{t_k^n}^{s} \left[a_2(s,u,x_u) - A_{2k}\left(s,u,x_u^n\right)\right] du \right|^p \right)^{1/p} (\Delta_s)^{(p-1)/p}\, ds \\
&\leqslant \frac{p-1}{p} \int_{t_k^n}^{t} \Delta_s\, ds \\
&\quad + \frac{1}{p} \int_{t_k^n}^{t} \left(s - t_k^n\right)^{p-1} \int_{t_k}^{s} E\left|a_2(s,u,x_u) - A_{2k}\left(s,u,x_u^n\right)\right|^p du\, ds \\
&\leqslant \frac{p-1}{p} \int_{t_k^k n}^{t} \Delta_s\, ds \\
&\quad + \frac{2^{p-1}}{p} \int_{t_k^n}^{t} \left(s - t_k^n\right)^{p-1} \int_{t_k^n}^{s} \left[L^p\, \Delta_u + \frac{L_2^p\, D_{(m_2+1)p}}{[(m_2+1)!]^p}\, \delta_n^{(m_2+1)p/2} \right] du\, ds \\
&\leqslant \left[\frac{2^{p-1}\, L^p}{p}\, \frac{(t-t_k^n)^p}{p} + \frac{p-1}{p} \right] \int_{t_k^n}^{t} \Delta_s\, ds \\
&\quad + \frac{2^{p-1}\, L_2^p\, D_{(m_2+1)p}}{p[(m_2+1)!]^p}\, \frac{(t-t_k^n)^{p+1}}{p+1} \delta_n^{(m_2+1)p/2} \\
&\leqslant \alpha_2 \int_{t_k^n}^{t} \Delta_s\, ds + \beta_2\, \delta_n^{(m_2+1)p/2} \left(t - t_k^n\right).
\end{aligned}
\tag{13}
$$

Similarly,

$$
\begin{aligned}
I_3(t) &= E \int_{t_k^n}^{t} \left| \int_{t_k^n}^{s} \left[a_3(s,x_s) - A_{3k}\left(s,x_s^n\right)\right] dw_u \right| \left|x_s - x_s^n\right|^{p-1} ds \\
&\leqslant \frac{p-1}{p} \int_{t_k^n}^{t} \Delta_s\, ds \\
&\quad + \frac{c_p}{p} \int_{t_k^n}^{t} \left(s - t_k^n\right)^{p/2-1} \int_{t_k^n}^{s} E\left|a_3(s,u,x_u) - A_{3k}\left(s,u,x_u^n\right)\right|^p du\, ds \\
&\leqslant \left[\frac{2^{p-1}\, c_p\, L^p}{p}\, \frac{(t-t_k^n)^{p/2}}{p/2} + \frac{p-1}{p} \right] \int_{t_k^n}^{t} \Delta_s\, ds
\end{aligned}
$$

$$+\frac{2^{p-1}c_pL_3^pD_{(m_3+1)p}}{p\,[(m_3+1)!]^p}\,\frac{(t-t_k^n)^{p/2+1}}{p/2+1}\,\delta_n^{(m_3+1)p/2}$$
$$\leqslant \alpha_3\int_{t_k^n}^t \Delta_s\,ds+\beta_3\,\delta_n^{(m_3+1)p/2}\big(t-t_k^n\big). \tag{14}$$

By repeating the previous procedures and by using Hölder inequality for $\mu=p/2$, $\nu=p/(p-2)$, we easily come to the estimations of the remaining integrals in (11). Thus,

$$I_4(t)=E\int_{t_k^n}^t\big|b_1(s,x_s)-B_{1k}\big(s,x_s^n\big)\big|^2\big|x_s-x_s^n\big|^{p-2}\,ds$$
$$\leqslant 2L^2\int_{t_k^n}^t\Delta_s\,ds+2E\int_{t_k^n}^t\big|b_1\big(s,x_s^n\big)-B_{1k}\big(s,x_s^n\big)\big|^2\,\big|x_s-x_s^n\big|^{p-2}\,ds$$
$$\leqslant\left[2L^2+\frac{2(p-2)}{p}\right]\int_{t_k^n}^t\Delta_s\,ds+\frac{4\bar{L}_1^2\,D_{(n_1+1)p}}{[(n_1+1)!]^2\,p}\,\delta_n^{(n_1+1)p/2}\big(t-t_k^n\big)$$
$$\equiv\alpha_4\int_{t_k^n}^t\Delta_s\,ds+\beta_4\,\delta_n^{(n_1+1)p/2}\big(t-t_k^n\big), \tag{15}$$

$$I_5(t)=E\int_{t_k^n}^t\left|\int_{t_k^n}^s\big[b_2(s,x_s)-B_{3k}\big(s,x_s^n\big)\big]\,du\right|^2\big|x_s-x_s^n\big|^{p-2}\,ds$$
$$\leqslant\left[\frac{2^pL^p}{p}\,\frac{(t-t_k^n)^p}{p}+\frac{p-2}{p}\right]\int_{t_k^n}^t\Delta_s\,ds$$
$$+\frac{2^p\,\bar{L}_2^p\,D_{(n_2+1)p}}{p\,[(n_2+1)!]^p}\,\frac{(t-t_k^n)^{p+1}}{p+1}\,\delta_n^{(n_2+1)p/2}$$
$$\leqslant\alpha_5\int_{t_k^n}^t\Delta_s\,ds+\beta_5\,\delta_n^{(n_2+1)p/2}\big(t-t_k^n\big), \tag{16}$$

$$I_6(t)=E\int_{t_k^n}^t\left|\int_{t_k^n}^s\big[b_3(s,x_s)-B_{3k}\big(s,x_s^n\big)\big]\,dw_u\right|^2\big|x_s-x_s^n\big|^{p-2}\,ds$$
$$\leqslant\left[\frac{2^pc_pL^p}{p}\,\frac{(t-t_k^n)^{p/2}}{p/2}+\frac{p-2}{p}\right]\int_{t_k^n}^t\Delta_s\,ds$$

$$+ \frac{2^p c_p \bar{L}_3^p D_{(n_3+1)p}}{p[(n_3+1)!]^p} \frac{(t-t_k^n)^{p/2+1}}{p/2+1} \delta_n^{(n_3+1)p/2}$$

$$\leqslant \alpha_6 \int_{t_k^n}^{t} \Delta_s \, ds + \beta_6 \delta_n^{(n_3+1)p/2} \big(t - t_k^n\big), \tag{17}$$

where α_j, β_j are generic constants independent of n and δ_n.

By taking (12)–(17) to (11), we come to the following relation:

$$\Delta_t \leqslant \Delta_{t_k^n} + \alpha \int_{t_k^n}^{t} \Delta_s \, ds + \beta \, \delta_n^{(m+1)p/2} \big(t - t_k^n\big), \quad t \in \big[t_k^n, t_{k+1}^n\big], \tag{18}$$

where $m = \min\{m_j, n_j, j = 1, 2, 3\}$ and α and β are some constants, generic by α_i and β_i and independent of n and δ_n. An application of the well-known Gronwall–Bellman inequality [3] leads to the estimation

$$\Delta_t \leqslant \big[\Delta_{t_k^n} + \beta \delta_n^{(m+1)p/2} \big(t - t_k^n\big)\big] e^{\alpha(t - t_k^n)}, \quad t \in \big[t_k^n, t_{k+1}^n\big], \ 0 \leqslant k \leqslant q_n - 1. \tag{19}$$

By taking $t = t_{k+1}^n$ and by applying the iterative procedure earlier used in [2,6,8], we conclude that

$$\Delta_{t_k^n} \leqslant \beta e^{\alpha} \delta_n^{(m+1)p/2}, \quad 0 \leqslant k \leqslant q_n - 1.$$

Hence, from (19) it follows that there exists a constant $H > 0$, so that $\sup_{t \in [0,1]} \Delta_t \leqslant H \delta_n^{(m+1)p/2}$. Thus, this part of the proof is completed.

For $p = 2$, we have

$$\Delta_t \leqslant 7 \Bigg[\Delta_{t_k^n} + E \Bigg| \int_{t_k^n}^{t} \big[a_1(s, x_s) - A_{1k}\big(s, x_s^n\big)\big] \, ds \Bigg|^2$$

$$+ E \Bigg| \int_{t_k^n}^{t} \int_{t_k^n}^{s} \big[a_2(s, u, x_u) - A_{2k}\big(s, u, x_u^n\big)\big] \, du \, ds \Bigg|^2$$

$$+ \cdots + E \Bigg| \int_{t_k^n}^{t} \int_{t_k^n}^{s} \big[b_3(s, u, x_u) - B_{3k}\big(s, u, x_u^n\big)\big] \, dw_u \, dw_s \Bigg|^2 \Bigg].$$

By applying the usual Ito integral isometry and Proposition 1, we easily deduce that the relation (18) holds for $p = 2$, which completes the proof of this assertion. □

Finally, on the basis of Proposition 2 we can expect that the sequence of the approximate solutions $\{x^n, n \in N\}$ converges to the solution x as $n \to \infty$ and $\delta_n \to 0$, in the sense of the L^p-norm. This conclusion immediately follows from the next theorem, in which the rate of the closeness between x and x^n is also given.

Theorem 1. *Let the conditions of Proposition 2 be satisfied. Then, for $p \geqslant 2$,*

$$E\Big\{\sup_{t\in[0,1]}\big|x_t - x_t^n\big|^p\Big\} \leqslant K\,\delta_n^{(m+1)p/2}, \tag{20}$$

where K is a generic constant independent of n and δ_n.

Proof. Let us denote that

$$A_1\big(s,x_s^n\big) = A_{1k}\big(s,x_s^n\big), \quad s\in\big[t_k^n, t_{k+1}^n\big],$$
$$A_2\big(s,u,x_u^n\big) = A_{2k}\big(s,u,x_u^n\big), \quad u\in\big[t_k^n, t_{k+1}^n\big]$$

and similarly for A_3, B_1, B_2, B_3. Then,

$$\begin{aligned} x_t^n = x_0 &+ \int_0^t \bigg[A_1\big(s,x_s^n\big) + \int_0^s A_2\big(s,u,x_u^n\big)\,du + \int_0^s A_3\big(s,u,x_u^n\big)\,dw_u\bigg]ds \\ &+ \int_0^t \bigg[B_1\big(s,x_s^n\big) + \int_0^s B_2\big(s,u,x_u^n\big)\,du + \int_0^s B_3\big(s,u,x_u^n\big)\,dw_u\bigg]dw_s, \\ & t\in[0,1]. \end{aligned} \tag{21}$$

From (4) and (21) we find that

$$\begin{aligned} &E\sup_{t\in[0,1]}\big|x_t - x_t^n\big|^p \\ &\leqslant 2^{p-1}\bigg\{E\sup_{t\in[0,1]}\bigg|\int_0^t \bigg[a_1(s,x_s) - A_1\big(s,x_s^n\big) + \int_0^s \big[a_2(s,u,x_u) - A_2\big(s,u,x_u^n\big)\big]\,du \\ &\quad + \int_0^s \big[a_3(s,u,x_u) - A_3\big(s,u,x_u^n\big)\big]\,dw_u\bigg]ds\bigg|^p \\ &\quad + E\sup_{t\in[0,1]}\bigg|\int_0^t \bigg[b_1(s,x_s) - B_1\big(s,x_s^n\big) + \int_0^s \big[b_2(s,u,x_u) - B_2\big(s,u,x_u^n\big)\big]\,du \\ &\quad + \int_0^s \big[b_3(s,u,x_u) - B_3\big(s,u,x_u^n\big)\big]\,dw_u\bigg]dw_s\bigg|^p\bigg\} \\ &\leqslant 2^{p-1}\bigg\{E\bigg|\int_0^1 \bigg[a_1(s,x_s) - \cdots + \int_0^s \big[a_3(s,u,x_u) - A_3\big(s,u,x_u^n\big)\big]\,dw_u\bigg]ds\bigg|^p \\ &\quad + c_p\int_0^1 E\bigg|b_1(s,x_s) - \cdots + \int_0^s \big[b_3(s,u,x_u) - B_3\big(s,u,x_u^n\big)\big]\,dw_u\bigg|^p\,ds\bigg\} \\ &\equiv 2^{p-1}(S_1 + c_pS_2), \end{aligned} \tag{22}$$

where S_1 and S_2 are the corresponding integrals. Let us estimate S_1:

$$S_1 \leqslant 6^{p-1} \int_0^1 \Bigg[E\big|a_1(s,x_s) - a_1\big(s,x_s^n\big)\big|^p + E\big|a_1\big(s,x_s^n\big) - A_1\big(s,x_s^n\big)\big|^p$$
$$+ E\left|\int_0^s \big[a_2(s,u,x_u) - a_2\big(s,u,x_u^n\big)\big]\,du\right|^p$$
$$+ E\left|\int_0^s \big[a_2\big(s,u,x_u^n\big) - A_2\big(s,u,x_u^n\big)\big]\,du\right|^p$$
$$+ E\left|\int_0^s \big[a_s(s,u,x_u) - a_2\big(s,u,x_u^n\big)\big]\,dw_u\right|^p$$
$$+ E\left|\int_0^s \big[a_3\big(s,u,x_u^n\big) - A_3\big(s,u,x_u^n\big)\big]\,dw_u\right|^p \Bigg]\,ds. \tag{23}$$

Because $m_1 \geqslant m$, from (7) and Proposition 1 we find that, for $s \in [t_k^n, t_{k+1}^n]$,

$$E\big|a_1\big(s,x_s^n\big) - A_1\big(s,x_s^n\big)\big|^p \leqslant \frac{L_1^p}{[(m_1+1)!]^p} E\big|x_s^n - x_{t_k}^n\big|^{(m_1+1)p} \leqslant h_1 \delta_n^{(m+1)p/2},$$

where h_1 is a constant. Then,

$$\int_0^1 E\big|a_1\big(s,x_s^n\big) - A_1\big(s,x_s^n\big)\big|^p\,ds \leqslant h_1 \delta_n^{(m+1)p/2},$$

and analogously for the fourth and sixth integrals in (23). Finally, by repeating the preceding procedures and stochastic integral isometry, without special emphasizing any step, we come to the following estimation:

$$S_1 \leqslant 6^{p-1} \Bigg\{ L^p \int_0^1 \Bigg[E\big|x_s - x_s^n\big|^p + s^{p-1} \int_0^s E\big|x_u - x_u^n\big|^p\,du$$
$$+ c_p s^{p/2-1} \int_0^s E\big|x_u - x_u^n\big|^p\,du \Bigg]\,ds + h\delta_n^{(m+1)p/2} \Bigg\},$$

where h is a constant. Now, from Proposition 2 it follows that

$$E\big|x_s - x_s^n\big|^p \leqslant \sup_{s\in[0,1]} E\big|x_s - x_s^n\big|^p \leqslant H\,\delta_n^{(m+1)p/2},$$

so that we finally have

$$S_1 \leqslant K_1 \delta_n^{(m+1)p/2},$$

where K_1 is a constant independent on n and δ_n. Since S_2 can be estimated by the same way, the proof of this theorem follows from (22). $\square$

Therefore, $x^n \xrightarrow{L^p} x$ when $n \to \infty$ and $\delta_n \to 0$, uniformly on $[0, 1]$, and the rate of this convergence is given by (20). Likewise, from Theorem 1 it follows that the rate of the closeness, in the sense of the L^p-norm, between the solutions x and x^n decreases if the degrees of Taylor approximations of the functions a_i, b_i increase, which is similar to Taylor approximation in real analysis.

Moreover, by using Theorem 1, one can easily prove the following important result, the convergence with probability one of the sequence of the approximate solutions $\{x^n, n \in N\}$ to the solution x of Eq. (4).

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied and* $\sum_{n=1}^{\infty} \delta_n \xi_n^{-2} < \infty$ *for any sequence of positive numbers* $\xi_n \downarrow 0$, $n \to \infty$. *Then the sequence* $\{x^n, n \in N\}$ *of the approximate solutions converges with probability one to the solution* x *of Eq.* (4) *as* $n \to \infty$, *uniformly on* $[0, 1]$.

Proof. By using Chebyshev inequality and Theorem 1, we get

$$\sum_{n=1}^{\infty} P\Big\{\sup_{t\in[0,1]} \big|x_t - x_t^n\big|^{p/2} \geqslant \xi_n\Big\} \leqslant \sum_{n=1}^{\infty} E \sup_{t\in[0,1]} \big|x_t - x_t^n\big|^p \xi_n^{-2}$$
$$\leqslant K \sum_{n=1}^{\infty} \delta_n^{(m+1)p/2} \xi_n^{-2} < \infty,$$

so that Borel–Cantelli lemma enables us to conclude that $P\{\sup_{t\in[0,1]} |x_t - x_t^n| \geqslant \xi_n^{2/p}$ infinitely often$\} = 0$, i.e., $\sup_{t\in[0,1]} |x_t - x_t^n| < \xi_n^{2/p}$ with probability one for all large n, and, therefore, $\{x^n, n \in N\}$ converges to x with probability one, uniformly on $[0, 1]$. $\square$

In particular, if the partition points are uniformly disposed, i.e., $\delta_n = 1/n$, it is enough to take, for example, $\xi_n = n^{-1/3}$ for $p = 2$ and $m \geqslant 1$, or $\xi_n = n^{-\eta}$, $0 < \eta < (p/2 - 1)/2$, for $p > 2$ and $m \geqslant 0$.

Let us note some remarks:

First, note that for $a_j = b_j \equiv 0$, $j = 2, 3$, the present paper contains the results from paper [1] for $m_1 = n_1 = 0$, the results from paper [2] for $m_1 = n_1 = 1$, and the ones from paper [8] for any m_1 and n_1.

Likewise, note that because it is almost impossible to solve explicitly Eq. (4), it would be convenient to find its numerical solution based on stochastic numerical analysis, by applying the described analytic method in the construction of various time discrete approximations of Ito processes, and combining it with the Ito–Taylor expansion described, above all, by Kloeden and Platen [10,11]. We also mention here paper [16] by C. Tudor and M. Tudor, in which a general one-step numerical approximate scheme is considered for multidimensional Ito–Volterra stochastic equations, and which is an extension of an analogous approximation given by G.I. Milstein [14] for Ito equations. The comparison of the main result (Theorem 3.1) from paper [16] and, for $p = 2$, of Theorem 1 from the present paper, shows that both approximations have mean square errors of the same order. Specially, it is shown in [16] that Euler, Milstein and Platen–Wagner schemes, which are based on Taylor expansions of zero, first and second degrees, respectively, have order $1/2, 1, 3/2$, respectively, which is exactly the case with the analytic approximations described in the present paper obtained by applying Taylor expansions of zero, first and second degrees, respectively. These facts indicate that numerical approximations based on Taylor expansions of higher degrees could be improved by combining them with analytic approximations presented in this paper. However, it requires additional investigations and could be the object of forthcoming studies.

References

[1] M.A. Atalla, Finite-difference approximations for stochastic differential equations, in: Probabilistic Methods for the Investigation of Systems with an Infinite Number of Degrees of Freedom, Inst. of Math. Acad. of Science USSR, Kiev, 1986, pp. 11–16 (in Russian).

[2] M.A. Atalla, On one approximating method for stochastic differential equations, in: Asymptotic Methods for the Theory of Stochastic Processes, Inst. of Math. Acad. of Science USSR, Kiev, 1987, pp. 15–22 (in Russian).

[3] D. Bainov, P. Simeonov, Integral Inequalities and Applications, Kluwer Academic, Dordrecht, 1992.

[4] M.A. Berger, V.J. Mizel, Volterra equations with Ito integrals I, J. Integral Equations 2 (1980) 187–245.

[5] I.I. Gihman, A.V. Skorohod, Theory of Stochastic Processes III, Springer-Verlag, Berlin, 1979.

[6] D. Ilić, S. Janković, L^p-approximation of solutions of stochastic integrodifferential equations, Publ. Elektrotehn. Fak., Ser. Math., Univ. Beograd. 12 (2001) 52–60.

[7] K. Ito, On stochastic differential equations, Mem. Math. Soc. 4 (1951) 1–51.

[8] S. Janković, D. Ilić, An analytic approximation of solutions of stochastic integrodifferential equations, Comput. Math. Appl. 47 (2004) 903–912.

[9] S. Kanagawa, The rate of convergence for approximate solutions of stochastic differential equations, Tokyo Math. J. 12 (1989) 33–48.

[10] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, third ed., Springer-Verlag, Berlin, 1999.

[11] P.E. Kloeden, E. Platen, Numerical Solution of SDE Throught Computer Experiments, third ed., Springer-Verlag, Berlin, 1999.

[12] G. Ladde, V. Lakshmikantham, Random Differential Inequalities, Academic Press, New York, 1980.

[13] R. Liptser, A. Shiryaev, Statistics of Random Processes I, Springer-Verlag, New York, 1977.

[14] G.I. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic, Dordrecht, 1995.

[15] B. Oksendal, Stochastic Differential Equations: An Introduction with Applications, fifth ed., Springer-Verlag, Berlin, 1998.

[16] C. Tudor, M. Tudor, Approximation schemes for Ito–Volterra stochastic equations, Bol. Soc. Mat. Mexicana 3 (1995) 73–85.