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Asymptotic approximation of degenerate fiber integrals ☆

Brice Camus

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstr. 150, D-44780 Bochum, Germany

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Abstract

We study asymptotics of fiber integrals depending on a large parameter. When the critical fiber is singular, full-asymptotic expansions are established in two different cases: local extremum and isolated real principal type singularities. The main coefficients are computed and invariantly expressed. In the most singular cases, it is shown that the leading term of the expansion is related to invariant measures on the spherical blow-up of the singularity. The results can be applied to certain degenerate oscillatory integrals which occur in spectral analysis and quantum mechanics.

Keywords: Asymptotics; Fiber integrals; Degenerate oscillatory integrals

1. Introduction and statement of the main result

In [3] J. Brüning and R. Seeley have studied asymptotic expansions of integrals:

$$H(z)=\int_0^{\infty}\sigma(xz,x)\,dx,\qquad z\to\infty,\tag{1}$$

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E-mail address: brice.camus@univ-reims.fr.

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where $\sigma(x,\xi)$ is a singular symbol. The result of [3] is quite remarkable, in particular because it can be directly applied to spectral analysis. Many asymptotic questions can be reduced to the study of the previous problem but it is also interesting to consider a generalization

$$I(z)=\int_X g\big(zf(x),x\big)\,dx,\quad z\to\infty,\tag{2}$$

where $g:\mathbb{R}\times X\to\mathbb{R}$, $f:X\to\mathbb{R}$ are smooth and X is a smooth differentiable manifold equipped with the C^∞ strictly positive density dx.

General assumptions. Throughout this work, we will assume that $|f|$ is strictly positive outside of a compact set and that the Fourier transform $\hat{g}$ w.r.t. t exists with $\partial_t^k\hat{g}(t,x)\in L^1(\mathbb{R}\times X)$, $\forall k$. For this reason g will be called a symbol.

This assumption on g is strong but can be weakened, see, e.g., [6]. Mainly, this condition will be used to reach integrals with compact supports. As $z\to\infty$, the asymptotic behavior of $I(z)$ is related to the critical fiber:

$$\mathfrak{S}=f^{-1}\big(\{0\}\big)=\big\{x\in X:\ f(x)=0\big\}.\tag{3}$$

This can easily be viewed with the Fourier inversion formula:

$$I(z)=\int_X\int_{\mathbb{R}} e^{iztf(x)}\hat{g}(t,x)\,dt\,dx,\quad z\to\infty,\tag{4}$$

where $\hat{g}(t,x)$ is the normalized Fourier transform of g w.r.t. t:

$$\hat{g}(t,x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau t}g(\tau,x)\,d\tau.\tag{5}$$

In Eq. (4), the stationary points w.r.t. t are precisely given by $\mathfrak{S}$, i.e., $I(z)$ is asymptotically supported by $\mathfrak{S}$. Since f is smooth $\mathfrak{S}$ is closed and according to the general assumptions above we obtain:

(H_0) *The fiber $\mathfrak{S}$ is compact.* (6)

The next elementary result provides a comfortable formulation of the problem.

Lemma 1. *If $\partial_t^k\hat{g}\in L^1(\mathbb{R}\times X)$, $\forall k\in\mathbb{N}$, modulo terms $\mathcal{O}(z^{-\infty})$, asymptotics of Eq.* (2) *are not changed by assuming that $\hat{g}$ is compactly supported near $\mathfrak{S}$.*

Proof. With $\mathfrak{S}$ compact we choose a cut-off function $\Theta\in C_0^\infty(X)$ such that $\Psi=1$ near $\mathfrak{S}$ and $0\leqslant\Psi\leqslant 1$. We shall estimate the error integral:

$$E(z)=\int_X\int_{\mathbb{R}} e^{iztf(x)}\hat{g}(t,x)\big(1-\Psi(x)\big)\,dt\,dx,\quad z\to\infty.\tag{7}$$

With $L = -(i/zf(x))\partial_t$, we have $L^k e^{iztf(x)} = e^{iztf(x)}$, $\forall k \in \mathbb{N}$. By integration by parts and since $|f(x)| \geqslant C$ on $\operatorname{supp}(1-\Psi)$, we obtain

$$\bigl|E(z)\bigr| \leqslant (Cz)^{-k} \bigl\|\partial_t^k \hat{g}(t,x)\bigr\|_{L^1(\mathbb{R}\times X)} = \mathcal{O}\bigl(z^{-k}\bigr), \quad \forall k \in \mathbb{N}. \tag{8}$$

This gives the desired result, with our hypothesis on g. □

Lemma 1 allows to consider only integrals with compact support w.r.t. x which simplifies all questions of convergence. Notice that we can weaken the condition on g to $\partial_t^k \hat{g} \in L^1(\mathbb{R}\times X)$, $\forall k \leqslant k_0$, with an error $\mathcal{O}(z^{-k_0})$. We are mainly interested in the situation where $\mathfrak{S}$ has an isolated singularity. If $x_0 \in \mathfrak{S}$ is such a critical point, let $\Theta \in C_0^\infty(X)$ be a cut-off microlocally supported near x_0. We split-up our integral as $I(z) = I_r(z) + I_s(z)$, where:

$$I_r(z) = \int_X g\bigl(zf(x), x\bigr)(1-\Theta)(x)\,dx, \tag{9}$$

$$I_s(z) = \int_X g\bigl(zf(x)t, x\bigr)\Theta(x)\,dx. \tag{10}$$

The regular part I_r can be treated by the generalized stationary phase method, with non-degenerate normal Hessian, which we recall below. Since I_s is a local object and the main contributions below concern invariant objects, there is no loss of generality to assume that $\operatorname{supp}(\Theta)$ is an open of $\mathbb{R}^n$, $n = \dim(X)$. For $x_0 \in \mathfrak{S}$ a singularity of finite order we can write the germ of f as

$$f(x) = f_k(x) + \mathcal{O}\bigl(\bigl\|(x-x_0)\bigr\|^{k+1}\bigr), \tag{11}$$

where $f_k \neq 0$ is homogeneous of degree $k \geqslant 2$ w.r.t. $(x - x_0)$. The first elementary result concerns extremum attached to such homogeneous germs:

Theorem 2. *If f has a local extremum x_0 on $\mathfrak{S}$ whose jet is given by Eq.* (11) *(a fortiori k is even), we obtain a full-asymptotic expansion*:

$$I_s(z) \sim \sum_{j\in\mathbb{N}} c_j z^{-\frac{j}{k}}. \tag{12}$$

If $\dim(X) = n$*, the leading term is given by*

$$I_s(z) = z^{-\frac{n}{k}} \bigl\langle t_e^{\frac{n-k}{k}}, g(t, x_0)\bigr\rangle \frac{1}{k} \int_{\mathbb{S}^{n-1}} \bigl|f_k(\theta)\bigr|^{-\frac{n}{k}}\,d\theta + \mathcal{O}\bigl(z^{-\frac{n+1}{k}}\bigr), \tag{13}$$

with $t_e = \max(t,0)$ if x_0 is a minimum and $\max(-t,0)$ for a maximum.

The reader can observe that Theorem 2 includes the case of a non-integrable singularity on $\mathfrak{S}$ for $k > n$. Accordingly, always for $k > n$, the contribution of the critical point is bigger than the regular contribution (see below). The case of non-extremum degenerate critical points is more difficult. Since this problem can be very complicated in general position we impose:

(H_1) *$\mathfrak{S}$ has a unique critical point x_0. Moreover, f_k defined in Eq.* (11) *is non-degenerate in the sense that*

$$\nabla f_k \neq 0 \quad \text{on } C(f_k) = \mathbb{S}^{n-1} \cap \{f_k = 0\}. \tag{14}$$

Observe that this condition is very close to Hörmander's real principal type condition for distributions. We define the integrated density of f_k on $\mathbb{S}^{n-1}$ as

$$\mathrm{LVol}(w) = \int_{\{f_k(\theta)=w\}} |dL|(\theta), \qquad dL(\theta) \wedge df_k(\theta) = d\theta, \tag{15}$$

where $|dL|$ is the $(n-2)$-dimensional Liouville measure induced by f_k on $\mathbb{S}^{n-1}$, i.e., the Riemannian density induced by f_k on the standard density of $\mathbb{S}^{n-1}$. Note that (H_1) insures that $\mathrm{LVol}(w)$ is well defined and smooth near the origin.

Theorem 3. *Under the previous assumptions and if x_0 satisfies* (H_1)*, the singular part of our integral admits a full asymptotic expansion*:

$$I_s(z) \sim \sum_{j=0}^{\infty} c_j z^{-\frac{j}{k}} + \sum_{j=0}^{\infty} d_j z^{-j} \log(z). \tag{16}$$

(a) *If $k > n$* (*non-integrable singularity*)*, the leading term is*:

$$I_s(z) = C_0 z^{-\frac{n}{k}} + \mathcal{O}\big(z^{-\frac{n+1}{k}} \log(z)\big), \tag{17}$$

where the distributional coefficient C_0 is given by

$$\frac{1}{k}\left(\big\langle t_+^{\frac{n}{k}-1}, g(t,x_0)\big\rangle \int_{\{f_k \geqslant 0\}} \big|f_k(\theta)\big|^{-\frac{n}{k}}\, d\theta + \big\langle t_-^{\frac{n}{k}-1}, g(t,x_0)\big\rangle \int_{\{f_k \leqslant 0\}} \big|f_k(\theta)\big|^{-\frac{n}{k}}\, d\theta\right).$$

(b) *If $n = kp$, $p \in \mathbb{N}^*$, the leading term is logarithmic*:

$$I_s(z) = z^{-p} \log(z)\left(\frac{d^{p-1}}{dw^{p-1}}\mathrm{Lvol}(0)\right)\frac{1}{k}\int_{\mathbb{R}} |t|^{p-1} g(t,x_0)\, dt + \mathcal{O}\big(z^{-p}\big). \tag{18}$$

(c) *If $n > k$ and $n/k \notin \mathbb{N}$* (*integrable singularity*) *we obtain the same result as in* (a) *but with the modified distributions*:

$$\big\langle t_+^{\frac{n}{k}-1}, g(t,x_0)\big\rangle \left\langle \frac{\tilde{d}^n}{\tilde{d}w^n} w_+^{n-\frac{n}{k}}, \mathrm{Lvol}\right\rangle + \big\langle t_-^{\frac{n}{k}-1}, g(t,x_0)\big\rangle \left\langle \frac{\tilde{d}^n}{\tilde{d}w^n} w_-^{n-\frac{n}{k}}, \mathrm{Lvol}\right\rangle,$$

where the derivatives w.r.t. w are normalized distributional derivatives.

The meaning of normalized derivative is that one choose the normalization:

$$\left\langle \frac{\tilde{d}^n}{\tilde{d}w^n} w_\pm^{n-\frac{n}{k}}, f(w)\right\rangle := \big\langle w_\pm^{-\frac{n}{k}}, f(w)\big\rangle, \tag{19}$$

for all $f \in C_0^\infty$ with $f = 0$ in a neighborhood of the origin. The distributional bracket involving Lvol is detailed in the proof. Results (c) and (b) for $p \geqslant 2$ are not intuitive and are certainly difficult to be reached without geometry. In particular, for applications to oscillatory integrals (see below) one has to work in the dual since both Fourier transforms w.r.t. t in (c) and (b) are distributional. In (c), the nth derivative is arbitrary and the result is the same for any normalized derivative of order greater than $\mathrm{E}(n/k)$. Results (a) and (b) for $p=1$ are interesting for spectral analysis since these contributions are bigger than $I_r(z) = \mathcal{O}(z^{-1})$. As in Theorem 2, non-integrable singularities have a dominant contribution and the leading term of $I(z)$ is always an invariant.

Finally, to treat the regular part I_r, we recall the classical result:

Proposition 1. *Under the previous assumptions and if $\mathfrak{S}$ is a regular surface, $I(z)$ admits a full asymptotic expansion in powers of z^{-1} with*

$$I(z) = \frac{1}{z} \int\limits_{x\in\mathfrak{S}} \int\limits_{\mathbb{R}} g(t,x)\,dt\,d_{\mathfrak{S}}(x) + \mathcal{O}\big(z^{-2}\big), \tag{20}$$

where $d_{\mathfrak{S}}$ is the invariant surface measure of $\mathfrak{S}$. The same result holds for the integral $I_r(z)$ with insertion of the cut-off in the integral.

This result is a direct consequence of [6, Lemma 7.7.3, vol. 1]. Here $d_{\mathfrak{S}}$ is the Liouville measure of classical mechanics or Gelfand–Leray measure in theory of singularities. The oscillatory representation of delta-Dirac distributions, by mean of Schwartz kernels, provides a natural definition of this object.

Application to oscillatory integrals

A typical application of Theorems 2 and 3 can be the asymptotic expansion of distributional traces of quantum propagators. Hence, it is interesting to remark that our results can be extended to asymptotic integrals

$$\tilde{I}(z) = \int\limits_X G\big(z, zf(x), x\big)\,dx, \quad z \to +\infty,$$

if G admits an asymptotic expansion with a priori estimates, i.e.:

$$G(z,t,x) = \sum_{j\leqslant l} z^{-\alpha_j} g_j(t,x) + R_l(z,t,x),$$

$$\forall k \in \mathbb{N}^*: \quad \big\|R_k(z,t,x)\big\|_{L^1(\mathbb{R}\times X)} = \mathcal{O}\big(z^{-(\alpha_k+\varepsilon)}\big), \quad \varepsilon > 0,$$

where $(\alpha_j)_j$ is a strictly increasing sequence. Similarly, we can consider expansions in term of $z^{-\alpha_j}\log(z)^m$. This graduation w.r.t. z allows to apply our results but, to simplify, in this work we just consider the case of an integral of a symbol $g(t,x)$. We can treat degenerate oscillatory integrals

$$O(z) = \int\limits_{\mathbb{R}\times X} e^{iztf(x)} a(t,x,z)\,dt\,dx, \quad z \to +\infty, \tag{21}$$

providing that f satisfies the conditions of Theorems 2 or 3. An important application in quantum mechanics is the case $X = T^*\mathbb{R}^n$ where, after some technical modifications, the localized (distributional) trace of h-pseudors

$$\operatorname{Tr} u_h(A_h - E) := \operatorname{Tr} \int_{\mathbb{R}} \hat{u}(t) e^{\frac{i}{h} t(A_h - E)}\, dt, \quad \hat{u} \in C_0^\infty(\mathbb{R}),\ E \in \mathbb{R},$$

can be written as a locally finite sum of oscillatory integrals:

$$\int_{\mathbb{R}\times T^*\mathbb{R}^n} e^{\frac{i}{h}(S(t,y,\eta) - \langle y,\eta\rangle - tE)} b(h,t,y,\eta)\, dt\, dy\, d\eta, \tag{22}$$

where $b(h,\bullet) \sim \sum h^{-k} b_k$ satisfies a priori estimates as above and S is the local generating function of the group of diffeomorphism of the principal symbol of A_h. Here $z = h^{-1}$ is the parameter and, after a discussion based on classical mechanics, Eq. (22) can be reformulated as in Eq. (21) where $\mathfrak{S}$ is the energy surface of level E. For more details, we refer to [2,4,5].

2. Proof of the main results

To simplify notations we identify x_0 with the origin by mean of local coordinates. If X is Riemannian this can always be achieved by mean of the exponential $\exp_{x_0}$ and a cut-off χ on $T_{x_0}X$ with $\operatorname{supp}(\chi) \subset B(0, r_0/2)$ where r_0 is the injectivity radius at x_0.

2.1. Local minimum

By the extremum condition, x_0 is isolated on $\mathfrak{S}$ and $\mathfrak{S} \cap \operatorname{supp}(\Theta) = \{x_0\}$ for $\operatorname{supp}(\Theta)$ small enough. We use polar coordinates, by Taylor we have

$$f(r\theta) = r^k\big(f_k(\theta) + R(r,\theta)\big), \qquad R(0,\theta) = 0. \tag{23}$$

If $\operatorname{supp}(\Theta) \subset B(0, r_0)$ is chosen small enough we have

$$f_k(\theta) + R(r,\theta) \neq 0, \quad \forall \theta \in \mathbb{S}^{n-1},\ \forall r \in [0, r_0[.$$

Hence, with the homogeneous coordinates $v = (u, \theta)$ where

$$u(r,\theta) = r\big(f_k(\theta) + R(r,\theta)\big)^{1/k}, \tag{24}$$

we can express our integral as

$$I_s(z) = \int_{\mathbb{R}_+} G\big(zu^k, u\big)\, du. \tag{25}$$

The new symbol G is obtained by pullback and integration:

$$G(t,u) = \int_{\mathbb{S}^{n-1}} v^*\big(g(t, r\theta)\Theta(r\theta) r^{n-1} \big|J(v)\big|\big)\, d\theta. \tag{26}$$

The next lemma (see [5] for a proof) gives the existence of the expansion.

Lemma 4. *For a in $C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, the following asymptotic expansion holds*:

$$\int_{\mathbb{R}_+} a\big(zu^k, u\big)\,du \sim \frac{1}{k}\sum_{j\geqslant 0} z^{-\frac{j+1}{k}}\frac{1}{j!}\big\langle \tau_+^{\frac{j+1-k}{k}}\otimes\delta_0^{(j)}, a\big\rangle, \quad z\to\infty. \tag{27}$$

This expansion holds also for a pullback by $-u^k$ if we replace τ_+^α by τ_-^α. This allows to treat the case of a local maximum in Theorem 2. Also for an application to oscillatory integrals we obtain a nice formulation via the Fourier transform of the distributions $\tau_\pm^\alpha$, which avoids any "regularization."

We apply Lemma 4 to Eq. (25) to prove the existence of the asymptotic expansion and it remains to express invariantly the leading term. With the polar coordinates G vanishes up to the order $n-1$. Consequently, we have

$$I_s(z) = \frac{z^{-\frac{n}{k}}}{k}\frac{1}{(n-1)!}\big\langle t_+^{\frac{n-k}{k}}\otimes\delta_0^{n-1}, G\big\rangle + \mathcal{O}\big(z^{-\frac{n+1}{k}}\big).$$

Starting form Eq. (26) and since $|Jv|(0,\theta) = |f_k(\theta)|^{-\frac{1}{k}}$, by elementary manipulations on delta-Dirac distributions, we obtain that:

$$I_s(z) = z^{-\frac{n}{k}}\big\langle t_+^{\frac{n-k}{k}}, g(t,0)\big\rangle\frac{1}{k}\int_{\mathbb{S}^{n-1}}\big|f_k(\theta)\big|^{-\frac{n}{k}}\,d\theta + \mathcal{O}\big(z^{-\frac{n+1}{k}}\big). \tag{28}$$

Finally, for a local maximum we replace the distributions $t_+^{\frac{n-k}{k}}$ by $t_-^{\frac{n-k}{k}}$.

2.1.1. On the integral on the sphere

The integrals on $\mathbb{S}^{n-1}$ of Eq. (28) can be reformulated. We have

$$I(f_k) = \int_{\mathbb{R}_+\times\mathbb{S}^{n-1}} e^{-r^k f_k(\theta)} r^{n-1}\,dr\,d\theta = \int_0^\infty e^{-u^k}u^{n-1}\,du \int_{\mathbb{S}^{n-1}}\big|f_k(\theta)\big|^{-n/k}\,d\theta.$$

Hence our integral is given by

$$\frac{1}{k}\int_{\mathbb{S}^{n-1}}\big|f_k(\theta)\big|^{-n/k}\,d\theta = \frac{I(f_k)}{\Gamma(n/k)}. \tag{29}$$

$I(f_k)$ can be computed as a product of gamma or hypergeometric factors.

2.2. Case of non-extremum critical points

To perform a blow-up of the singularity we use polar coordinates and the next lemma gives a resolution of the singularity w.r.t. $C(f_k)$.

Lemma 5. *In a micro-local neighborhood of the origin there exists local coordinates y, on the blow-up of the critical point, such that*

$$f(x) \simeq \pm y_1^k, \quad \textit{respectively in directions where } f_k(\theta) > 0 \textit{ and } f_k(\theta) > 0,$$
$$f(x) \simeq y_1^k y_2, \quad \textit{locally near } C(f_k).$$

Proof. By Taylor, there exists R continuous in $r = 0$ such that

$$f(x) \simeq f(r\theta) = r^k\big(f_k(\theta) + R(r,\theta)\big). \tag{30}$$

If $\theta_0 \notin C(f_k)$ and θ is close to θ_0, we simply choose:

$$(y_2,\dots,y_n)(r,\theta) = (\theta_1,\dots,\theta_{n-1}),$$
$$y_1(r,\theta) = r\big|f_k(\theta) + R(r,\theta)\big|^{1/k}.$$

In these coordinates the phase becomes y_1^k if $f_k(\theta_0)$ is positive (respectively $-y_1^k$ for a negative value) and the Jacobian satisfies $|Jy|(0,\theta) = |f_k(\theta)|^{1/k} \neq 0$ locally. Now, let $\theta_0 \in C(f_k)$. Up to a permutation, we can suppose that $\partial_{\theta_1} f_k(\theta_0) \neq 0$. We accordingly choose the new local coordinates:

$$(y_1, y_3,\dots,y_n)(r,\theta) = (r,\theta_2,\dots,\theta_{n-1}),$$
$$y_2(r,\theta) = f_k(\theta) + R(r,\theta).$$

Since we have $|Jy|(0,\theta_0) = |\partial_{\theta_1} f_k(\theta_0)| \neq 0$, lemma follows. □

To use Lemma 5 we introduce an adapted partition of unity on $\mathbb{S}^{n-1}$. We pick cut-off functions $\Psi_j \in C_0^\infty(\mathbb{S}^{n-1})$, $0 \leqslant \Psi_j \leqslant 1$, $\sum \Psi_j = 1$ in a tubular neighborhood of $C(f_k)$, with supports chosen so that normal forms of Lemma 5 exist, for $r < r_0$, in a conic neighborhood of $\operatorname{supp}(\Psi_j)$. By compactness this set of functions can be chosen finite and we obtain a partition of unity on $\mathbb{S}^{n-1}$ by adding $\Psi_0 = 1 - \sum \Psi_j$ to our family. The support of Ψ_0 is not connected and we define Ψ_0^+, with $f_k(\theta) > 0$ on $\operatorname{supp}(\Psi_0^+)$. Similarly, we define Ψ_0^- where $f_k < 0$, so that $\Psi_0 = \Psi_0^+ + \Psi_0^-$. We accordingly split up $I_s(z)$ to obtain

$$I_s^\pm(z) = \int\limits_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} \Psi_0^\pm(\theta) g\big(zf(r\theta), r\theta\big)\Theta(r\theta) r^{n-1}\, dr\, d\theta$$
$$= \int\limits_{\mathbb{R}_+} G_0^\pm\big(\pm z y_1^k, y_1\big)\, dy_1,$$

respectively for the directions where $f_k(\theta) > 0$ and $f_k(\theta) < 0$, also

$$I_s^{0,j}(z) = \int\limits_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{n-1}} \Psi_j(\theta) g\big(zf(r\theta), r\theta\big)\Theta(r\theta) r^{n-1}\, dr\, d\theta$$
$$= \int\limits_{\mathbb{R}_+\times\mathbb{R}} G_j\big(z y_1^k y_2, y_1, y_2\big)\, dy_1\, dy_2,$$

for the set $C(f_k)$. The new symbols are respectively given by

$$G_0^\pm(t, y_1) = \int y^*\big(\Psi_0^\pm(\theta) g(t, r\theta)\Theta(r\theta) r^{n-1}|Jy|\big)\, dy_2 \dots dy_n, \tag{31}$$

$$G_j(t, y_1, y_2) = \int y^*\big(\Psi_j(\theta) g(t, r\theta)\Theta(r\theta) r^{n-1}|Jy|\big)\, dy_3 \dots dy_n. \tag{32}$$

Hence, the singular part of our integral can be written as a finite sum:

$$I_s(z) = I_s^-(z) + I_s^+(z) + \sum_j I_s^{0,j}(z). \tag{33}$$

Note that $I_s^-(z)$ and $I_s^+(z)$ can be treated as previously.

Remark 6. Since $y_1(r,\theta) = r$, our new symbols satisfy $G_j(t, y_1, y_2) = \mathcal{O}(y_1^{n-1})$, near $y_1 = 0$. Since the asymptotic expansion involves delta-Dirac distributions w.r.t y_1, cf. Lemma 7 below, the dimension will cause a shift in the expansion.

For $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$, we define the family of elementary fiber integrals:

$$I_{n,k}(z) = \int_0^\infty \left(\int_{\mathbb{R}} a\big(zy_1^k y_2, y_1, y_2\big)\, dy_2 \right) y_1^{n-1}\, dy_1. \tag{34}$$

Lemma 7. *There exists a sequence of distributions $(D_{j,p})$ such that*

$$I_{n,k}(z) \sim \sum_{p=0,1} \sum_{j\in\mathbb{N},\ j\geqslant n} D_{j,p}(a) z^{-\frac{j}{k}} \log(z)^p, \quad \text{as } z \to \infty, \tag{35}$$

where the logarithms only occur when (j/k) is an integer. As concerns the leading term, if $(n/k) \notin \mathbb{N}^*$ we obtain

$$I_{n,k}(z) = z^{-\frac{n}{k}} d(a) + \mathcal{O}\big(z^{-\frac{n+1}{k}} \log(z)\big), \tag{36}$$

with

$$d(a) = C_{n,k} \int_0^\infty \int_0^\infty t^{\frac{n}{k}-1} y_2^{n-\frac{n}{k}} \big(\partial_{y_2}^n a(t,0,y_2) + \partial_{y_2}^n a(-t,0,-y_2)\big)\, dy_2\, dt.$$

But when $n/k = p \in \mathbb{N}^*$, we have

$$I_{n,k}(z) = \frac{1}{k} z^{-\frac{n}{k}} \log(z) \int_{\mathbb{R}} |t|^{p-1} \partial_{y_2}^{p-1} a(t,0,0)\, dt + \mathcal{O}\big(z^{-\frac{n}{k}}\big).$$

Remark 8. The remainder of Eq. (36) can be optimized to $\mathcal{O}(z^{-\frac{n+1}{k}})$ when $(n+1)/k$ is not an integer, as shows the proof below.

Proof. By a standard density argument we can assume that the amplitude is of the form $a(s, y_1, y_2) = f(s)b(y_1, y_2)$. The Melin transforms of f are:

$$M_\pm(\xi) = \int_0^\infty s^{\xi-1} f(\pm s)\, ds. \tag{37}$$

We split-up $I_{n,k}$ as I_+ and I_- by separating integrations $y_2 > 0$ and $y_2 < 0$. Via Melin's inversion formula, we accordingly obtain

$$I_+(z) = \frac{1}{2i\pi} \int_\gamma M_+(\xi) z^{-\xi} \int_{\mathbb{R}_+^2} \left(y_1 y_2^k\right)^{-\xi} b(y_1, y_2) y_1^{n-1}\, dy_1\, dy_2\, d\xi, \tag{38}$$

where $\gamma = c + i\mathbb{R}$ and $0 < c < k^{-1}$. Similarly we have

$$I_-(z) = \frac{1}{2i\pi} \int_\gamma M_-(\xi) z^{-\xi} \int_{\mathbb{R}_+^2} \left(y_1 y_2^k\right)^{-\xi} b(y_1, -y_2) y_1^{n-1}\, dy_1\, dy_2\, d\xi. \tag{39}$$

Lemma 9. *The family of distributions $\xi \mapsto (y_1 y_2^k)^{-\xi}$ on $C_0^\infty(\mathbb{R}_+^2)$ initially defined in the domain $\Re(\xi) < k^{-1}$ is meromorphic on $\mathbb{C}$ with poles: $\xi_{j,k} = j/k,\ j \in \mathbb{N}^*$. These poles are of order 2 when $\xi_{j,k} \in \mathbb{N}^*$ and of order 1 otherwise.*

Proof. We form the Bernstein–Sato polynomial b_k attached to our problem:

$$T\left(y_2 y_1^k\right)^{1-\xi} := \frac{\partial}{\partial y_2} \frac{\partial^k}{\partial y_1^k} \left(y_2 y_1^k\right)^{1-\xi} = b_k(\xi)\left(t y_2 y_1^k\right)^{-\xi},$$

$$b_k(\xi) = (1-\xi) \prod_{j=1}^k (j - k\xi).$$

If $\Re(\xi) < k^{-1}$, $(k+1)$-integrations by parts yield

$$\int_{\mathbb{R}_+^2} \left(y_1 y_2^k\right)^{-\xi} f(y_1, y_2)\, dy_1\, dy_2 = \frac{(-1)^{k+1}}{b_k(\xi)} \int_{\mathbb{R}_+^2} \left(y_1 y_2^k\right)^{1-\xi} (Tf)(y_1, y_2)\, dy_1\, dy_2.$$

Now the integral in the r.h.s. is analytic in $\Re(\xi) < 1 + k^{-1}$. After m iterations the poles, with their orders, can be read off the rational functions:

$$\mathfrak{R}_m(\xi) = \prod_{p=1}^m \frac{1}{b_k(\xi - p)}. \tag{40}$$

This gives the result since m can be chosen arbitrary large. □

Accordingly, the following functions are meromorphic on $\mathbb{C}$:

$$\mathfrak{g}^\pm(\xi) = \int_{\mathbb{R}_+^2} \left(y_1 y_2^k\right)^{-\xi} b(y_1, \pm y_2)\, dy_1\, dy_2. \tag{41}$$

A classical result, see, e.g., [1], is that $M_\pm(c + ix) \in \mathcal{S}(\mathbb{R}_x)$ when $c \notin -\mathbb{N}$. If we shift the path of integration γ to the right in our integral representation, Cauchy's residue method provides the asymptotic expansion. In fact for any $d > c$, outside of the poles, we have

$$\int_{c+i\mathbb{R}} z^{-\xi} M_+(\xi)\mathfrak{g}^+(\xi)\, d\xi - \int_{d+i\mathbb{R}} z^{-\xi} M_+(\xi)\mathfrak{g}^+(\xi)\, d\xi = \sum_{c<\xi_{j,k}<d} \operatorname{res}\left(z^{-\xi} M_+ \mathfrak{g}\right)(\xi_{j,k}).$$

Since d is not a pole, the last integral can be estimated via

$$\left| \int\limits_{d+i\mathbb{R}} z^{-\xi} M_+(\xi)\mathfrak{g}^+(\xi)\, d\xi \right| \leqslant C(f,b)z^{-d} = \mathcal{O}\big(z^{-d}\big), \tag{42}$$

where, for each d, the constant C involves the L^1-norm of a finite number derivatives of b. This will indeed lead to an asymptotic expansion with precise remainders. Applying this method to $I_+(z)$ and $I_-(z)$, we obtain

$$I(z) \sim \sum_{p=0,1} \sum_{j\in\mathbb{N}^*} C_{j,p} z^{-\frac{j}{k}} \log(z)^p. \tag{43}$$

Moreover, by Lemma 9, these logarithms only occur when j/k is integer.

For our problem, we can commute the polynomial weight via

$$T\big((y_1y_2)^{1-\xi} y_1^{n-1}\big) = \mathfrak{b}(\xi)\big(y_2y_1^k\big)^{-\xi} y_1^{n-1}, \tag{44}$$

$$\mathfrak{b}(\xi) = (1-\xi)\prod_{j=1}^{k}(j - k\xi + n - 1). \tag{45}$$

By iteration, we obtain that the poles are the rational numbers:

$$\xi_{p,j,k,n} = p + \frac{j+n-1}{k}, \quad j \in [1,\dots,k],\ p \in \mathbb{N}.$$

By elementary considerations, all residuum are zero before:

$$\xi_0 = \frac{n}{k}. \tag{46}$$

To compute the first effective residue, we must distinguish out the case where ξ_0 is an integer or not. The optimal number of iterations to reach ξ_0 is $\mathrm{E}(n/k) + 1$ but, by analytic continuation, any integer bigger than this one is acceptable. A fortiori we can use n iterations and our starting point will be

$$z^{-\xi} M_+(\xi)(-1)^{n(k+1)}\mathfrak{B}_n(\xi) \int\limits_{\mathbb{R}_+^2} \big(y_1^k y_2\big)^{n-\xi} y_1^{n-1} T^n b(y_1,y_2)\, dy_1\, dy_2, \tag{47}$$

$$\mathfrak{B}_n(\xi) = \prod_{l=0}^{n-1} \frac{1}{\mathfrak{b}(\xi - l)}. \tag{48}$$

2.2.1. Case of ξ_0 simple pole

In this case our residue is simply given by

$$Cz^{-\frac{n}{k}} M_+\left(\frac{n}{k}\right) \int\limits_{\mathbb{R}_+^2} \big(y_1^k y_2\big)^{n-\frac{n}{k}} y_1^{n-1} T^n b(y_1,y_2)\, dy_1\, dy_2,$$

$$C = \lim_{\xi\to\frac{n}{k}} (-1)^{n(k+1)}\left(\xi - \frac{n}{k}\right)\mathfrak{B}_n(\xi).$$

In particular, we can compute the integral w.r.t. y_1 via

$$\int_0^\infty y_1^{kn-1}\partial_{y_1}^{kn}\big(\partial_{y_2}^n b(y_1,y_2)\big)\,dy_1=(-1)^{kn}(kn-1)!\partial_{y_2}^n b(0,y_2).$$

A similar result holds for I_- and we obtain

$$I_+(z)=z^{-\frac{n}{k}}C_{n,k}M_+\left(\frac{n}{k}\right)\int_0^\infty y_2^{n-\frac{n}{k}}\big(\partial_{y_2}^n b\big)(0,y_2)\,dy_2+R_1(z), \tag{49}$$

$$I_-(z)=z^{-\frac{n}{k}}C_{n,k}M_-\left(\frac{n}{k}\right)\int_0^\infty y_2^{n-\frac{n}{k}}\big(\partial_{y_2}^n b\big)(0,-y_2)\,dy_2+R_2(z). \tag{50}$$

Here $C_{n,k}$ is the canonical constant:

$$C_{n,k}=\frac{1}{k}\prod_{j=1}^n\frac{-1}{j-\frac{n}{k}}. \tag{51}$$

Also, according to the analysis above, each remainder is of order $\mathcal{O}(z^{-\frac{n+1}{k}})$ if $(n+1)/k\notin\mathbb{N}$ and $\mathcal{O}(z^{-\frac{n+1}{k}}\log(z))$ otherwise.

2.2.2. Case of ξ_0 double pole

If h is meromorphic with a pole of order 2 in ξ_0 we have

$$\operatorname{res}(h)(\xi_0)=\frac{1}{2}\lim_{\xi\to\xi_0}\frac{\partial}{\partial\xi}(\xi-\xi_0)^2h(\xi).$$

Applying this principle to our residue, we obtain, via Leibnitz's rule, that

$$I_+(z)=B\log(z)z^{-\frac{n}{k}}+\mathcal{O}\big(z^{-n/k}\big). \tag{52}$$

We can compute the distribution B as before and we find:

$$B=-\frac{1}{2}D_{n,k}M_+\left(\frac{n}{k}\right)\int_0^\infty y_2^{n-\frac{n}{k}}\big(\partial_{y_2}^n b\big)(0,y_2)\,dy_2,$$

$$D_{n,k}=(-1)^{n(k+1)}\lim_{\xi\to\frac{n}{k}}\left(\xi-\frac{n}{k}\right)^2\mathfrak{B}_n(\xi).$$

Since $p=n/k$ is an integer, by integration by parts we obtain

$$\int_0^\infty y_2^{n-p}\big(\partial_{y_2}^n b\big)(0,y_2)\,dy_2=(-1)^{n-p+1}(n-p)!\partial_{y_2}^{p-1}b(0,0). \tag{53}$$

With a similar identity for I_-, we obtain the result by gathering all the constants and summation. Finally, since all our coefficients are given by

$$\langle T^j, f\otimes b\rangle = \langle T_1^j, f\rangle\langle T_2^j, b\rangle, \quad T_{1,2}^j \in \mathcal{D}'(\mathbb{R}),$$

by linearity and continuity, the results hold for a symbol $a(t, y_1, y_2)$. $\square$

Taking Remark 6 into account, to avoid unnecessary calculations we define:

$$G_0^{\pm}(t, y_1) = y_1^{n-1}\tilde{G}_0^{\pm}(t, y_1), \tag{54}$$

$$G_j(t, y_1, y_2) = y_1^{n-1}\tilde{G}_j(t, y_1, y_2). \tag{55}$$

2.2.3. Directions where $f_k(\theta)\neq 0$

By Lemma 4, the first non-zero coefficient, obtained for $l = n-1$, is

$$\frac{z^{-\frac{n}{k}}}{k}\frac{1}{(n-1)!}\big\langle t_+^{\frac{n-k}{k}}\otimes\delta_0^{(n-1)}, G_0^+(t, y_1)\big\rangle = \frac{z^{-\frac{n}{k}}}{k}\int\limits_{\mathbb{R}} t_+^{\frac{n-k}{k}}\tilde{G}_0^+(t, 0)\,dt.$$

By construction, we have

$$\tilde{G}_0^+(t, 0) = \int\limits_{\mathbb{S}^{n-1}} g(t, 0)\Psi_0^+(\theta)\big|f_k(\theta)\big|^{-n/k}\,d\theta.$$

A similar computation gives the contribution of $\operatorname{supp}(\Psi_0^-)$, and we obtain

$$I_s^+(z) = z^{-\frac{n}{k}}\big\langle t_+^{\frac{n-k}{k}}, g(t, 0)\big\rangle\frac{1}{k}\int\limits_{\mathbb{S}^{n-1}}\Psi_0^+(\theta)\big|f_k(\theta)\big|^{-\frac{n}{k}}\,d\theta + \mathcal{O}\big(z^{-\frac{n+1}{k}}\big), \tag{56}$$

$$I_s^-(z) = z^{-\frac{n}{k}}\big\langle t_-^{\frac{n-k}{k}}, g(t, 0)\big\rangle\frac{1}{k}\int\limits_{\mathbb{S}^{n-1}}\Psi_0^-(\theta)\big|f_k(\theta)\big|^{-\frac{n}{k}}\,d\theta + \mathcal{O}\big(z^{-\frac{n+1}{k}}\big). \tag{57}$$

2.2.4. Microlocal contribution of $C(f_k)$

(a) *Case of $k > n$, non-integrable singularity on $\mathfrak{S}$.* Here $n/k \in]0, 1[$, so that the singularity on the blow-up is integrable. Via Lemma 7, the contribution of $I_s^{0,j}(z)$ is given by

$$\frac{1}{k}z^{-\frac{n}{k}}\int\limits_{\mathbb{R}_+^2}|t|^{\frac{n}{k}-1}|y_2|^{-\frac{n}{k}}\big(\tilde{G}_j(t, 0, y_2) + \tilde{G}_j(-t, 0, -y_2)\big)\,dt\,dy_2 + \mathcal{O}\big(z^{-\frac{n+1}{k}}\log(z)\big).$$

Reminding that $y_2(t, 0, \theta) = f_k(\theta)$, we obtain

$$\int\limits_{\mathbb{R}_+}|y_2|^{-\frac{n}{k}}\tilde{G}_j(t, 0, y_2)\,dy_2 = g(t, 0)\int\limits_{\{f_k(\theta)\geqslant 0\}}\big|f_k(\theta)\big|^{-\frac{n}{k}}\Psi_j(\theta)\,d\theta.$$

Since $\Psi_0^{\pm}, \Psi_j$ is a partition of unity on $\mathbb{S}^{n-1}$, by summation of all local contributions $I_s(z)$ is asymptotically equivalent to

$$\frac{z^{-\frac{n}{k}}}{k}\left(\left\langle t_+^{\frac{n}{k}-1}, g(t,0)\right\rangle \int\limits_{\{f_k\geqslant 0\}} \left|f_k(\theta)\right|^{-n/k} d\theta + \left\langle t_-^{\frac{n}{k}-1}, g(t,0)\right\rangle \int\limits_{\{f_k\leqslant 0\}} \left|f_k(\theta)\right|^{-\frac{n}{k}} d\theta\right).$$

(b) *Case of $p = n/k$ integer.* Here the contribution of $I_s^{0,j}(z)$ is dominant and we obtain

$$I_s^{0,j}(z) \sim \frac{1}{k}\log(z) z^{-p} \int\limits_{\mathbb{R}} |t|^{p-1} \partial_{y_2}^{p-1} \tilde{G}_j(t,0,0)\, dt + \mathcal{O}\big(z^{-p}\big).$$

Unless $p = 1$, there is no way to take the limit directly, and the geometric properties are still hidden in the Jacobian. But we will reach the result by the Schwartz kernel technic. Clearly, it is enough to evaluate our derivative and to integrate w.r.t. t. With $s = (s_1, s_2) \in \mathbb{R}^2$, we write the evaluation as

$$\partial_{y_2}^{p-1}\tilde{G}_j(t,0,0) = \frac{1}{(2\pi)^2}\int e^{i\langle s,(y_1,y_2)\rangle}(is_2)^{p-1}\tilde{G}_j(t,y_1,y_2)\, dy_1\, dy_2\, ds.$$

This integral formulation allows to inverse our diffeomorphism to obtain

$$\partial_{y_2}^{p-1}\tilde{G}_j(t,0,0) = \frac{1}{(2\pi)^2}\int e^{i\langle s,(r,y_2(r,\theta))\rangle}(is_2)^{p-1} g(t,r\theta)\Psi_j(\theta)\, dr\, d\theta\, ds.$$

Extending the integrand by 0 for $r < 0$, the normalized integral w.r.t. (r, s_1) provides δ_r. By construction $y_2(0,\theta) = f_k(\theta)$, hence

$$\partial_{y_2}^{p-1}\tilde{G}_j(t,0,0) = g(t,0)\frac{1}{(2\pi)} \int\limits_{\mathbb{R}\times\mathbb{S}^{n-1}} e^{iuf_k(\theta)}(iu)^{p-1}\Psi_j(\theta)\, d\theta\, du. \tag{58}$$

This Fourier integral makes sense with $\mathbb{S}^{n-1}$ compact. We recall the density

$$J_j(w) = \int\limits_{\{f_k(\theta)=w\}} \Psi_j(\theta)\, dL_w(\theta), \tag{59}$$

where J_j is compactly supported, and hence in $L^2(\mathbb{R})$. Since J_j is smooth near the origin, the sum over all the Ψ_j provides

$$\frac{1}{(2\pi)}\int\limits_{\mathbb{R}^2} e^{iuw}(iu)^{p-1}\sum_j J_j(w)\, dw\, du = \frac{d^{p-1}\mathrm{Lvol}}{dw^{p-1}}(0). \tag{60}$$

Note that $n = k$ is directly accessible. In this case I_s dominates I_r with

$$I_s(z) = z^{-1}\log(z)\mathrm{LVol}(0)\frac{1}{k}\int\limits_{\mathbb{R}} g(t,0)\, dt + \mathcal{O}\big(z^{-1}\big).$$

(c) *$k < n$ and simple pole, integrable singularity on $\mathfrak{S}$.* Finally, we treat the case of a simple pole with a non-integrable singularity on $\mathbb{S}^{n-1}$. Clearly, we can use the same globalization technic as above. The sum of all positives contributions, completed with the main term of J_s^+, provides

$$\langle T_+, g\rangle = C_{n,k} \int_0^\infty |t|^{\frac{n}{k}-1} g(t,0)\,dt \int_0^\infty w^{n-\frac{n}{k}} \frac{\partial^n L(w)}{\partial w^n}\,dw. \tag{61}$$

A similar result holds for the directions where $f_k(\theta) < 0$. Now, we detail the construction of the distributional bracket. The key point is that we can put in duality distributions $\partial_u^n |u|_\pm^{n-\alpha}$ and $\mathrm{Lvol}(u)$ since their singular supports are disjoints. Let be $\chi \in C_0^\infty$, $0 \leqslant \chi \leqslant 1$ on $\mathbb{R}$, chosen such that $\chi = 1$ near the origin and $\chi(u) = 0$ for $|u| \geqslant \varepsilon$. If $\varepsilon > 0$ is small enough, Lvol is smooth on $]-\varepsilon, \varepsilon[$. We write the geometric contribution as

$$\langle T, \mathrm{Lvol}\rangle = \langle T, \chi \mathrm{Lvol}\rangle + \big\langle T, (1-\chi)\mathrm{Lvol}\big\rangle.$$

Away from the origin, e.g., for $u > 0$, we obtain directly

$$C_{n,k}\left\langle \frac{d^n}{du^n} u_+^{n-n/k}, (1-\chi)(u)\mathrm{Lvol}(u)\right\rangle = \frac{1}{k} \int_{\{f_k(\theta)>0\}} (1-\chi)\big(f_k(\theta)\big)\big|f_k(\theta)\big|^{-n/k}\,d\theta.$$

Note that Eq. (51) for $C_{n,k}$ justifies the normalization of Eq. (19). On $\mathrm{supp}(\chi)$, we use the local regularity of $\mathrm{Lvol}(u)$ near $u = 0$ and integrations by parts to conclude. Finally, since $C(f_k)$ is compact and f_k is continuous we can choose our partition of unity such that $\sum \Psi_j = 1$ for $|f_k(\theta)| \leqslant 2\varepsilon$. $\square$

References

[1] N. Bleistein, R. Handelsman, Asymptotic Expansion of Integrals, Dover, New York, 1986.

[2] R. Brummelhuis, T. Paul, A. Uribe, Spectral estimate near a critical level, Duke Math. J. 78 (1995) 477–530.

[3] J. Brüning, R. Seeley, Regular singular asymptotics, Adv. Math. 58 (1985) 133–148.

[4] B. Camus, A semi-classical trace formula at a non-degenerate critical level, J. Funct. Anal. 208 (2004) 446–481.

[5] B. Camus, A semi-classical trace formula at a totally degenerate critical level. Contributions of extremums, Comm. Math. Phys. 207 (2004) 513–526.

[6] L. Hörmander, The Analysis of Linear Partial Operators 1, 2, 3, 4, Springer-Verlag, 1985.