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The Berezin transform and Laplace–Beltrami operator

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Abstract

We study the Berezin transform of bounded operators on the Bergman space on a bounded symmetric domain Ω in $\mathbb{C}^n$. The invariance of range of the Berezin transform with respect to $\mathcal{G} = \mathrm{Aut}(\Omega)$, the automorphism group of biholomorphic maps on Ω, is derived based on the general framework on invariant symbolic calculi on symmetric domains established by Arazy and Upmeier. Moreover we show that as a smooth bounded function, the Berezin transform of any bounded operator is also bounded under the action of the algebra of invariant differential operators generated by the *Laplace–Beltrami* operator on the unit disk and even on the unit ball of higher dimensions.

Keywords: Berezin transform; Laplace–Beltrami operator; Bergman space

1. Introduction

The Segal–Bargmann space $H^2(\mathbb{C}^n, d\mu)$ consists of Gaussian square integrable entire functions on $\mathbb{C}^n$. It is well known that it is a subspace of $L^2(\mathbb{C}^n, d\mu)$ as the L^2-closure of the polynomials, and a reproducing-kernel Hilbert space with the Bergman kernel $K(z, w) = e^{\langle z,w\rangle/2}$, which satisfies that for any f in $H^2(\mathbb{C}^n, d\mu)$,

$$f(z) = \int_{\mathbb{C}^n} f(w)K(z, w)\, d\mu(w),$$

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where $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ and

$$d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2}\,d\upsilon(z)$$

with $d\upsilon$ the Lebesgue volume measure.

For the algebra $\mathcal{B}(H^2(\mathbb{C}^n, d\mu))$ of all bounded linear operators on the Segal–Bargmann space, the Berezin transform of $X \in \mathcal{B}(H^2(\mathbb{C}^n, d\mu))$ is defined by $Ber: X \to \tilde{X}$ where

$$\tilde{X}(z) = \langle Xk_z, k_z\rangle = \operatorname{trace}(XP_z),$$

where $P_z = k_z \otimes k_z$ is rank one projection, and $k_z(w) = K(w,z)K(z,z)^{-1/2}$ is the normalized reproducing kernel with $\|k_z\|_{H^2(\mathbb{C}^n,d\mu)} = 1$ for any $z \in \mathbb{C}^n$. Generally, $\tilde{X}(z)$ is real analytic and bounded by the operator norm of X. Moreover $\tilde{X}(z)$ is uniquely determined by X, that is, $\tilde{X}(z) = 0$ if and only if $X = 0$. So the mapping *Ber* is linear and one-to-one. Note that *Ber* can also be defined for Bergman space on any bounded domain $\Omega \subset \mathbb{C}^n$.

Recently, in [5] Coburn obtained a Lipschitz estimate for *Ber*. More precisely, it was shown that

$$\left|\tilde{X}(z) - \tilde{X}(w)\right| \leqslant \sqrt{2}\|X\||z - w| \tag{1.1}$$

for any $X \in \mathcal{B}(H^2(\mathbb{C}^n, d\mu))$ and $z, w \in \mathbb{C}^n$. The corresponding Lipschitz estimate was also obtained there for the analogous Bergman space $A^2(\Omega)$ on any bounded domain $\Omega \subset \mathbb{C}^n$, with $|z-w|$ replaced by $\beta(z,w)$ in (1.1), where $\beta(\cdot,\cdot)$ is the distance function with respect to Bergman metric on Ω. It was also pointed out in [5] (without proof) that the Berezin transform of any bounded operator on $H^2(\mathbb{C}^n, d\mu)$ is also bounded under the action of the algebra of linear differential operators with constant coefficients, after the translation invariance of the range of Berezin transform on $\mathbb{C}$ was established there.

We show in this note that the range of Berezin transform is Möbius-invariant (see Theorem 2.1) for the Bergman space $A^2(\Omega)$ on bounded symmetric domain Ω in $\mathbb{C}^n$. Furthermore for two particular cases of Ω of rank 1, namely $\Omega = \mathbb{D}$ unit disk on the complex plane and $\Omega = \mathbb{B}$ unit ball in $\mathbb{C}^n$ with $n > 1$, we obtain that the Berezin transform of any bounded operator on $A^2(\Omega)$ is bounded under the action of the algebra of invariant differential operators.

Our motivation is due to the following observations. First note that it is easy to show that the range of Berezin transform on $\mathbb{C}^n$ is also unitary-invariant, that is, $f \circ U \in Ber(\mathcal{B}(H^2(\mathbb{C}^n, d\mu)))$ for any unitary transformation U on $\mathbb{C}^n$ if $f \in Ber(\mathcal{B}(H^2(\mathbb{C}^n, d\mu)))$. In fact, we can define $(R_U f)(z) = f(U^{-1}z)$ on $H^2(\mathbb{C}^n, d\mu)$, then R_U is unitary operator and the adjoint $R_U^* = R_{U^{-1}}$ and for $X \in \mathcal{B}(H^2(\mathbb{C}^n, d\mu))$,

$$\langle R_U^* X R_U k_z, k_z\rangle = \langle Xk_{Uz}, k_{Uz}\rangle.$$

Hence combining with Coburn's result, we conclude that the range of Berezin transform on $\mathbb{C}^n$ is invariant under the action of the group of all orientation-preserving rigid motions of $\mathbb{C}^n$. Next note that for $\mathbb{C}^n$, the Bergman metric induced by Bergman kernel function of $H^2(\mathbb{C}^n, d\mu)$ gives the distance function $\beta(z,w) = |z - w|$ which is invariant with respect to the group of all orientation-preserving rigid motions (generated by translations and unitary transformations) of $\mathbb{C}^n$, whereas the counterpart for bounded symmetric domain Ω is the Möbius group $\mathcal{G} = \operatorname{Aut}(\Omega)$, which makes us conceive the Möbius invariance of the range of Berezin transform on Ω. Finally we notice that for bounded symmetric domain of rank 1, the algebra of invariant differential operator with respect to $\mathcal{G}$ is generated by *Laplace–Beltrami* operator. We take fully advantage of this fact and explicit formulas of Bergman kernel function and *Laplace–Beltrami* operator on $\mathbb{D}$ and $\mathbb{B}$ in our proof.

This note is organized as follows. In the next section, we give an overview of some relevant facts needed for our purpose from the theory of invariant symbolic calculi in [1], and derive the Möbius invariance of the range of Berezin transform for bounded symmetric domain. In Sections 3 and 4, we show that the Berezin transform of any bounded operator on Bergman space on $\mathbb{D}$ or $\mathbb{B}$ is bounded under the action of any power of *Laplace–Beltrami* operator, respectively, from which our general results follow easily. We discuss some related results in the last section.

2. The invariance of range of Berezin transform

For Ω a bounded symmetric (Cartan) domain in $\mathbb{C}^n$ with normalized Lebesgue measure $d\upsilon(z)$ on it, we assume that Ω is in its Harish-Chandra realization so that $0 \in \Omega$, and $K(z,0) = 1$ for all $z \in \Omega$, then Ω could be realized as the quotient $\Omega = \mathcal{G}/\mathcal{K}$, where $\mathcal{K}$ is the stabilizer in $\mathcal{G}$ of the origin $0 \in \Omega$. For each a in Ω, there is $\varphi_a \in \mathcal{G}$ with the properties $\varphi_a(a) = 0$ and $\varphi_a \circ \varphi_a = I$ the identity map, just indicated in [4] where a lot of properties of Bergman kernel function relating to φ_a are obtained and listed under the above assumptions, such as $|(J_{\mathbf{c}}\varphi_a)(z)|^2 = |k_a(z)|^2$ and $k_a(\varphi_a(z))k_a(z) = 1$, where $(J_{\mathbf{c}}\varphi_a)(z)$ is the complex Jacobian of φ_a at z, of which we will make essential use.

In [1], Arazy and Upmeier established the general theory of invariant symbolic calculi in the context of weighted Bergman spaces on bounded symmetric domains, which is closely related to the intertwinement of the action of $\mathcal{G}$ on some class of $\mathcal{G}$-invariant functions on Ω and the action of $\mathcal{G}$ on the algebra of bounded operators on the weighted Bergman spaces. In particular, for the Bergman space $A^2(\Omega)$, the irreducible projective unitary representation $g \to U_g$ of $\mathcal{G}$ on $A^2(\Omega)$ is given by (see identity (12) in [1])

$$U_g(f)(z) = \left(J_{\mathbf{c}} g^{-1}\right)(z) f\left(g^{-1}(z)\right) \tag{2.1}$$

for any $g \in \mathcal{G}$, $f \in A^2(\Omega)$ and $z \in \Omega$, where $(J_{\mathbf{c}} g^{-1})(z)$ is the complex Jacobian of g^{-1}. And the action of $\mathcal{G}$ on $\mathcal{B}(A^2(\Omega))$ is given by

$$\pi(g)T = U_g T U_g^{-1}.$$

It turns out that π is also a genuine representation of $\mathcal{G}$ since $\pi(g_1 \circ g_2) = \pi(g_1)\pi(g_2)$ for $g_1, g_2 \in \mathcal{G}$.

It follows from (2.1) and the transformation laws of reproducing kernels [8, p. 44] that

$$U_g k_z = c(g,z) k_{g(z)}, \tag{2.2}$$

where $c(g,z)$ is a constant of modulus 1 depending on g and z. Recall that $P_z = k_z \otimes k_z$ is the rank one projection onto the span of k_z, then (2.2) implies that

$$\pi(g)P_z = U_g P_z U_g^{-1} = k_{g(z)} \otimes k_{g(z)} = P_{g(z)}.$$

Then for any $X \in \mathcal{B}(A^2(\Omega))$ and $g \in \mathcal{G}$,

$$\begin{aligned}\tilde{X}\big(g(z)\big) &= \operatorname{trace}(X P_{g(z)}) = \operatorname{trace}\big(X U_g P_z U_g^{-1}\big) \\ &= \operatorname{trace}\big(U_g^{-1} X U_g P_z\big) = \widetilde{U_g^{-1} X U_g}(z).\end{aligned} \tag{2.3}$$

So the preceding overview and discussion lead to the following result:

Theorem 2.1. *The range $Ber(\mathcal{B}(A^2(\Omega)))$ is Möbius-invariant under the action of $\mathcal{G}$.*

For $h \in L^\infty(\Omega)$, *Toeplitz* operator defined on $A^2(\Omega)$ with symbol h is $T_h(f) = P(hf)$ for $f \in A^2(\Omega)$, where P is the orthogonal (Bergman) projection from $L^2(\Omega)$ onto $A^2(\Omega)$. We define the Berezin transform of h by

$$\tilde{h}(a) = \tilde{T}_h(a) = \langle T_h k_a, k_a\rangle = \langle P(hk_a), k_a\rangle = \langle hk_a, k_a\rangle.$$

It is pointed out in Example 2.1 of [1] that

$$U_g^{-1} T_h U_g = T_{h\circ g}$$

for any $h \in L^\infty(\Omega)$ and $g \in \mathcal{G}$, which has also been obtained in Lemma 8 of [2] for $\Omega =$ the unit disk. This fact combined with (2.3) yields that

Corollary 2.2. *For $h \in L^\infty(\Omega)$, $(\tilde{h} \circ g)(a) = \widetilde{h \circ g}(a)$, $\forall g \in \mathcal{G}$ and $a \in \Omega$. In other words, $\tilde{T}_h \circ g = \tilde{T}_{h\circ g}$.*

For $g = \varphi_a \in \mathcal{G}$, we denote $U_a = U_{\varphi_a}$. It is easy to see that U_a is a selfadjoint unitary operator on $A^2(\Omega)$ by involutive property of φ_a for $a \in \Omega$. For A a bounded operator on $A^2(\Omega)$, we can now define an averaging operation by

$$\hat{A} = \int_\Omega U_a^* A U_a \, d\upsilon(a). \tag{2.4}$$

Note that $\hat{A}$ is determined by

$$\langle \hat{A} f, g\rangle = \int_\Omega \langle U_a^* A U_a f, g\rangle \, d\upsilon(a)$$

for any $f, g \in A^2(\Omega)$. That is, the above operator integral is understood in the weak sense.

It is easy to obtain that for any $A \in \mathcal{B}(A^2(\Omega))$,

$$\tilde{\hat{A}}(z) = \int_\Omega \tilde{A} \circ \varphi_a(z) \, d\upsilon(a)$$

and

$$\tilde{T}_{\tilde{A}}(z) = \int_\Omega \tilde{A} \circ \varphi_z(a) \, d\upsilon(a).$$

Note that $\tilde{\hat{A}}$ and $\tilde{T}_{\tilde{A}}$ may not coincide generally. Let $\Omega = \mathbb{D}$ the unit disk, for instance, and $A = T_z = M_z$, then we know $\tilde{A}(z) = z$, so $\tilde{T}_{\tilde{A}}(z) = z$ but $\tilde{\hat{A}}(z) = -z/2$ for any $z \in \mathbb{D}$. However, it was shown in Theorem 6 of [3] that for Segal–Bargmann space $H^2(\mathbb{C}^n, d\mu)$ case, $\tilde{\hat{A}} = \tilde{T}_{\tilde{A}}$, so $\hat{A} = T_{\tilde{A}}$, where the corresponding $(U_a f)(z) = k_a(z) f(z-a)$, $\forall a \in \mathbb{C}^n$ which is unitary but selfadjoint. The reason why this difference occurs may be due to the flat structure of $H^2(\mathbb{C}^n, d\mu)$.

3. The unit disk case

Let $\Omega \subset \mathbb{C}$ be a circular domain containing the origin (i.e., a disk or the entire plane), and the usual Laplace operator is $\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}$ defined on $C^2(\Omega)$. For the sake of simplicity of notation, we let $D = \frac{\partial}{\partial z}$ and $\bar{D} = \frac{\partial}{\partial \bar{z}}$, then $\Delta = D\bar{D}$.

Let E_0 be the evaluation functional on $C^\infty(\Omega)$ at the origin, that is, $E_0(f) = f(0)$ for $f \in C^\infty(\Omega)$. For any unitary transformation on $\mathbb{C}$, that is, a rotation $z \to \alpha z$ for $|\alpha| = 1$, we can define the operator $U_\alpha(f)(z) = f(\alpha z)$ on $C^\infty(\Omega)$ due to the circularity of Ω.

Lemma 3.1. *Let $T = \sum_{j=1}^m \sum_{k=1}^n c_{j,k} D^j \bar{D}^k$ be a constant coefficient differential operator on $C^\infty(\Omega)$. If $E_0 T U_\alpha = E_0 U_\alpha T$ on $C^\infty(\Omega)$ for any $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, then*

$$T = \sum_{j=1}^{\min(m,n)} c_j \Delta^j,$$

where $c_j = c_{j,k}$ for $j = k$.

Proof. Taking $g(z) = z^j \bar{z}^k$ for $1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n$, we know by a direct calculation that

$$E_0 T U_\alpha(g) = \alpha^{j-k} c_{j,k} j! k!$$

and

$$E_0 U_\alpha T(g) = c_{j,k} j! k!.$$

Our assumption yields that

$$c_{j,k} = 0 \quad \text{for } j \neq k,$$

then the desired result follows from it. $\square$

Remark 3.2. In fact our preceding result implies the fact that any rotation-invariant constant coefficient differential operator on $\Omega \subset \mathbb{C}$ must be a polynomial of the Laplacian Δ.

Let $f \in C^\infty(\Omega)$, $M_f : C^\infty(\Omega) \to C^\infty(\Omega)$ defined by

$$(M_f g)(z) = f(z) g(z), \quad \forall g \in C^\infty(\Omega),$$

is the usual multiplication operator with symbol f. Now we denote $\Delta_f = M_f \circ \Delta$ the composition of multiplication operator and Laplace operator, and Δ_f^n is the composition of Δ_f with itself n times, which are all linear operators on $C^\infty(\Omega)$ for all $n \in \mathbf{N}$.

Proposition 3.3. *Suppose that f is a radial function in $C^\infty(\Omega)$ (i.e., $f(z) = f(|z|)$). Then for any $g \in C^\infty(\Omega)$ and $n \in \mathbf{N}$,*

$$\left(\Delta_f^n g\right)(0) = \sum_{i=1}^n c_i \left(\Delta^i g\right)(0), \tag{3.1}$$

where c_i's are constants depending only on f and n.

Proof. It is easy to see by induction that

$$\Delta_f^n = \sum_{j,k=1}^n c_{j,k}(z) D^j \bar{D}^k,$$

where $c_{j,k}(z) \in C^\infty(\mathbb{D})$ depending on f and n. Now we define that

$$T = \sum_{j,k=1}^n c_{j,k}(0) D^j \bar{D}^k$$

by freezing the coefficients of Δ_f^n at the origin, then it is a constant coefficient differential operator and $E_0 T = E_0 \Delta_f^n$.

For any U_α defined above, we know that $\Delta U_\alpha = U_\alpha \Delta$, and $M_f U_\alpha = U_\alpha M_f$ since f is radial, then $\Delta_f U_\alpha = U_\alpha \Delta_f$, and $\Delta_f^n U_\alpha = U_\alpha \Delta_f^n$ further. It follows that

$$E_0 T U_\alpha = E_0 \Delta_f^n U_\alpha = E_0 U_\alpha \Delta_f^n = U_\alpha E_0 \Delta_f^n = U_\alpha E_0 T = E_0 U_\alpha T,$$

where we have applied the simple fact $E_0 U_\alpha = U_\alpha E_0 = E_0$. Thus our assertion follows immediately from Lemma 3.1. □

Remark 3.4.

(i) Our assumption that f is radial is also necessary for $\Delta_f U_\alpha = U_\alpha \Delta_f$.
(ii) Our proof of Proposition 3.3 makes use of the circularity of domain Ω. For general domain containing the origin in $\mathbb{C}$, we can show by direct calculations that (3.1) still holds, provided that we put some additional restrictions on f, for instance, assuming that $D^k \bar{D}^j f(0) = 0$ for $(k, j) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \setminus \{(i, i)\colon\ i = 0, 1, 2\}$.

Corollary 3.5. *For the invariant Laplace–Beltrami operator $\tilde{\Delta} = 4(1 - |z|^2)^2 \Delta$ on the unit disk $\mathbb{D}$, and for any $g \in C^\infty(\mathbb{D})$ and $n \in \mathbf{N}$,*

$$\big(\tilde{\Delta}^n g\big)(0) = \sum_{i=1}^{n} c_i \big(\Delta^i g\big)(0), \tag{3.2}$$

where c_i's are constants depending only on $f(z) = 4(1 - |z|^2)^2$ and n.

Proof. It follows from Proposition 3.3 if we write $\tilde{\Delta} = \Delta_f = M_f \circ \Delta$ with $f(z) = 4(1 - |z|^2)^2$. □

Now we are ready to prove the main result of this section.

Theorem 3.6. *For any $n \in \mathbf{N}$, $\tilde{\Delta}^n \tilde{X} \in L^\infty(\mathbb{D})$ for any $X \in \mathcal{B}(A^2(\mathbb{D}))$. Moreover*

$$\big\| \tilde{\Delta}^n \tilde{X} \big\|_\infty \leqslant m \|X\|$$

for some constant m depending only on $\tilde{\Delta}$, n and $\mathbb{D}$.

Proof. We know that for $A^2(\mathbb{D})$, $K(w, z) = \frac{1}{(1 - w\bar{z})^2}$, so $k_z(w) = \frac{1 - |z|^2}{(1 - w\bar{z})^2}$. By binomial expansion formula,

$$k_z(w) = \big(1 - |z|^2\big) \sum_{k=0}^{\infty} (k + 1)(w\bar{z})^k,$$

where the series converges absolutely and uniformly on compact subset of $\mathbb{D}$ in terms of w or z and also converges in $A^2(\mathbb{D})$ for fixed $z \in \mathbb{D}$. By the continuity of inner product $\langle \cdot, \cdot \rangle$ and $X \in \mathcal{B}(A^2(\mathbb{D}))$, we have

$$\begin{aligned} \tilde{X}(z) &= \langle X k_z, k_z \rangle \\ &= \sum_{j,k=0}^{\infty} (j + 1)(k + 1)\big(1 - |z|^2\big)^2 z^j \bar{z}^k \big\langle X w^k, w^j \big\rangle \end{aligned}$$

$$= \sum_{j,k=0}^{\infty} (j+1)(k+1)\big(z^{j+2}\bar{z}^{k+2} + z^j\bar{z}^k - 2z^{j+1}\bar{z}^{k+1}\big)\big\langle Xw^k, w^j\big\rangle. \tag{3.3}$$

Let $g_{j,k}(z,\bar{z}) = z^{j+2}\bar{z}^{k+2} + z^j\bar{z}^k - 2z^{j+1}\bar{z}^{k+1}$, which is a polynomial in z and $\bar{z}$, then $(\Delta^n g_{j,k})(0) = (D^n\bar{D}^n g_{j,k})(0) = 0$ if $(j,k) \notin \{(n-2,n-2),(n-1,n-1),(n,n)\}$ for $n \geqslant 2$. Noting this fact, we differentiate the right side term by term in (3.3), which is legal by the uniform convergence on compact subsets of $\mathbb{D}$ of the above series, and evaluate at 0, then

$$\begin{aligned}\big(\Delta^n\tilde{X}\big)(0) &= (n-1)^2(n!)^2\big\langle Xw^{n-2}, w^{n-2}\big\rangle + (n+1)^2(n!)^2\big\langle Xw^n, w^n\big\rangle \\ &\quad - 2n^2(n!)^2\big\langle Xw^{n-1}, w^{n-1}\big\rangle.\end{aligned}$$

Using the fact that for $i \geqslant 1$, $\|w^i\|^2_{A^2(\mathbb{D})} = \int_{\mathbb{D}} |w|^{2i}\,d\upsilon(w) = \frac{1}{i+1}$, it is easy to see by a direct calculation that

$$\big|\big(\Delta^n\tilde{X}\big)(0)\big| \leqslant 4n(n!)^2\|X\|, \tag{3.4}$$

which also holds for $n = 1$.

Now $\forall a \in \mathbb{D}$, let $X_a = U_a^* X U_a$ as in (2.4), then $(\tilde{\Delta}\tilde{X}_a)(z) = \tilde{\Delta}(\tilde{X}\circ\varphi_a)(z) = (\tilde{\Delta}\tilde{X})\circ\varphi_a(z)$ by (2.3) and the Möbius invariant property of $\tilde{\Delta}$. Furthermore, $\tilde{\Delta}^n(\tilde{X}_a)(z) = (\tilde{\Delta}^n\tilde{X})(\varphi_a(z))$. Thus

$$\big|\tilde{\Delta}^n(\tilde{X}_a)(0)\big| = \big|\big(\tilde{\Delta}^n\tilde{X}\big)(a)\big|. \tag{3.5}$$

Now applying Corollary 3.5 to $\tilde{X}_a$, and using (3.4), (3.5) and the fact $\|X\| = \|X_a\|$, we know there exists a constant m, which depends only on $\tilde{\Delta}$, n and $\mathbb{D}$, such that for any $a \in \mathbb{D}$,

$$\big|\big(\tilde{\Delta}^n\tilde{X}\big)(a)\big| = \big|\big(\tilde{\Delta}^n\tilde{X}_a\big)(0)\big| \leqslant m\|X\|,$$

which completes the proof. $\square$

4. The unit ball case of $\mathbb{C}^n$ for $n > 1$

We will imitate what we have done in Section 3 to deal with the unit ball case in this part, where the manipulation of multi-indices is necessary. For that purpose, let $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ where $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1,\dots,\alpha_n)$ with α_i's non-negative integers. Denote $\alpha! = \alpha_1!\cdots\alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, then $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n = \sum_{|\alpha|=1} z^\alpha\bar{w}^\alpha$. We denote $D_i = \frac{\partial}{\partial z_i}$, $\bar{D}_i = \frac{\partial}{\partial \bar{z}_i}$ for $i = 1,2,\dots,n$, and $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $\bar{D}^\alpha = \bar{D}_1^{\alpha_1}\cdots\bar{D}_n^{\alpha_n}$. For $k = (k_1,\dots,k_n)$, $\binom{k}{\alpha} = \binom{k_1}{\alpha_1}\cdots\binom{k_n}{\alpha_n}$.

The *Laplace–Beltrami* operator $\tilde{\Delta}$ defined on $C^2(\mathbb{B})$ is

$$\tilde{\Delta} = \sum_{i,j=1}^{n} 4\big(1-|z|^2\big)(\delta_{i,j} - z_i\bar{z}_j)D_i\bar{D}_j,$$

where $\delta_{i,j} = 1$ if $i = j$ and is 0 otherwise. Let $f_{i,j}(z) = 4(1-|z|^2)(\delta_{i,j} - z_i\bar{z}_j)$, then $f_{i,j}$ is a polynomial in z and $\bar{z}$ of degree 4. If we define $\gamma_i = (0,\dots,1,\dots,0)$ with 1 on the ith slot for $i = 1,\dots,n$, then with above notations we can rewrite

$$\tilde{\Delta} = \sum_{i,j=1}^{n} f_{i,j}D^{\gamma_i}\bar{D}^{\gamma_j}.$$

Moreover we denote $\Delta = \sum_{i=1}^{n} D_i \bar{D}_i$, which is exactly the usual *Laplace* operator.

Similar to the case of $n = 1$, let E_0 be the evaluation functional at the origin and $U(n)$ be the group of unitary transformations on $\mathbb{C}^n$. For any $U \in U(n)$, we define $Uf(z) = f(Uz)$ for $f \in C^\infty(\mathbb{C}^n)$.

Lemma 4.1. *Let $T = \sum_{\alpha\in I}\sum_{\beta\in J} c_{\alpha,\beta} D^\alpha \bar{D}^\beta$ be a constant coefficient differential operator on $C^\infty(\mathbb{C}^n)$. If $E_0TU = E_0UT$ on $C^\infty(\mathbb{C}^n)$ for any $U \in U(n)$, then*

$$T = \sum_{\alpha\in I\cap J} c_\alpha D^\alpha \bar{D}^\alpha,$$

where $c_\alpha = c_{\alpha,\beta}$ for $\alpha = \beta \in I \cap J$.

Proof. Taking $g(z) = z^\alpha \bar{z}^\beta$ for $\alpha \in I$, $\beta \in J$, and $U \in U(n)$ such that $U(z_1, \dots, z_n) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$ where $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, we know by a direct calculation that

$$E_0TU(g) = \left(\prod_{j=1}^{n} e^{i\theta_j(\alpha_j-\beta_j)}\right) c_{\alpha,\beta}\alpha!\beta!$$

and

$$E_0UT(g) = c_{\alpha,\beta}\alpha!\beta!.$$

Our assumption yields that

$$c_{\alpha,\beta} = 0 \quad \text{for } \alpha \neq \beta,$$

then the desired result follows from it. $\square$

Remark 4.2. Note that the unitary group $U(n)$ could be replaced by its subgroup circle group in the above lemma.

The prior lemma is intimately related to the fact that a constant–coefficient differential operator on $\mathbb{C}^n$ invariant under unitary rotations must be a polynomial in Δ, that is, for $T = \sum_{\alpha,\beta} c_{\alpha,\beta} D^\alpha \bar{D}^\beta$ a constant coefficient differential operator, if $TU = UT$ for any $U \in U(n)$, then $T = \sum_i a_i \Delta^i$.

In fact, let $f(z,\bar{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$ be the polynomial corresponding to T, and $w^t = Uz^t$ where t means transpose, then $D_w = (D_{w_1}, \dots, D_{w_n})^t = UD_z = U(D_{z_1}, \dots, D_{z_n})^t$. So the rotation invariance of T ($TU = UT$) is equivalent to $f(UD_z, \bar{U}\bar{D}_z) = f(D_z, \bar{D}_z)$ for any $U \in U(n)$. Furthermore, it reduces to the fact that the polynomial $h(z,\bar{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$, satisfying $h(Uz, \bar{U}\bar{z}) = h(z,\bar{z})$ for any $U \in U(n)$, must be of the form $h(z,\bar{z}) = \sum_i a_i \langle z, z\rangle^i$, which could be proved by using the homogeneous expansion of h and the invariance of homogeneous polynomial under linear transformations after we apply the circle group first, similar to the proof of Lemma 4.1, to conclude that $h(z,\bar{z}) = \sum_\alpha c_\alpha z^\alpha \bar{z}^\alpha$. Thus we give a sketch of proof of the fact mentioned in the preceding paragraph so far, while the formal proof of it might have appeared elsewhere already.

Now we come back to consider the particular *Laplace–Beltrami* operator on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$.

Proposition 4.3. *For any $m \in \mathbf{N}$ and $g \in C^\infty(\mathbb{B})$,*

$$\big(\tilde{\Delta}^m g\big)(0) = \sum_{|\gamma|=1}^{m} a_\gamma D^\gamma \bar{D}^\gamma g(0),$$

where a_γ's are constants depending on $\tilde{\Delta}$ and m.

Proof. It is easy to see that

$$\tilde{\Delta}^m = \sum_{|\alpha|=1}^{m} \sum_{|\beta|=1}^{m} c_{\alpha,\beta}(z) D^\alpha \bar{D}^\beta,$$

where $c_{\alpha,\beta}(z) \in C^\infty(\mathbb{B})$. Define $T = \sum_{|\alpha|=1}^{m} \sum_{|\beta|=1}^{m} c_{\alpha,\beta}(0) D^\alpha \bar{D}^\beta$ by freezing the coefficients of $\tilde{\Delta}^m$ at the origin, then $E_0\tilde{\Delta}^m = E_0 T$. We know that $\tilde{\Delta}U = U\tilde{\Delta}$ for any $U \in U(n)$ by the Möbius invariance of $\tilde{\Delta}$. Thus

$$E_0 TU = E_0\tilde{\Delta}^m U = E_0 U\tilde{\Delta}^m = E_0 T = E_0 UT.$$

So the assertion follows from the application of Lemma 4.1 to T. □

Theorem 4.4. *For any $p \in \mathbf{N}$, $\tilde{\Delta}^p \tilde{X} \in L^\infty(\mathbb{B})$ for any $X \in \mathcal{B}(A^2(\mathbb{B}))$. Moreover*

$$\big\|\tilde{\Delta}^p \tilde{X}\big\|_\infty \leqslant l\|X\|$$

for some constant l depending only on $\tilde{\Delta}$, p and $\mathbb{B}$.

Proof. The normalized Bergman kernel function $k_z(w)$ on $\mathbb{B}$ can be written as

$$k_z(w) = \big(1 - |z|^2\big)^{\frac{n+1}{2}} \sum_{|\alpha|\geqslant 0} \frac{(|\alpha| + n)!}{n!\alpha!} w^\alpha \bar{z}^\alpha.$$

And

$$\big(1 - |z|^2\big)^{n+1} = \sum_{m=0}^{n+1} \sum_{|\gamma|=m} (-1)^m \binom{n+1}{m} \frac{m!}{\gamma!} z^\gamma \bar{z}^\gamma.$$

Thus for any $X \in \mathcal{B}(A^2(\mathbb{B}))$,

$$\begin{aligned}
\tilde{X}(z) &= \langle Xk_z, k_z\rangle \\
&= \sum_{|\alpha|\geqslant 0} \sum_{|\beta|\geqslant 0} \frac{(|\alpha| + n)!(|\beta| + n)!}{(n!)^2\alpha!\beta!} \big(1 - |z|^2\big)^{n+1} \bar{z}^\alpha z^\beta \big\langle Xw^\alpha, w^\beta\big\rangle \\
&= \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} \frac{m!}{(n!)^2} \sum_{|\alpha|\geqslant 0} \sum_{|\beta|\geqslant 0} \sum_{|\gamma|=m} \frac{(|\alpha| + n)!(|\beta| + n)!}{\alpha!\beta!\gamma!} \\
&\quad \times \big\langle Xw^\alpha, w^\beta\big\rangle \bar{z}^{\alpha+\gamma} z^{\beta+\gamma}.
\end{aligned}$$

Then

$$\begin{aligned}
&\big[\big(D^k\bar{D}^k\big)\tilde{X}\big](z)\\
&\quad=\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}\frac{m!}{(n!)^2}\sum_{|\alpha|\geqslant 0}\sum_{|\beta|\geqslant 0}\sum_{|\gamma|=m}\frac{(|\alpha|+n)!(|\beta|+n)!}{\alpha!\beta!\gamma!}\\
&\qquad\times\big\langle Xw^\alpha, w^\beta\big\rangle D^k\bar{D}^k\big(\bar{z}^{\alpha+\gamma}z^{\beta+\gamma}\big)\\
&\quad=\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}\frac{m!}{(n!)^2}\sum_{|\alpha|\geqslant 0}\sum_{|\beta|\geqslant 0}\sum_{|\gamma|=m}\frac{(|\alpha|+n)!(|\beta|+n)!}{\alpha!\beta!\gamma!}\\
&\qquad\times\big\langle Xw^\alpha, w^\beta\big\rangle\frac{(\beta+\gamma)!(\alpha+\gamma)!}{(\beta+\gamma-k)!(\alpha+\gamma-k)!}\bar{z}^{\alpha+\gamma-k}z^{\beta+\gamma-k}.
\end{aligned}$$

Evaluating both sides of the above equality at 0, then we have

$$\begin{aligned}
&\big[\big(D^k\bar{D}^k\big)\tilde{X}\big](0)\\
&\quad=\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}\frac{m!}{(n!)^2}\sum_{|\gamma|=m}\frac{((|k-\gamma|+n)!)^2(k!)^2}{((k-\gamma)!)^2\gamma!}\big\langle Xw^{k-\gamma}, w^{k-\gamma}\big\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big|\big(D^k\bar{D}^k\big)\tilde{X}(0)\big|\\
&\quad\leqslant\sum_{m=0}^{n+1}\binom{n+1}{m}\frac{m!}{(n!)^2}\sum_{|\gamma|=m}\frac{((|k-\gamma|+n)!)^2(k!)^2}{((k-\gamma)!)^2\gamma!}\big|\big\langle Xw^{k-\gamma}, w^{k-\gamma}\big\rangle\big|.
\end{aligned}$$

Since

$$\big|\big\langle Xw^{k-\gamma}, w^{k-\gamma}\big\rangle\big|\leqslant\|X\|\big\|w^{k-\gamma}\big\|^2_{A^2(\mathbb{B})}=\frac{n!(k-\gamma)!}{(n+|k-\gamma|)!}\|X\|,$$

we have

$$\big|\big(D^k\bar{D}^k\big)\tilde{X}(0)\big|\leqslant\left(\sum_{m=0}^{n+1}\sum_{|\gamma|=m}\frac{(n+1)(|k-\gamma|+n)!(k!)^2}{(n+1-m)!(k-\gamma)!\gamma!}\right)\|X\|. \tag{4.1}$$

For any $c\in\mathbb{B}$, $g=\varphi_c\in\mathcal{G}$, let $X_c=U_{\varphi_c}^{-1}XU_{\varphi_c}$, then X_c and X are unitarily equivalent with $\|X_c\|=\|X\|$, and $\tilde{X}_c(z)=(\tilde{X}\circ\varphi_c)(z)$ as shown in Section 2. So

$$\big(\tilde{\Delta}^p\tilde{X}_c\big)(z)=\big(\tilde{\Delta}^p\tilde{X}\big)\circ\varphi_c(z) \tag{4.2}$$

by the Möbius invariance of $\tilde{\Delta}$. Therefore by Proposition 4.3, and (4.1) and (4.2), there is a constant $l>0$ such that

$$\big|\big(\tilde{\Delta}^p\tilde{X}\big)(c)\big|=\big|\big(\tilde{\Delta}^p\tilde{X}_c\big)(0)\big|\leqslant l\|X_c\|=l\|X\|,$$

where constant l depends only on $\tilde{\Delta}$, p and $\mathbb{B}$. $\square$

5. Invariant differential operators

For $\Omega = \mathbb{D}$ or $\mathbb{B}$, a differential operator T on $C^\infty(\Omega)$ is said to be invariant if

$$T(f \circ g) = (Tf) \circ g, \quad \forall g \in \mathcal{G}.$$

Let $\mathcal{A}$ be the algebra of invariant differential operators on $C^\infty(\Omega)$, then we know from [7] that for any operator $T \in \mathcal{A}$, $T = \sum_{i=0}^{n} c_i \tilde{\Delta}^i$ where c_i's are constants. That is, *Laplace–Beltrami* operator $\tilde{\Delta}$ is the generator of $\mathcal{A}$, which force the definitions of weakly harmonic function and strongly harmonic function on them to coincide. So we have following result based on this fact and Theorems 3.6 and 4.4.

Theorem 5.1. *For any $T \in \mathcal{A}$, $\|T\tilde{X}\|_\infty \leqslant l\|X\|$ for any $X \in \mathcal{B}(A^2(\Omega))$, where the constant l depends only on T and Ω.*

It is known that the range of Berezin transform belongs to the subalgebra $L^\infty(\Omega) \cap C^\infty(\Omega)$ of $L^\infty(\Omega)$. Our main results show that if for any $T \in \mathcal{A}$ we define $\bar{T} = T \circ Ber : \mathcal{B}(A^2(\Omega)) \to L^\infty(\Omega)$, then $\bar{T}$ is bounded.

Clearly our main results strongly rely on the Möbius invariance of range of Berezin transform, which has been established for much more general domains, bounded symmetric domains, not unit disk or unit ball merely. So it is quite natural to ask how to extend our results to the general case.

As pointed out in the Introduction, unit disk and unit ball are both particularly bounded symmetric domains of rank 1, which make the structure of $\mathcal{A}$ the algebra of invariant differential operators much simpler than others of rank more than 1. Generally, it is known [7] that for any non-compact bounded symmetric domain of rank $r \geqslant 1$, the algebra of invariant differential operators is a polynomial algebra in r algebraically independent commuting operators, where one of the generators can be chosen as the *Laplace–Beltrami* operator.

It is pleased to know, when most of our work is done, that Engliš and Zhang in [6] have obtained the similar results for the general bounded symmetric domains. They considered the lift of the Berezin transform and the *strong derivative* of Banach vector-valued function, and proved the commutativity of differential operators and trace operator. However, it is still an interesting question raised by them naturally that whether the similar result also holds in the context of any bounded domain of $\mathbb{C}^n$.

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