J. Math. Anal. Appl. 348 (2008) 366–382

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the $L^p - L^q$ norm of the Hankel transform and related operators

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ARTICLE INFO

Article history:
Received 25 March 2007
Available online 10 July 2008
Submitted by R.H. Torres

Keywords:
Fourier–Bessel transform
Hankel transform
Convolution operator
Bessel functions
Laguerre polynomials
Best constants

ABSTRACT

We investigate the $L^p(0,\infty) - L^q(0,\infty)$ mapping properties of the operators

$$\mathcal{L}^{\alpha}_{\nu,\mu} f(y) = y^{\mu} \int_0^{\infty} (xy)^{\nu} f(x) J_{\alpha}(xy)\, dx, \quad f \in C_0^{\infty}(0,+\infty),$$

for suitable values of the parameters, and we evaluate the operator norm of $\mathcal{L}^{\alpha}_{\nu,\mu}$ in some special and significant cases.

1. Introduction

We consider the class of operators

$$\mathcal{L} = \left\{ \mathcal{L}^{\alpha}_{\nu,\mu} f(y) = y^{\mu} \int_0^{\infty} (xy)^{\nu} f(x) J_{\alpha}(xy)\, dx, \;\; f \in C_0^{\infty}(0,+\infty) \right\}, \tag{1.1}$$

where $J_{\alpha}(r)$ denotes the usual Bessel function of the first kind, $\alpha \geqslant -\frac{1}{2}$, ν and μ are real parameters.

These operators are interesting because they generalize a number of important operators in Analysis. For example, the restriction of the Fourier transform to radial functions and the Hankel transform belong to $\mathcal{L}$ (see the next section).

One of the main results of this paper is the following

Theorem 1.1. *$\mathcal{L}^{\alpha}_{\nu,\mu}$ is bounded from $L^p(0,\infty)$ to $L^q(0,\infty)$ whenever $\alpha \geqslant -\frac{1}{2}$, $1 \leqslant p \leqslant q \leqslant \infty$, and if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \quad \text{and} \quad -\alpha - \frac{1}{p'} < \nu \leqslant \frac{1}{2} - \max\left\{ \frac{1}{p'} - \frac{1}{q}, 0 \right\}. \tag{1.2}$$

To prove this theorem we use an interpolation argument of Stein and Weiss. In Appendix A we provide examples that show that these bounds are best possible.

We are interested in the optimal constants $C = C^{\alpha}_{\nu,p,q}$ for which

$$\frac{\|\mathcal{L}^{\alpha}_{\nu,\mu} f\|_{L^q(0,\infty)}}{\|f\|_{L^p(0,\infty)}} \leqslant C.$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.06.053

In Section 2 we will show that this problem can be solved for special values of the parameters with the aid of a celebrated theorem of W. Beckner [2], but the proof seems to be very difficult in the general case.

In this paper we evaluate $C^{\alpha}_{\nu,p,q}$ for $\nu=\frac{1}{2}$ and $q=p'$. Our main result is the following

Theorem 1.2. *The following inequality holds for every* $1<p\leqslant 2$, $\alpha\geqslant -\frac{1}{2}$ *and* $f\in C_0^{\infty}(0,\infty)$,

$$\frac{\|\mathcal{L}^{\alpha}_{\frac{1}{2},0} f\|_{L^{p'}(0,\infty)}}{\|f\|_{L^p(0,\infty)}}\leqslant 2^{\frac{1}{p}-\frac{1}{2}}\,\frac{p^{\frac{1}{2}(\alpha+\frac{1}{2}+\frac{1}{p})}}{(p')^{\frac{1}{2}(\alpha+\frac{1}{2}+\frac{1}{p'})}}\,\frac{\Gamma((\alpha+\frac{1}{2})\frac{p'}{2}+\frac{1}{2})^{\frac{1}{p'}}}{\Gamma((\alpha+\frac{1}{2})\frac{p}{2}+\frac{1}{2})^{\frac{1}{p}}}. \tag{1.3}$$

The constant on the right-hand side of (1.3) *is best possible and coincides with the* $L^p(0,\infty)\to L^{p'}(0,\infty)$ *norm of the Hankel transform* (*see the next section*).

By continuity, $\mathcal{L}^{\alpha}_{\frac{1}{2},0}$ extends to a bounded operator from $L^p(0,\infty)$ to $L^{p'}(0,\infty)$, and it is possible to prove that the equality in (1.3) is attained by the functions $f_s(x)=x^{\alpha+\frac{1}{2}}e^{-sx^2}$, $s>0$. We will prove this in Section 3.

Theorem 1.2 is interesting because the Hankel transform is a remarkable operator who shares many similarities with the Fourier transform. However, the steps toward the proof of Theorem 1.2 are as interesting as the result itself. We prove the theorem using an extension of Beckner's techniques. We divide the proof into a series of lemmas, some of them crucial, some of them of technical nature. All the steps of the proof, except the last, are valid for every operators in $\mathcal{L}$, and can be applied toward the solution of other problems in Analysis.

In particular, we show in Section 5.2 how the full solution to the best constant problem for the operators in $\mathcal{L}$ is related to the hypercontractivity of the Laguerre semigroup.

2. The Hankel transforms

Our interest in the class $\mathcal{L}$ was originally motivated by the Fourier transform and the Hankel transform.

The Fourier transform $\hat{f}(\zeta)=\int_{\mathbf{R}^n}e^{-i(x_1\zeta_1+\cdots+x_n\zeta_n)}f(x)\,dx$ is well defined when $f\in C_0^{\infty}(\mathbf{R}^n)$, and can be extended to a bounded linear operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ if and only if $1\leqslant p\leqslant 2$ and $q=p'$ (see e.g. [15]). Furthermore,

$$\|\hat{f}\|_{L^{p'}(\mathbf{R}^n)}\leqslant (2\pi)^{\frac{n}{p'}}\big(p^{\frac{1}{p}}(p')^{-\frac{1}{p'}}\big)^{\frac{n}{2}}\|f\|_{L^p(\mathbf{R}^n)},\quad f\in L^p\big(\mathbf{R}^n\big). \tag{2.1}$$

The constant on the right-hand side of (2.1) is best possible, as W. Beckner proved in a celebrated paper [2].

The Gaussian functions $f_s(x)=e^{-s(x_1^2+\cdots+x_n^2)}$, with $s>0$, attain the equality in (2.1). E.H. Lieb proved in [11] that the f_s are the only function for which the equality is attained. Since the Fourier transform of a radial function is radial, we can state the following important observation: *The Fourier transform has the same* $L^p(\mathbf{R}^n)\to L^{p'}(\mathbf{R}^n)$ *norm of its restriction to the radial functions of* $L^p(\mathbf{R}^n)$.

The restriction of the Fourier transform to the space of radial functions can be rewritten as a constant multiple of an operator of the class $\mathcal{L}$. In fact the Fourier transform of $f(|x|)$ is

$$\begin{aligned}\hat{f}\big(|\zeta|\big)&=(2\pi)^{\frac{n}{2}}|\zeta|^{-\frac{n}{2}+1}\int_0^{+\infty}f(r)r^{\frac{n}{2}}J_{\frac{n}{2}-1}\big(r|\zeta|\big)\,dr=(2\pi)^{\frac{n}{2}}|\zeta|^{-n+1}\int_0^{+\infty}f(r)\big(|\zeta|r\big)^{\frac{n}{2}}J_{\frac{n}{2}-1}\big(r|\zeta|\big)\,dr\\&=(2\pi)^{\frac{n}{2}}\mathcal{L}^{\frac{n}{2}-1}_{\frac{n}{2},1-n}f\big(|\zeta|\big).\end{aligned} \tag{2.2}$$

Following [3], we will refer to $\mathcal{L}^{\alpha}_{\alpha+1,-2\alpha-1}$, $\alpha>-1$, as to the *Fourier–Bessel transform* of order α, even though this operator, which H. Hankel introduced in 1875 (see [8]), is sometimes referred to as Hankel transform in the literature. We let

$$\widetilde{\mathcal{H}}_{\alpha}f(x)=\mathcal{L}^{\alpha}_{\alpha+1,-2\alpha-1}f(x)=\int_0^{+\infty}f(t)(xt)^{-\alpha}J_{\alpha}(xt)t^{2\alpha+1}\,dt. \tag{2.3}$$

From (2.2) it follows that

$$\hat{f}\big(|\zeta|\big)=(2\pi)^{\frac{n}{2}}\widetilde{\mathcal{H}}_{\frac{n}{2}-1}f\big(|\zeta|\big),\quad f\in C_0^{\infty}(0,\infty),\ \zeta\in\mathbf{R}^n.$$

The Fourier–Bessel transform of order α shares a lot of properties with the Fourier transform. H. Hankel proved the following inversion formula

$$\widetilde{\mathcal{H}}_{\alpha}(\widetilde{\mathcal{H}}_{\alpha}f)(x)=f(x),\quad f\in C_0^{\infty}(0,+\infty). \tag{2.4}$$

A short and elegant proof of (2.4) is in [3]. It is easy to prove that the Fourier–Bessel transform extends to an isometry on $L^2((0,\infty), x^{2\alpha+1}\,dx)$. Moreover, $|\widetilde{\mathcal{H}}_\alpha f(x)| \leqslant b_\alpha \|f\|_{L^1((0,\infty),t^{2\alpha+1}\,dt)}$, where $b_\alpha = \sup_{t\in(0,\infty)} |t^{-\alpha} J_\alpha(t)|$. By Riesz interpolation theorem, the Fourier–Bessel transform extends to a bounded linear operator from $L^p((0,\infty), t^{2\alpha+1}\,dt)$ to $L^{p'}((0,\infty), t^{2\alpha+1}\,dt)$, and

$$\frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}((0,\infty),t^{2\alpha+1}\,dt)}}{\|f\|_{L^p((0,\infty),t^{2\alpha+1}\,dt)}} \leqslant b_\alpha^{1-\frac{2}{p'}}. \tag{2.5}$$

Note that the $L^p((0,\infty), t^{2\alpha+1}\,dt) - L^{p'}((0,\infty), t^{2\alpha+1}\,dt)$ norm of $\widetilde{\mathcal{H}}_\alpha$ is the same as the $L^p(0,\infty) - L^{p'}(0,\infty)$ norm of $\mathcal{L}^\alpha_{\frac{2\alpha+1}{p'}-\alpha,0}$. Indeed, if we let $F(t) = t^{\frac{2\alpha+1}{p}} f(t)$, and we observe that $F(t) \in L^p(0,\infty)$ if and only if $f(t) \in L^p((0,\infty), t^{2\alpha+1}\,dt)$, using (2.3) we can see that

$$\frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}((0,\infty),t^{2\alpha+1}\,dt)}}{\|f\|_{L^p((0,\infty),t^{2\alpha+1}\,dt)}} = \frac{\|\mathcal{L}^\alpha_{\frac{2\alpha+1}{p'}-\alpha,0} F\|_{L^{p'}(0,\infty)}}{\|F\|_{L^p(0,\infty)}}. \tag{2.6}$$

$\mathcal{L}^\alpha_{\frac{1}{2},0}$ is the so-called *Hankel transform of order α*. This is a well-studied operator with remarkable properties. We will let

$$\mathcal{H}_\alpha f(x) = \mathcal{L}^\alpha_{\frac{1}{2},0} f(x) = \int_0^{+\infty} f(t)(xt)^{\frac{1}{2}} J_\alpha(xt)\,dt. \tag{2.7}$$

The Hankel transform shares many properties with the Fourier transform as well. The following inversion formula for the Hankel transform is proved e.g. in [4]

$$\mathcal{H}_\alpha(\mathcal{H}_\alpha f)(x) = f(x), \quad f \in C_0^\infty(0,+\infty). \tag{2.8}$$

From (2.8) it follows that the Hankel transform extends to an isometry on $L^2(0,\infty)$. Moreover, $|\mathcal{H}_\alpha f(x)| \leqslant c_\alpha \|f\|_{L^1(0,\infty)}$, where $c_\alpha = \sup_{t\in(0,+\infty)} |t^{\frac{1}{2}} J_\alpha(t)|$.

By the M. Riesz convexity theorem, the Hankel transform extends to a bounded linear operator from $L^p(0,\infty)$ to $L^{p'}(0,\infty)$ for every $1 \leqslant p \leqslant 2$, and

$$\frac{\|\mathcal{H}_\alpha f\|_{L^{p'}(0,\infty)}}{\|f\|_{L^p(0,\infty)}} \leqslant c_\alpha^{1-\frac{2}{p'}}. \tag{2.9}$$

In this paper we evaluate the $L^p(0,\infty) \to L^{p'}(0,\infty)$ norm of this operator (see Theorem 1.2). Unfortunately the techniques that we used to compute this norm cannot be used to compute the $L^p((0,\infty), t^{2\alpha+1}\,dt) - L^{p'}((0,\infty), t^{2\alpha+1}\,dt)$ norm of the Fourier–Bessel transform for general α's. When $\alpha = \frac{n}{2} - 1$, where n is a positive integer, the norm of the Fourier–Bessel transform can be computed with the aid of the theorems of Beckner and Lieb.

The following proposition is proved in Appendix B.

Proposition 2.1. *The following inequality holds for every $1 < p \leqslant 2$, $n \geqslant 1$ and $f \in L^p(0,\infty)$,*

$$\frac{\|\mathcal{L}^{\frac{n}{2}-1}_{1-\frac{n}{2}+\frac{n-1}{p'},0} f\|_{L^{p'}(0,\infty)}}{\|f\|_{L^p(0,\infty)}} \leqslant \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{p^{\frac{n}{2p}}}{(p')^{\frac{n}{2p'}}} 2^{(\frac{n}{2}-1)(\frac{1}{p'}-\frac{1}{p})}. \tag{2.10}$$

The constant in (2.10) is best possible, and equals the $L^p((0,\infty), t^{2\alpha+1}\,dt) - L^{p'}((0,\infty), t^{2\alpha+1}\,dt)$ operator norm of $\widetilde{\mathcal{H}}_{\frac{n}{2}-1}$. The equality in (2.10) is attained by the functions $f_s(x) = e^{-sx^2}$, $s > 0$. These are the only functions for which the maximum in (2.10) is attained.

3. Open problems and conjectures

The problem of evaluating the $L^p - L^q$ operator norm of $\mathcal{L}^\alpha_{\nu,\mu}$ for general values of the parameters seems to be very difficult. At the moment we can only evaluate it when $q = p'$, $\alpha \geqslant -\frac{1}{2}$ and $\nu = \frac{1}{2}$ (Theorem 1.2), and when $q = p'$, $\alpha = \frac{n}{2} - 1$ and $\nu = 1 - \frac{n}{2} + \frac{n-1}{p'}$ (Proposition 2.1).

The theorem of E.H. Lieb implies that the Gaussian functions are the only maximizers for the operator norm of the Fourier–Bessel transform when $\alpha = \frac{n}{2} - 1$. We conjecture that this is true in general, that is, that the Gaussian functions are the only maximizers of the ratio $\frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}((0,\infty),r^{n-1}\,dr)}}{\|\tilde{f}\|_{L^p((0,\infty),r^{n-1}\,dr)}}$ for every $\alpha \geqslant -\frac{1}{2}$. If that is the case, then the $L^p((0,\infty), r^{2\alpha+1}\,dr) - L^{p'}((0,\infty), r^{2\alpha+1}\,dr)$ norm of the Fourier–Bessel transform is

$$\sup \frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}((0,\infty),r^{2\alpha+1}\,dr)}}{\|\tilde{f}\|_{L^p((0,\infty),r^{2\alpha+1}\,dr)}} = \sup \frac{\|\mathcal{L}^\alpha_{\frac{2\alpha+1}{p'}-\alpha,0}(F)\|_{L^{p'}(0,\infty)}}{\|F\|_{L^p(0,\infty)}} = 2^{\alpha(\frac{1}{p'}-\frac{1}{p})} \frac{p^{\frac{\alpha+1}{p}}}{(p')^{\frac{\alpha+1}{p'}}} \Gamma(\alpha+1)^{\frac{1}{p'}-\frac{1}{p'}}.$$

More in general, we conjecture the following:

Conjecture 1. *The $L^p \to L^q$ norm of $\mathcal{L}^\alpha_{\nu,\mu}$ is finite if $\alpha \geqslant -\frac{1}{2}$, $1 \leqslant p \leqslant q \leqslant \infty$, and if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \quad \text{and} \quad -\alpha - \frac{1}{p'} < \nu \leqslant \frac{1}{2} - \max\left\{\frac{1}{p'} - \frac{1}{q}, 0\right\}. \tag{3.1}$$

For these values of the parameters,

$$\sup \frac{\|\mathcal{L}^\alpha_{\nu,\mu} f\|_{L^q(0,\infty)}}{\|f\|_{L^p(0,\infty)}} = C^\alpha_{\nu,p,q}, \tag{3.2}$$

where we have let

$$C^\alpha_{\nu,p,q} = 2^{\nu-\frac{1}{q}} \frac{p^{\frac{1}{2}(1+\alpha-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\alpha+\nu+\frac{1}{p'})}} \frac{\Gamma(\frac{\alpha+\nu+\mu}{2}q+\frac{1}{2})^{\frac{1}{q}}}{\Gamma(\frac{1+\alpha-\nu}{2}p+\frac{1}{2})^{\frac{1}{p}}}. \tag{3.3}$$

Conjecture 2. *The functions $f_s(x) = e^{-sx^2}x^{1-\nu+\alpha}$, $s > 0$, are the only maximizers of the ratio in* (3.2).

Both conjectures hold true in the special and significant cases of Proposition 2.1, and Theorem 1.2 validates the first conjecture. In Appendix A we prove that the bounds for μ and ν are optimal.

Proving that $\frac{\|L^\alpha_{\nu,\mu} f\|_{L^q(0,\infty)}}{\|f\|_{L^p(0,\infty)}} \geqslant C^\alpha_{\nu,p,q}$ is easy; the functions $f_s(x) = e^{-sx^2}x^{1-\nu+\alpha}$ are in $L^p(0,\infty)$ because, by (3.1), $1-\nu+\alpha \geqslant \frac{1}{2}+\alpha \geqslant 0 > -\frac{1}{p}$. A change of variables shows that the ratio $\frac{\|\mathcal{L}^\alpha_{\nu,\mu}(f_s)\|_{L^q(0,\infty)}}{\|f_s\|_{L^p(0,\infty)}}$ is independent of s. When $s=\frac{1}{2}$, $\mathcal{L}^\alpha_{\nu,\mu}(f_{\frac{1}{2}})(y) = y^\mu \int_0^\infty e^{-\frac{x^2}{2}} J_\alpha(xy)x^{\alpha+1}\,dx$ can be explicitly computed (see e.g. [4, p. 29, no. 10]), and is $y^{\alpha+\nu+\mu}e^{-\frac{y^2}{2}}$. Thus, by the well-known identity

$$\int_0^\infty e^{-sx^2}x^m\,dx = \frac{s^{-\frac{1+m}{2}}}{2}\Gamma\left(\frac{1+m}{2}\right), \quad m > -1, \tag{3.4}$$

it follows that

$$\frac{\|\mathcal{L}^\alpha_{\nu,\mu}(f_s)\|_{L^q(0,\infty)}}{\|f_s\|_{L^p(0,\infty)}} = \frac{\|y^{\nu+\alpha+\mu}e^{-\frac{y^2}{2}}\|_{L^q(0,\infty)}}{\|x^{1-\nu+\alpha}e^{-\frac{x^2}{2}}\|_{L^p(0,\infty)}} = \frac{(\int_0^\infty y^{(\nu+\alpha+\mu)q}e^{-\frac{qy^2}{2}})^{\frac{1}{q}}}{(\int_0^\infty x^{(1-\nu+\alpha)p}e^{-\frac{px^2}{2}})^{\frac{1}{p}}}$$
$$= 2^{\nu-\frac{1}{q}} \frac{p^{\frac{1}{2}(1+\alpha-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\alpha+\nu+\frac{1}{p'})}} \frac{\Gamma(\frac{\alpha+\nu+\mu}{2}q+\frac{1}{2})^{\frac{1}{q}}}{\Gamma(\frac{1+\alpha-\nu}{2}p+\frac{1}{2})^{\frac{1}{p}}} = C^\alpha_{\nu,p,q}.$$

4. Proof of Theorem 1.1

From now we will use $\| \ \|_p$ instead of $\| \ \|_{L^p(0,\infty)}$. In this section, C is a constant that can change from line to line.

The proof of Theorem 1.1 relies on a theorem on interpolation of operators with change of measure proved by E.M. Stein and G. Weiss in [12].

We let $M = N = (0,\infty)$. Let $i = 0, 1$, and let $\beta_i = y^{m_i}dy$ be measures on $\mathcal{N}$, and $\alpha_i = x^{n_i}dx$ be measures on $\mathcal{M}$.

We let $\beta = \beta_0 + \beta_1$, $\alpha = \alpha_0 + \alpha_1$. With this position, for any measurable E, $F \subset (0,\infty)$,

$$\beta_i(E) = \int_E h_i(y)\,d\beta(y), \qquad \alpha_i(F) = \int_F k_i(x)\,d\alpha(x),$$

where $h_i = \frac{y^{m_i}}{y^{m_1}+y^{m_0}}$ and $k_i = \frac{x^{n_i}}{x^{n_1}+x^{n_0}}$.

For every r, $s \in [0,1]$ we can define the following measures on $\mathcal{M}$ and $\mathcal{N}$,

$$\beta_s(E) = \int_E h_1^s h_0^{1-s}(x)\,d\beta = \int_E y^{sm_1+(1-s)m_0}\,dy,$$
$$\alpha_r(F) = \int_F k_1^r k_0^{1-r}(x)\,d\alpha = \int_F x^{rn_1+(1-r)n_0}\,dx.$$

Let $1 \leqslant p_0 \neq p_1 \leqslant \infty$, $1 \leqslant q_0 \neq q_1 \leqslant \infty$. For every $t \in (0,1)$, we consider the exponents q_t and p_t that satisfy the following relations:

$$\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_0}, \qquad \frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_0}.$$

We also let $s(t) = \frac{tq_t}{q_1}$ (so that $1 - s(t) = (1-t)\frac{q_t}{q_0}$) and $r(t) = \frac{tp_t}{p_1}$. With this position

$$d\beta_{s(t)} = y^{q_t(t\frac{m_1}{q_1} + (1-t)\frac{m_0}{q_0})}\,dy \quad \text{and} \quad d\alpha_{r(t)} = x^{p_t(t\frac{n_1}{p_1} + (1-t)\frac{n_0}{p_0})}\,dx.$$

The following theorem is a consequence of the main theorem in [12].

Theorem 4.1. *Let T be a sublinear operator satisfying*

$$\|Tf\|_{L^{q_i}(N,d\beta_i)} \leqslant K_i \|f\|_{L^{p_i}(M,d\alpha_i)}$$

(or $\|Tf\|_{L^{q_i,\infty}(N,d\beta_i)} \leqslant K_i \|f\|_{L^{p_i}(M,d\alpha_i)}$), for every $f \in L^{p_i}(M, d\alpha_i)$ and $i = 0, 1$. Then T is defined also in $L^{p(t)}(M, d\alpha_{r(t)})$ for every $t \in (0,1)$, and

$$\|Tf\|_{L^{q(t)}(N,d\beta_{s(t)})} \leqslant K_t \|f\|_{L^{q(t)}(M,d\alpha_{r(t)})},$$

where K_t is independent of f.

We first prove Theorem 1.1 when $q \leqslant p'$ and $\nu = \frac{1}{2}$. We let

$$TF(y) = \int_0^\infty F(x) J_\alpha(xy)\,dx, \quad F \in C_0^\infty(0,\infty).$$

Then

$$\mathcal{L}^\alpha_{\frac{1}{2},\mu} f(y) = y^\mu \int_0^\infty (xy)^{\frac{1}{2}} f(x) J_\alpha(xy)\,dx = y^{\mu+\frac{1}{2}} \int_0^\infty x^{\frac{1}{2}} f(x) J_\alpha(xy)\,dx = y^{\mu+\frac{1}{2}} T\big(x^{\frac{1}{2}} f\big)(y).$$

If we let

$$d\beta = y^{q(\mu+\frac{1}{2})}\,dy, \qquad d\alpha = x^{-\frac{p}{2}}\,dx, \tag{4.1}$$

and $F(x) = x^{\frac{1}{2}} f(x)$, we can see at once that the inequalities

$$\|TF\|_{L^q((0,\infty),d\beta)} \leqslant C\|F\|_{L^p((0,\infty),d\alpha)}$$

and

$$\|TF\|_{L^{q,\infty}((0,\infty),d\beta)} \leqslant C\|F\|_{L^p((0,\infty),d\alpha)}$$

are equivalent to

$$\big\|\mathcal{L}^\alpha_{\frac{1}{2},\mu} f\big\|_q \leqslant C\|f\|_p$$

and

$$\big\|\mathcal{L}^\alpha_{\frac{1}{2},\mu} f\big\|_{L^{q,\infty}(0,\infty)} \leqslant C\|f\|_p.$$

When $q = p'$ we are in the case of the Hankel transform, so we assume $q < p'$. The point $(\frac{1}{p}, \frac{1}{q})$ is above the duality line $\frac{1}{q} = 1 - \frac{1}{p}$.

We let $(\frac{1}{p_0}, \frac{1}{q_0})$ be the intersection of the duality line and the line that joins the points $(1,1)$ and $(\frac{1}{p}, \frac{1}{q})$.

It is easy to verify that $p_0 = 1 + \frac{q'}{p'}$ (and of course $q_0 = p_0'$). With this position,

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1-t)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + t(1,1)$$

when $t = \frac{1}{q} - \frac{1}{p'}$.

To apply Theorem 4.1 we argue as follows: We prove first that $\mathcal{L}^{\alpha}_{\frac{1}{2},-1}$ is bounded from $L^1(0,\infty)$ to $L^{1,\infty}(0,\infty)$. By the observations above and (4.1) this is equivalent to proving that $\|TF\|_{L^{1,\infty}((0,\infty),d\beta_1)} \leqslant C\|F\|_{L^1((0,\infty),d\alpha_1)}$, where we have let

$$d\beta_1 = y^{-\frac{1}{2}}\,dy \quad \text{and} \quad d\alpha_1 = x^{-\frac{1}{2}}\,dx.$$

We know that $\mathcal{L}^{\alpha}_{\frac{1}{2},0}$ is bounded from $L^{p_0}(0,\infty)$ to $L^{q_0}(0,\infty)$ because it is the Hankel transform, and so, if we let $d\beta_0 = y^{\frac{p_0'}{2}}\,dy$ and $d\alpha_0 = x^{-\frac{p_0}{2}}\,dx$, we gather

$$\|TF\|_{L^{q_0}((0,\infty),d\beta_0)} \leqslant C\|F\|_{L^{p_0}((0,\infty),d\alpha_0)}.$$

Therefore, for every $0 < t < 1$,

$$\|TF\|_{L^{q_t}((0,\infty),d\beta_{s(t)})} \leqslant C\|F\|_{L^{p_t}((0,\infty),d\alpha_{r(t)})}; \tag{4.2}$$

in particular, for $t = \frac{1}{q} - \frac{1}{p'}$ (which is the value of t for which $p_t = p$ and $q_t = q$) and the definitions of $d\beta_{s(t)}$ and $d\alpha_{r(t)}$, we gather

$$d\beta_{s(t)} = y^{q(-\frac{t}{2}+\frac{(1-t)p_0'}{2p_0'})}\,dy = y^{q(\frac{1}{2}+\mu)}\,dy \quad \text{and} \quad d\alpha_{r(t)} = x^{-p(-\frac{t}{2}-\frac{(1-t)p_0}{2p_0})}\,dx = x^{-\frac{p}{2}}\,dx,$$

and (4.2) is equivalent to

$$\|L^{\alpha}_{\frac{1}{2},\mu} f\|_q \leqslant C_t \|f\|_p, \tag{4.3}$$

as required.

Let us prove that $\mathcal{L}^{\alpha}_{\frac{1}{2},-1}$ is bounded from $L^1(0,\infty)$ to $L^{1,\infty}(0,\infty)$. Indeed,

$$\mathcal{L}^{\alpha}_{\frac{1}{2},-1} f(y) = y^{-1}\int_0^\infty (xy)^{\frac{1}{2}} J_\alpha(xy) f(x)\,dx = y^{-1}\mathcal{H}_\alpha f(y),$$

and since

$$\left\|\mathcal{L}^{\alpha}_{\frac{1}{2},-1} f\right\|_{L^{1,\infty}(0,\infty)} = \sup_{t>0} t\left|\left\{y\colon\ y^{-1}\mathcal{H}_\alpha f(y) > t\right\}\right|$$

and we have recalled in Section 1 that $|\mathcal{H}_\alpha f(y)| \leqslant C\|f\|_1$, then

$$\left\|\mathcal{L}^{\alpha}_{\frac{1}{2},-1} f\right\|_{L^{1,\infty}(0,\infty)} \leqslant \sup_{t>0} t\left(t^{-1}C\|f\|_1\right) = C\|f\|_1, \tag{4.4}$$

as required.

To prove the theorem for $q \leqslant p'$ and $-\alpha - \frac{1}{p'} \leqslant \nu < \frac{1}{2}$ we let $\nu = \frac{1}{2} - \epsilon$, with $0 < \epsilon < \frac{1}{p'} + \alpha - \frac{1}{2}$.

We use the identity (A.1) in Appendix A to infer that $(xy)^{\frac{1}{2}-\epsilon} J_\alpha(xy) = \mathcal{H}_{\alpha-\epsilon}\psi_\epsilon(xy)$, where we have let $\psi_\epsilon(t) = 2^{1-\epsilon}\Gamma(\epsilon)^{-1}\chi_{(0,1)}(t)(1-t^2)^{\epsilon-1}t^{\alpha-\epsilon+\frac{1}{2}}$. With this position and the inversion formula for the Hankel transform

$$\mathcal{L}^{\alpha}_{\nu,\mu} f(y) = y^\mu \int_0^\infty f(x)\mathcal{H}_{\alpha-\epsilon}\psi_\epsilon(xy)\,dx = y^\mu \int_0^\infty \psi_\epsilon(z)\mathcal{H}_{\alpha-\epsilon} f(zy)\,dz = \int_0^\infty \psi_\epsilon(z) z^{-\mu}\mathcal{L}^{\alpha-\epsilon}_{\frac{1}{2},\mu} f(zy)\,dz.$$

Thus, by (4.3),

$$\begin{aligned}\left\|L^{\alpha}_{\nu,\mu} f\right\|_q &\leqslant \int_0^\infty z^{-\mu}\psi_\epsilon(z)\left\|L^{\alpha-\epsilon}_{\frac{1}{2},\mu} f(z)\right\|_q dz \leqslant C_t\|f\|_{L^p(0,\infty)}\int_0^\infty z^{-\mu-\frac{1}{q}}\psi_\epsilon(z)\,dz\\ &= 2^{1-\epsilon}\Gamma(\epsilon)^{-1}C_t\|f\|_{L^p(0,\infty)}\int_0^1 \left(1-z^2\right)^{\epsilon-1} z^{\alpha-\epsilon+\frac{1}{2}-\frac{1}{q}-\mu}\,dz.\end{aligned}$$

Recall that $\mu = \frac{1}{p'} - \frac{1}{q}$ and $0 < \epsilon < \frac{1}{p'} + \alpha - \frac{1}{2}$, and so the exponent of z is $\alpha - \epsilon + \frac{1}{2} - \frac{1}{p'} > 1 - \frac{2}{p'} > -1$.

Therefore, the integral is finite, and we have proved Theorem 1.1 also in this case.

To prove the theorem for $q > p'$ and $\nu \leqslant \frac{1}{2} - \frac{1}{p'} + \frac{1}{q}$, we observe that the adjoint of $\mathcal{L}^{\alpha}_{\nu,\mu}$ is $\mathcal{L}^{\alpha}_{\nu+\mu,-\mu}$. Therefore, the $L^p - L^q$ norm of $\mathcal{L}^{\alpha}_{\nu,\mu}$ is finite if and only if the same is true for the $L^{q'} - L^{p'}$ norm of $\mathcal{L}^{\alpha}_{\nu+\mu,-\mu}$.

Since $q = (q')' > p'$, we are in the case that we have already proved: the $L^{q'} - L^{p'}$ norm of $\mathcal{L}^{\alpha}_{\nu+\mu,-\mu}$ is finite if $\nu + \mu \leqslant \frac{1}{2}$, that is, $\nu \leqslant \frac{1}{2} - \frac{1}{p'} + \frac{1}{q}$. This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 is a generalization of Beckner's celebrated proof in [2], and will be performed with a series of steps which are important in their own because of the connection with other problems in Analysis, the hypercontractivity of the Laguerre semigroup being one of the most significant issues.

5.1. Preliminaries

In this section we collect together a few preliminary facts concerning the Laguerre polynomials and we state our main theorem. We refer to [13] or to [14] for details.

For $\alpha > -1$, $x > 0$ and $k = 0, 1, 2, \ldots$, the *Laguerre polynomials of type* α are defined by the formula

$$e^{-x} x^{\alpha} L^{\alpha}_k(x) = \frac{1}{k!} \frac{d^k}{dx^k} \left(e^{-x} x^{k+\alpha}\right). \tag{5.1}$$

Each L^{α}_k is a polynomial of degree k. The Laguerre polynomials satisfy the following orthogonality relations:

$$\int_0^{+\infty} L^{\alpha}_k(x) L^{\alpha}_j(x)\, d\tilde{\mu}_{\alpha}(x) = \begin{cases} 0 & \text{if } k \neq j, \\ \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)} & \text{if } k = j, \end{cases} \tag{5.2}$$

where we have let $\tilde{d}\mu_{\alpha}(x) = \frac{e^{-x} x^{\alpha}}{\Gamma(\alpha+1)} dx$.

A change of variables shows that the orthogonality relation (5.2) can be rewritten as

$$\int_{-\infty}^{+\infty} L^{\alpha}_k\left(\frac{x^2}{2}\right) L^{\alpha}_j\left(\frac{x^2}{2}\right) d\mu_{\alpha}(x) = \begin{cases} 0 & \text{if } k \neq j, \\ \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)} & \text{if } k = j, \end{cases} \tag{5.3}$$

where we have let $d\mu_{\alpha}(x) = \frac{e^{-\frac{x^2}{2}} x^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}$.

We will use the following important identity, often called *the Hille–Hardy identity*, which is valid for real or complex ω's such that $|\omega| < 1$ and for $x, y \in \mathbf{R}$,

$$\begin{aligned} K^{\alpha}_{\omega}\left(x^2, y^2\right) &= \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L^{\alpha}_k\left(x^2\right) L^{\alpha}_k\left(y^2\right) \omega^k \\ &= (1-\omega)^{-1} \left(|xy|\sqrt{-\omega}\right)^{-\alpha} \Gamma(\alpha+1) e^{-\frac{\omega}{1-\omega}(x^2+y^2)} J_{\alpha}\left(\frac{2|xy|(-\omega)^{\frac{1}{2}}}{1-\omega}\right). \end{aligned} \tag{5.4}$$

$K^{\alpha}_{\omega}(x^2, y^2)$ is the *Hille–Hardy kernel of order* α. By (5.3),

$$\int_{-\infty}^{\infty} K^{\alpha}_{\omega}\left(\frac{x^2}{2}, \frac{y^2}{2}\right) L^{\alpha}_k\left(\frac{x^2}{2}\right) d\mu_{\alpha}(x) = \omega^k L^{\alpha}_k\left(\frac{y^2}{2}\right). \tag{5.5}$$

In what follows we will let

$$T^{\alpha}_{\omega}(\psi)(y) = \int_0^{\infty} K^{\alpha}_{\omega}(x, y)\psi(x)\, d\tilde{\mu}_{\alpha}(x), \tag{5.6}$$

where $|\omega| < 1$ and ψ is a polynomial. This is the Laguerre semigroup (see e.g. [14]). After a change of variables and normalization, the latter is equivalent to

$$T^{\alpha}_{\omega}(\psi)\left(\frac{y^2}{2}\right) = \int_{-\infty}^{\infty} K^{\alpha}_{\omega}\left(\frac{x^2}{2}, \frac{y^2}{2}\right) \psi\left(\frac{x^2}{2}\right) d\mu_{\alpha}(x). \tag{5.7}$$

If $\psi(x) = L_k^\alpha(y)$, then by (5.5),

$$T_\omega^\alpha\big(L_k^\alpha\big)\left(\frac{y^2}{2}\right) = \omega^k\big(L_k^\alpha\big)\left(\frac{y^2}{2}\right). \tag{5.8}$$

5.2. Hypercontractivity of the Laguerre semigroup

The Hille–Hardy identity allows us to replace the Bessel function in the definition of $\mathcal{L}_{\nu,\mu}^\alpha$ with the Mehler kernel, and to establish a connection between the $L^p - L^q$ mapping properties of these operators and the continuity of the Laguerre semigroup in weighted L^p spaces. We recall the probability measures that we have defined in the previous section:

$$d\tilde{\mu}_k(x) = \frac{e^{-x}x^k\,dx}{\Gamma(k+1)} \quad \text{and} \quad d\mu_k(x) = \frac{e^{-\frac{x^2}{2}}x^{2k+1}\,dx}{2^{k+1}\Gamma(k+1)}, \quad k > -1.$$

We prove the following

Lemma 5.1. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $\alpha \geqslant -\frac{1}{2}$. Let μ and ν such that*

$$\mu = \frac{1}{p'} - \frac{1}{q} \quad \text{and} \quad -\alpha - \frac{1}{p'} < \nu \leqslant \frac{1}{2} - \max\left\{\frac{1}{p'} - \frac{1}{q}, 0\right\} \tag{5.9}$$

(see (3.1)). We let

$$q(\alpha + \nu + \mu) = 2\gamma + 1, \qquad (\alpha - \nu + 1)p = 2\beta + 1. \tag{5.10}$$

Let $C_{\nu,p,q}^\alpha$ be defined as in (3.3).

The inequality

$$\big\|\mathcal{L}_{\nu,\mu}^\alpha f\big\|_q \leqslant C_{\nu,p,q}^\alpha \|f\|_p \tag{5.11}$$

is valid for every $f \in C_0^\infty(0,\infty)$ if and only if the inequality

$$\left(\int_0^\infty \big|T_{\bar{\omega}}^\alpha(k)(y\epsilon)\big|^q\,d\tilde{\mu}_\gamma(y)\right)^{\frac{1}{q}} \leqslant \left(\int_0^\infty \big|k(x\epsilon)\big|^p\,d\tilde{\mu}_\beta(y)\right)^{\frac{1}{p}} \tag{5.12}$$

is valid for $\bar{\omega} = -pq^{-1}$, $\epsilon = \frac{p+q}{pq}$, and for every polynomial k.

It is known that the Laguerre semigroup $\{T_t^\alpha\}$ is hypercontractive for all $\alpha > -1/2$ and $t \in (0,1)$ (see [6,9,10]); that is, for every $p > 1$ there exists a strictly increasing function $q : \mathbf{R}^+ \to [q(0) = p, \infty)$, such that for every $t \geqslant 0$,

$$\left(\int_0^\infty \big|T_t^\alpha(k)(y)\big|^{q(t)}\,d\mu_\alpha(y)\right)^{\frac{1}{q}} \leqslant \left(\int_0^\infty \big|k(x)\big|^p\,d\mu_\alpha(x)\right)^{\frac{1}{p}}.$$

It would be very interesting to find the optimal range of $q > p$ and t's for which the inequality above still holds true.

Very little is known about the hypercontractivity of the Laguerre semigroup for negative or complex t's, and, to the best of our knowledge, there is little or no literature on the continuity of this semigroup in weighted L^p spaces. Proving (5.12) would provide a solution to this problem for a special t. Moreover, a proof of (5.12) could be key to the general solution of the problem: indeed, Beckner proved in [2] that the Hermite semigroup T_ω satisfies $\|T_\omega f\|_{L^p(d\mu_1)} \leqslant \|f\|_{L^q(d\mu_1)}$ when $1 < p \leqslant 2$, $q = p'$, and $\omega = \sqrt{-pq^{-1}}$; a few years later J. Epperson showed in [5] that a clever modification of Beckner's proof can be used to find the optimal set of parameters $p < q$, and $\omega \in \mathbf{C}$, for which the inequality above is still valid.

We hope that a proof of (5.12) and Epperson's technique will produce a full solution of the hypercontractivity problem for the Laguerre semigroup. The establishment of (5.12) in the optimal range of all parameters involved would also be of great interest.

Note that if $\mathcal{L}_{\nu,\mu}^\alpha$ is the Fourier–Bessel transform, then $\nu = \frac{2\alpha+1}{p'} - \alpha$ and $\mu = 0$, and so $2\beta + 1 = 2\gamma + 1 = 2\alpha + 1$.

Therefore, evaluating the $L^p - L^{p'}$ operator norm of the Fourier–Bessel transform is equivalent to proving that the Laguerre semigroup T_ω^α satisfies $\|T_{\bar{\omega}}^\alpha f\|_{L^p(d\tilde{\mu}_\alpha)} \leqslant \|f\|_{L^q(d\tilde{\mu}_\alpha)}$ for $1 < p \leqslant 2$, $q = p'$, and $\bar{\omega} = -pq^{-1}$.

If $\mathcal{L}_{\nu,\mu}^\alpha$ is the Hankel transform, then $\nu = \frac{1}{2}$ and $\mu = 0$, and $\frac{2\beta+1}{p} = \frac{2\gamma+1}{q}$. Observe also that the assumptions on μ and ν in (5.9) imply

$$-\frac{1}{q} < \frac{2\gamma+1}{q} \leqslant \frac{2\beta+1}{p} + \mu. \tag{5.13}$$

Proof of Lemma 5.1. We will let $q = p'$, since the proof is very similar in the other case. With this assumption, $\epsilon = 1$ and $\mu = 0$.

Recalling the definition of β and γ in (5.10), the constant $C^{\alpha}_{\nu,p,q}$ in (3.3) can be written as

$$C^{\alpha}_{\nu,p,q} = 2^{\nu-\frac{1}{q}} \frac{p^{\frac{1}{2}(1+\alpha-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\alpha+\nu+\frac{1}{p'})}} \frac{\Gamma(\frac{\alpha+\nu+\mu}{2}q+\frac{1}{2})^{\frac{1}{q}}}{\Gamma(\frac{1+\alpha-\nu}{2}p+\frac{1}{2})^{\frac{1}{p}}} = 2^{\frac{1}{p}-\frac{1}{q}} \frac{(\Gamma(\gamma+1)2^{\gamma+1}(q)^{-\gamma-1})^{\frac{1}{q}}}{(\Gamma(\beta+1)2^{\beta+1}p^{-\beta-1})^{\frac{1}{p}}},$$

and (5.11) can be rewritten in the following equivalent fashion:

$$2^{\frac{1}{q}} \left(\frac{q^{\gamma+1}}{\Gamma(\gamma+1)2^{\gamma+1}}\right)^{\frac{1}{q}} \left\| \mathcal{L}^{\alpha}_{\nu,\mu} f \right\|_q \leqslant 2^{\frac{1}{p}} \left(\frac{p^{\beta+1}}{\Gamma(\beta+1)2^{\beta+1}}\right)^{\frac{1}{p}} \|f\|_p. \tag{5.14}$$

With a change of variables we can rewrite (5.12) as:

$$\left(\int_{-\infty}^{\infty} \left|T^{\alpha}_{\bar{\omega}}(k)\left(\frac{y^2}{2}\right)\right|^q \frac{e^{-\frac{y^2}{2}}|y|^{2\gamma+1}\,dx}{\Gamma(\gamma+1)2^{\gamma+1}}\right)^{\frac{1}{q}} \leqslant \left(\int_{-\infty}^{\infty} \left|k\left(\frac{x^2}{2}\right)\right|^p \frac{e^{-\frac{x^2}{2}}|x|^{2\beta+1}\,dx}{\Gamma(\beta+1)2^{\beta+1}}\right)^{\frac{1}{p}}$$

and we can replace y with $\sqrt{q}y$ in the first integral and x with $\sqrt{p}x$ in the second integral. We obtain

$$q^{\frac{\gamma+1}{q}} \left(\int_{-\infty}^{\infty} \left|T^{\alpha}_{\bar{\omega}}(k)\left(\frac{qy^2}{2}\right)\right|^q \frac{e^{-\frac{qy^2}{2}}|y|^{2\gamma+1}\,dx}{\Gamma(\gamma+1)2^{\gamma+1}}\right)^{\frac{1}{q}} \leqslant p^{\frac{\beta+1}{p}} \left(\int_{-\infty}^{\infty} \left|k\left(\frac{px^2}{2}\right)\right|^p \frac{e^{-\frac{px^2}{2}}|x|^{2\beta+1}\,dx}{\Gamma(\beta+1)2^{\beta+1}}\right)^{\frac{1}{p}}. \tag{5.15}$$

Then, we use the identity (5.4) to replace the Hardy–Hille kernel in the definition of T^{α}_{ω} with a product of $J_\alpha(xy)$ and an exponential function. Indeed,

$$T^{\alpha}_{\bar{\omega}}(k)\left(\frac{qy^2}{2}\right) = \int_{-\infty}^{\infty} K^{\alpha}_{\bar{\omega}}\left(\frac{x^2}{2}, \frac{qy^2}{2}\right) k\left(\frac{x^2}{2}\right) d\mu_\alpha(x),$$

and recalling that $K^{\alpha}_{\omega}(x^2, y^2) = (1-\omega)^{-1}(|xy|\sqrt{-\omega})^{-\alpha}\Gamma(\alpha+1)e^{-\frac{\omega}{1-\omega}(x^2+y^2)} J_\alpha(\frac{2|xy|(-\omega)^{\frac{1}{2}}}{1-\omega})$, and $\bar{\omega} = -pq^{-1} = -(p-1)$, we gather

$$\begin{aligned} K^{\alpha}_{\bar{\omega}}\left(\frac{x^2}{2}, \frac{qy^2}{2}\right) &= p^{-1}\left(\left|\frac{xyq^{\frac{1}{2}}}{2}\right|\left(pq^{-1}\right)^{\frac{1}{2}}\right)^{-\alpha} \Gamma(\alpha+1) e^{\frac{1}{2q}(x^2+qy^2)} J_\alpha\left(|xy|p^{-\frac{1}{2}}\right) \\ &= p^{-1}\left(2p^{-\frac{1}{2}}\right)^{\alpha}\Gamma(\alpha+1)|xy|^{-\alpha} e^{\frac{1}{2q}(x^2+qy^2)} J_\alpha\left(|xy|p^{-\frac{1}{2}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} T^{\alpha}_{\bar{\omega}}(k)\left(\frac{qy^2}{2}\right) &= p^{-1}\left(2p^{-\frac{1}{2}}\right)^{\alpha}\Gamma(\alpha+1) \int_{-\infty}^{\infty} |xy|^{-\alpha} J_\alpha\left(|xy|p^{-\frac{1}{2}}\right) k\left(\frac{x^2}{2}\right) e^{\frac{1}{2q}(x^2+qy^2)}\, d\mu_\alpha(x) \\ &= \frac{1}{2} p^{-\frac{\alpha}{2}-1} \int_{-\infty}^{\infty} |xy|^{-\alpha} J_\alpha\left(|xy|p^{-\frac{1}{2}}\right) k\left(\frac{x^2}{2}\right) e^{\frac{y^2}{2}-\frac{x^2}{2p}} |x|^{2\alpha+1}\, dx \\ &= p^{-\frac{\alpha}{2}-1} |y|^{-2\alpha-1} e^{\frac{y^2}{2}} \int_0^{\infty} \left(x|y|\right)^{\alpha+1} J_\alpha\left(x|y|p^{-\frac{1}{2}}\right) e^{-\frac{x^2}{2p}} k\left(\frac{x^2}{2}\right) dx. \end{aligned}$$

If let $x = \sqrt{p}x'$ and recall that by (5.10), $\nu = \alpha + 1 - \frac{2\beta+1}{p} = \frac{2\gamma+1}{q} - \alpha$ (and so $\frac{2\gamma+1}{q} = 2\alpha + 1 - \frac{2\beta+1}{p}$), we obtain

$$T^{\alpha}_{\bar{\omega}}(k)\left(\frac{qy^2}{2}\right) = |y|^{-\frac{2\gamma+1}{q}} e^{\frac{y^2}{2}} \int_0^{\infty} \left(x'|y|\right)^{\nu} J_\alpha\left(x'|y|\right)(x')^{2\beta+1} e^{-\frac{(x')^2}{2}} k\left(\frac{p(x')^2}{2}\right) dx' = |y|^{-\frac{2\gamma+1}{q}} e^{\frac{y^2}{2}} \mathcal{L}^{\alpha}_{\nu,0}(h)(y),$$

where we have let $h(x) = e^{-\frac{x^2}{2}} x^{\frac{2\beta+1}{p}} e^{-\frac{x^2}{2}} k(\frac{px^2}{2})$.

Using the identity above and the definition of h, we can see at once that the inequality (5.15) is equivalent to

$$q^{\frac{\gamma+1}{q}} \left(\int_{-\infty}^{\infty} \left|\mathcal{L}^{\alpha}_{\nu,0}(h)(y)\right|^q \frac{dy}{\Gamma(\gamma+1)2^{\gamma+1}}\right)^{\frac{1}{q}} \leqslant p^{\frac{\beta+1}{p}} \left(\int_{-\infty}^{\infty} \left|h(x)\right|^p \frac{dx}{\Gamma(\beta+1)2^{\beta+1}}\right)^{\frac{1}{p}}, \tag{5.16}$$

which is (5.14).

There is no loss of generality if we replace $k(\frac{px^2}{2})$ with $k(x^2)$. Since the functions of the form of $h(x) = x^{\frac{2\beta+1}{p}} e^{-\frac{x^2}{2}} k(x^2)$ are dense in $L^p(0,\infty)$ (see e.g. [1]), it is enough to prove (5.11) for these functions. The proof of the lemma is concluded. □

5.3. Reduction to a discrete operator

Let us recall that (5.12) is equivalent to

$$\left(\int_{-\infty}^{\infty}\left|T_{\bar\omega}^{\alpha}(k)\left(\frac{\epsilon y^2}{2}\right)\right|^q d\mu_\gamma(y)\right)^{\frac{1}{q}} \leqslant \left(\int_{-\infty}^{\infty}\left|k\left(\frac{\epsilon x^2}{2}\right)\right|^p d\mu_\beta(x)\right)^{\frac{1}{p}}, \tag{5.17}$$

where $\epsilon = \frac{p+q}{pq}$, $\bar\omega = -pq^{-1}$ and k is a polynomial. In the next crucial step we will approximate the measures $d\mu_\gamma(y)$ and $d\mu_\beta(x)$ in (5.17) with sequences of discrete measures, and the Laguerre polynomials in $\frac{x^2}{2}$ with homogeneous functions in n variables. Then we will define a discrete analogue of $T_{\bar\omega}^{\alpha}$ and we will show that Lemma 5.1 can be proved as a consequence of the $L^p - L^q$ mapping properties of this operator. We will assume $\epsilon = 1$ (as in the case $q = p'$), since the proof can be easily adjusted when $\epsilon \neq 1$.

Before we state the next lemma we need some preliminaries. Let δ_{t_0} be the Dirac distribution on $\mathbf{R}$ with unitary mass at t_0. We let $d\nu(t)$ be the Bernoulli trial $\frac{1}{2}(\delta_1(t) + \delta_{-1}(t))$, and for every positive integer n, we let $\bar t = (t_1, \dots, t_n)$, $\sigma(\bar t) = t_1 + \dots + t_n$, and $d\nu_n(\bar t) = d\nu(t_1)\cdots d\nu(t_n)$. We also let $d\nu_{\beta,n}(\bar x)$, or $d\nu_\beta(\bar x)$ when there is no risk of confusion, be the probability measure

$$d\nu_{k,n}(\bar x) = \frac{|\sigma(\bar x)|^{2k+1}\, d\nu_n(\bar x)}{\int_{\mathbf{R}^n}|\sigma(\bar x)|^{2k+1}\, d\nu_n(\bar x)}.$$

$d\nu_{k,n}(\bar x)$ is the discrete analogue of the measure $d\tilde\mu_k(x)$ that we have defined in the previous section.

Let $\psi_{k,n}(\bar t) = k!\sum_{1\leqslant m_1<\dots<m_k\leqslant n} t_{m_1}\cdots t_{m_k}$ be the elementary symmetric function in n variables of degree k, and let $\mathbf{X_n}$ be the vector space which is generated by the functions $\sigma(\bar t)^j\psi_{k,n}(\bar t)$, with $k, j \geqslant 0$. These functions are homogeneous of degree $k + j$.

We prove the following

Lemma 5.2. *If there exists $N > 0$ for which the inequality*

$$\left(\int_{\mathbf{R}^n}\left|g(\sqrt{\omega}\bar s)\right|^q d\nu_{\gamma,n}(\bar s)\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}^n}\left|g(\bar s)\right|^p d\nu_{\beta,n}(\bar s)\right)^{\frac{1}{p}} \tag{5.18}$$

is valid for every $n > N$ and every function $g(\bar s) \in \mathbf{X_n}$, then (5.17) *is valid for every polynomial $k(x)$.*

Proof. By the central limit theorem, the sequence $d\nu^{(n)}(t)$, the n-fold convolutions of $d\nu(\sqrt{n}t)$ with itself, converges to $(2\pi)^{-\frac{1}{2}}e^{-\frac{t^2}{2}}$ in the weak topology of the continuous functions on $\mathbf{R}$, and furthermore, the moments of $d\nu^{(n)}(t)$ will converge to the moments of $e^{-\frac{t^2}{2}}(2\pi)^{-\frac{1}{2}}$. That is,

$$\int_{\mathbf{R}} f(t)|t|^m\, d\nu^{(n)}(t) \to \int_{\mathbf{R}} f(t)|t|^m \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt \tag{5.19}$$

whenever $m > -1$ and f is continuous. Thus

$$\begin{aligned}\int_{\mathbf{R}} f(t)|t|^{2m+1}\, d\nu^{(n)}(t) &= \int_{\mathbf{R}^n} f(t_1+\dots+t_n)|t_1+\dots+t_n|^{2m+1}\, d\nu(t_1\sqrt{n})\cdots d\nu(t_1\sqrt{n})\\ &= \int_{\mathbf{R}^n} f\big(\sigma(\bar t)\big)\left|\sigma(\bar t)\right|^{2m+1} d\nu_n(\sqrt{n}\bar t) \to \int_{\mathbf{R}} f(t)|t|^{2m+1}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt,\end{aligned}$$

and

$$\int_{\mathbf{R}^n}\left|\sigma(\bar t)\right|^{2m+1} d\nu_n(\sqrt{n}\bar t) \to \int_{\mathbf{R}} |t|^{2m+1}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt$$

when $n \to \infty$. Therefore

$$\int_{\mathbf{R}^n} f\big(\sigma(\bar{t})\big)\, d\nu_{m,n}(\bar{t}) \to \int_{\mathbf{R}} f(t)\, d\mu_m(t). \tag{5.20}$$

Observe that

$$\int_{\mathbf{R}^n} f\big(\sigma(\bar{t})\big)\big|\sigma(\bar{t})\big|^{2m+1}\, d\nu_n(\sqrt{n}\bar{t}) = \frac{1}{2^n}\sum f\left(\pm\frac{1}{\sqrt{n}} \pm \cdots \pm \frac{1}{\sqrt{n}}\right)\left|\pm\frac{1}{\sqrt{n}} \pm \cdots \pm \frac{1}{\sqrt{n}}\right|^{2m+1}, \tag{5.21}$$

where the sum is taken over all possible combinations of n signs (and thus the sum has 2^n terms). Therefore, the integral on the left-hand side of (5.20) equals to

$$\sum f\left(\pm\frac{1}{\sqrt{n}} \pm \cdots \pm \frac{1}{\sqrt{n}}\right)\frac{|\pm 1 \pm \cdots \pm 1|^{2m+1}}{\sum |\pm 1 \pm \cdots \pm 1|^{2m+1}}.$$

From (5.20) it follows that

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}\left|T^\alpha_\omega k\left(\frac{1}{2}\sigma^2(\bar{y})\right)\right|^q d\nu_{\gamma,n}(\bar{y}) = \int_{\mathbf{R}}\left|T^\alpha_\omega k\left(\frac{1}{2}y^2\right)\right|^q d\mu_\gamma(y)$$

and

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}\left|k\left(\frac{1}{2}\sigma^2(\bar{x})\right)\right|^p d\nu_{\beta,n}(\bar{x}) = \int_{\mathbf{R}}\left|k\left(\frac{1}{2}x^2\right)\right|^p d\mu_\beta(x).$$

Therefore,

$$\lim_{n\to\infty}\frac{(\int_{\mathbf{R}^n}|T^\alpha_\omega k(\frac{1}{2}\sigma^2(\bar{y}))|^q\, d\nu_{\gamma,n}(\bar{y}))^{\frac{1}{q}}}{(\int_{\mathbf{R}^n}|k(\frac{1}{2}\sigma^2(\bar{x}))|^p\, d\nu_{\beta,n}(\bar{x}))^{\frac{1}{p}}} = \frac{(\int_{\mathbf{R}}|T^\alpha_\omega k(\frac{1}{2}y^2)|^q\, d\mu_\gamma(x))^{\frac{1}{q}}}{(\int_{\mathbf{R}}|k(\frac{1}{2}x^2)|^p\, d\mu_\beta(x))^{\frac{1}{p}}}.$$

Recall that $k(x)$ is a finite linear combination of Laguerre polynomials, and that $T^\alpha_{\bar{\omega}}$ acts as a multiplier over the Laguerre polynomials of order α (see (5.8)).

We show that $L^\alpha_m(\frac{1}{2}\sigma(\bar{x})^2)$ can be $d\nu_n(\sqrt{n}\bar{x})$ approximated with a linear combination of homogeneous functions of degree $2m$.

We will need the following lemma, whose proof will be postponed to Appendix C.

Lemma 5.3. *For every $\alpha \geqslant -\frac{1}{2}$, $m \geqslant 0$, $n \geqslant 1$, and $\bar{x} = (\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}})$,*

$$L^\alpha_m\left(\frac{1}{2}\sigma^2(\bar{x})\right) = \Phi^\alpha_m(\bar{x}) + \frac{1}{n}\mathcal{R}^\alpha_m\big(\sigma(\bar{x})\big),$$

where $\mathcal{R}^\alpha_m(x)$ is a polynomial of degree $\leqslant 2m-2$ whose coefficients depend only on m and α, and $\Phi^\alpha_m(\bar{x})$ is a homogeneous function of degree $2m$ which is defined as follows:

$$\Phi^\alpha_m(\bar{x}) = \begin{cases} \frac{(-1)^m}{2^{2m}m!}\psi_{2m,n}(\bar{x}) & \text{if } \alpha = -\frac{1}{2}, \\ \sum_{j=0}^{2m}\eta^\alpha_{m,j}\sigma(\bar{x})^j\psi_{2m-j,n}(\bar{x}) & \text{if } \alpha \geqslant -\frac{1}{2}, \end{cases} \tag{5.22}$$

where the $\psi_{k,n}(\bar{x})$ are the elementary symmetric functions in $\bar{x} = (x_1, \ldots, x_n)$, and

$$\eta^\alpha_{m,j} = \frac{(-1)^m\sqrt{2}\Gamma(m+\alpha+1)}{\pi^{\frac{1}{2}}(2m)!\Gamma(\alpha+\frac{1}{2})}2^j\binom{2m}{j}\int_{-\sqrt{2}}^{\sqrt{2}}\big(1-2t^2\big)^{\alpha-\frac{1}{2}}(t-1)^j\, dt. \tag{5.23}$$

Furthermore, for every $l > -1$ and every $q \geqslant 1$,

$$\lim_{n\to\infty}\left(\int_{\mathbf{R}^n}\left|\Phi^\alpha_m(\bar{x}) - L^\alpha_m\left(\frac{1}{2}\sigma^2(\bar{x})\right)\right|^q d\mu_{l,n}(\bar{x})\right)^{\frac{1}{q}} = 0. \tag{5.24}$$

The functions $\Phi_m^\alpha(\bar{x})$ are the analogues of the Laguerre polynomials in $\frac{1}{2}x^2$. It is important to note that these functions are homogeneous of degree $2m$.

Recall that $\mathbf{X_n}$ is the vector space which is generated by the functions $\sigma(\bar{t})^j\psi_{k,n}(\bar{t})$, with $k, j \geqslant 0$; therefore, every finite linear combinations of the $\Phi_m^\alpha(\bar{x})$ is in $\mathbf{X_n}$.

We shall define an operator on $\mathbf{X_n}$ that approximates T_ω^α. By (5.8)

$$T_\omega^\alpha\left(\sum_{k=0}^{M} c_k L_k^\alpha\right)\left(\frac{y^2}{2}\right) = \sum_{k=0}^{M} c_k \omega^k L_k^\alpha\left(\frac{y^2}{2}\right).$$

Since $L_m^\alpha(\frac{1}{2}\sigma(\bar{x})^2)$ can be approximated, in the sense of the previous lemma, by $\Phi_m^\alpha(\bar{x})$, the natural replacement for T_ω^α is the operator $\mathcal{K}_\omega = \mathcal{K}_{\omega,n} : \mathbf{X_n} \to \mathbf{X_n}$,

$$\mathcal{K}_\omega\left(\sum_{k=0}^{M} c_k \phi_j\right)(\bar{s}) = \sum_{k=0}^{M} c_j \omega^j \phi_j(\bar{s}),$$

where the ϕ_j's are homogeneous generators of $\mathbf{X_n}$ of degree $2j$. Thus, $\mathcal{K}_\omega \phi_j(\bar{s}) = \omega^j \phi_j(\bar{s}) = \phi_j(\bar{s}\sqrt{\omega})$, and if we let $k(\frac{s^2}{2}) = \sum_{k=0}^{M} c_k L_k^\alpha(\frac{s^2}{2})$ and $g(\bar{s}) = \sum c_j \Phi_j^\alpha(\bar{s})$, we obtain $\mathcal{K}_\omega g(\bar{s}) = g(s\sqrt{\omega})$.

By Lemma 5.3,

$$\lim_{n\to\infty}\left(\int_{\mathbf{R}^n}\left|k\left(\frac{1}{2}\sigma^2(\bar{s})\right) - g(\bar{s})\right|^p d\nu_{\beta,n}(\bar{s})\right)^{\frac{1}{p}} = 0$$

and

$$\lim_{n\to\infty}\left(\int_{\mathbf{R}^n}\left|T^\alpha k\left(\frac{1}{2}\sigma^2(\bar{s})\right) - \mathcal{K}_\omega(g)(\bar{s})\right|^q d\nu_{\gamma,n}(\bar{s})\right)^{\frac{1}{q}} = 0.$$

Consequently,

$$\lim_{n\to\infty}\frac{(\int_{\mathbf{R}^n}|\mathcal{K}_\omega g(\bar{s})|^q\, d\nu_{\gamma,n}(\bar{s}))^{\frac{1}{q}}}{(\int_{\mathbf{R}^n}|g(\bar{s})|^p\, d\nu_{\beta,n}(\bar{s}))^{\frac{1}{p}}} = \frac{(\int_{\mathbf{R}}|T_\omega^\alpha k(\frac{s^2}{2})|^q\, d\mu_\gamma(s))^{\frac{1}{q}}}{(\int_{\mathbf{R}}|k(\frac{s^2}{2})|^p\, d\mu_\beta(s))^{\frac{1}{p}}} \tag{5.25}$$

and if we prove that, for every $n > 1$, the ratio on the left-hand side of (5.25) is < 1, then the same is true for the ratio on the left-hand side of (5.25).

Since we have observed that $\mathcal{K}_\omega g(\bar{s}) = g(\sqrt{\omega}\bar{s})$, we have proved the lemma. □

5.4. End of the proof of Theorem 1.2

Replacing T_ω^α with $\mathcal{K}_\omega$ is one of the most crucial steps of the proof because it allows to reduce the proof of the inequality (5.12) to the proof of the discrete inequality

$$\left(\int_{\mathbf{R}^n}\left|g(\bar{s}\sqrt{\bar{\omega}})\right|^q d\nu_{\gamma,n}(\bar{s})\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}^n}\left|g(\bar{s})\right|^p d\nu_{\beta,n}(\bar{s})\right)^{\frac{1}{p}}, \quad g \in \mathbf{X_n} \tag{5.26}$$

(see (5.18)), whenever $\bar{\omega} = -pq^{-1}$, and when $-\frac{1}{q} < \frac{2\gamma+1}{q} \leqslant \frac{2\beta+1}{p} + \frac{1}{p'} - \frac{1}{q}$, and n is sufficiently large. Recall that we have let $d\nu_{m,n}(\bar{s}) = \frac{|\sigma(\bar{s})|^{2m+1} d\nu_n(\sqrt{n}\bar{s})}{\int_{\mathbf{R}^n}|\sigma(\bar{s})|^{2m+1} d\nu_n(\sqrt{n}\bar{s})}$.

When $\beta = \gamma = -\frac{1}{2}$ and $q = p'$, (5.26) has been proved by Beckner in [2]. That is, Beckner proved the following unweighted inequality:

$$\left(\int_{\mathbf{R}^n}\left|g(\bar{s}\sqrt{\bar{\omega}})\right|^q d\nu_n(\bar{s})\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}^n}\left|g(\bar{s})\right|^p d\nu_n(\bar{s})\right)^{\frac{1}{p}}, \quad g \in \mathbf{X_n}. \tag{5.27}$$

Beckner proved (5.27) with iterated applications of the following "two-point inequality":

$$\left(\frac{|A\sqrt{\bar{\omega}} + B|^{p'} + |A\sqrt{\bar{\omega}} - B|^{p'}}{2}\right)^{\frac{1}{p'}} \leqslant \left(\frac{|A + B|^p + |A - B|^p}{2}\right)^{\frac{1}{p}}. \tag{5.28}$$

The weighted inequality (5.26) cannot be proved in the same manner, and its proof seems to be quite difficult.

We will need the following observation: We have observed that $\int_{\mathbf{R}^n} |\sigma(\bar{s})|^{2m+1}\, d\nu_n(\sqrt{n}\bar{s})$ converges to $(2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} |x|^{2m+1} \times e^{-\frac{x^2}{2}}\, dx$ when $n \to \infty$. Therefore, if n is sufficiently large, we can replace $\int_{\mathbf{R}^n} |\sigma(\bar{s})|^{2m+1}\, d\nu_n(\sqrt{n}\bar{s})$ with $c_m = (2\pi)^{-\frac{1}{2}} \times \int_{\mathbf{R}} |x|^{2m+1} e^{-\frac{x^2}{2}}\, dx$, and, instead of (5.26), we can prove the following inequality:

$$\left(\int_{\mathbf{R}^n} \left|g(\bar{s}\sqrt{\bar{\omega}})\right|^q \frac{|\sigma(\bar{s})|^{2\gamma+1}\, d\nu_n(\sqrt{n}\bar{s})}{c_\gamma}\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}^n} \left|g(\bar{s})\right|^p \frac{|\sigma(\bar{s})|^{2\beta+1}\, d\nu_n(\sqrt{n}\bar{s})}{c_\beta}\right)^{\frac{1}{p}}, \quad g \in \mathbf{X_n}.$$

Recalling (5.21), we can see at once that when $\frac{2\beta+1}{p} = \frac{2\gamma+1}{q}$, the latter is equivalent to

$$\left(\int_{\mathbf{R}^n} \left|g(\bar{s}\sqrt{\bar{\omega}})\right|^q \frac{|\sigma(\bar{s})|^{2\gamma+1}\, d\nu_n(\bar{s})}{c_\gamma}\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}^n} \left|g(\bar{s})\right|^p \frac{|\sigma(\bar{s})|^{2\beta+1}\, d\nu_n(\bar{s})}{c_\beta}\right)^{\frac{1}{p}}. \tag{5.29}$$

For these values of β and γ we can actually prove that (5.29) holds for every $n \geqslant 1$ and for the class of functions $g(\bar{s})$ for which $g(\bar{s})|\sigma(\bar{s})|^{\frac{2\beta+1}{p}}$ is in $L^p(d\nu_n(\bar{s}))$. That is, we require that

$$\frac{1}{2^n} \sum \left|g(\pm 1, \ldots, \pm 1)\right|^p |\pm 1 \pm \cdots \pm 1|^{2\beta+1} < \infty,$$

where the sum is taken over all possible combinations of n signs. Since this sum is finite, this is equivalent to assume that $|g(\pm 1, \ldots, \pm 1)| < \infty$ for every choice of $(\pm 1, \ldots, \pm 1)$.

The functions in $\mathbf{X_n}$ belong to this class.

The proof of (5.29) concludes the proof of Theorem 1.2 since, by (5.10), $\frac{2\gamma+1}{q} = \frac{2\beta+1}{q}$ implies $\nu = \frac{1}{2}$ and $\mu = 0$, as in the Hankel transform.

We argue by induction on n. When $n = 1$, $\bar{s} = s = \pm 1$, and $\sigma(s)$ takes only the values ± 1, (5.29) is equivalent to

$$\left(\int_{\mathbf{R}} \left|g(\sqrt{\bar{\omega}} s)\right|^q d\nu(s)\right)^{\frac{1}{q}} \leqslant (c_\beta)^{-\frac{1}{p}} (c_\gamma)^{\frac{1}{q}} \left(\int_{\mathbf{R}} \left|g(s)\right|^p d\nu(s)\right)^{\frac{1}{p}}.$$

By (5.27) the following inequality holds true:

$$\left(\int_{\mathbf{R}} \left|g(\sqrt{\bar{\omega}} s)\right|^q d\nu(s)\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{R}} \left|g(s)\right|^p d\nu(s)\right)^{\frac{1}{p}}.$$

So, if $1 \leqslant (c_\gamma)^{\frac{1}{q}} (c_\beta)^{-\frac{1}{p}}$, or equivalently, if

$$\left((2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} |x|^{2\beta+1} e^{-\frac{x^2}{2}}\, dx\right)^{\frac{1}{p}} \leqslant \left((2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} |x|^{2\gamma+1} e^{-\frac{x^2}{2}}\, dx\right)^{\frac{1}{q}}, \tag{5.30}$$

then (5.29) follows. By Hölder's inequality

$$\left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{2\beta+1}\, dx\right)^{\frac{1}{p}} \leqslant \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{\frac{q}{p}(2\beta+1)}\, dx\right)^{\frac{1}{q}} \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}}\, dx\right)^{\frac{1}{p}-\frac{1}{q}}.$$

Since $\int_{\mathbf{R}} e^{-\frac{x^2}{2}}\, dx = (2\pi)^{-\frac{1}{2}}$ and $\frac{2\beta+1}{p} = \frac{2\gamma+1}{q}$, then

$$\left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{2\beta+1}\, dx\right)^{\frac{1}{p}} \leqslant (2\pi)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{2\gamma+1}\, dx\right)^{\frac{1}{q}},$$

which is (5.30).

We now assume that (5.29) is valid when $n \geqslant 1$, and we prove that the same is true for $n+1$.

Let $g(\bar{s})$ be such that $g(\bar{s})|\sigma(\bar{s})|^{\frac{2\beta+1}{p}}$ is in $L^p(d\nu_{n+1}(\bar{s}))$. We let $\bar{s} = (\bar{s}', s_{n+1})$, with $\bar{s}' \in \mathbf{R}^n$, and $d\nu_{n+1}(\bar{s}) = d\nu_n(\bar{s}')\, d\nu(s_{n+1})$. We also let

$$g_1(\bar{s}) = \begin{cases} g(\bar{s}) \left(\frac{|\sigma(\bar{s})|}{|\sigma(\bar{s}')|}\right)^{\frac{2\gamma+1}{q}} & \text{if } |\sigma(\bar{s}')| \neq 0, \\ 0 & \text{if } \sigma(\bar{s}') = 0, \end{cases}$$

and

$$g_2(\bar{s}) = \begin{cases} g(\bar{s}) & \text{if } |\sigma(\bar{s}')| = 0, \\ 0 & \text{if } \sigma(\bar{s}') \neq 0. \end{cases}$$

With this position,

$$g(\bar{s})\left|\sigma(\bar{s})\right|^{\frac{2\gamma+1}{q}} = g_1(\bar{s})\left|\sigma(\bar{s}')\right|^{\frac{2\gamma+1}{q}} + g_2(\bar{s}).$$

This is because $\sigma(\bar{s})$ only takes the values ± 1 in the set where $\sigma(\bar{s}') = 0$.

$g_2(\bar{s})$ is in $L^p(d\nu_{n+1}(\bar{s}))$ because $g(\bar{s})|\sigma(\bar{s})|^{\frac{2\beta+1}{p}}$ is; for the same reason, the restrictions of $g_1(\bar{s})|\sigma(\bar{s}')|^{\frac{2\beta+1}{p}}$ to the sets where s_{n+1} is constant are in $L^p(d\nu_n(\bar{s}'))$ and the restrictions of $g_1(\bar{s})$ to the sets where s' is constant are in $L^p(d\nu(s_{n+1}))$.

Thus,

$$\begin{aligned}\left(\int_{\mathbf{R}^{n+1}} \left|g(\sqrt{\bar{\omega}}\bar{s})\right|^q \frac{|\sigma(\bar{s})|^{2\gamma+1}}{c_\gamma}\, d\nu_{n+1}(\bar{s})\right)^{\frac{1}{q}} &= \left(\int_{\mathbf{R}^{n+1}} \left|g_1(\sqrt{\bar{\omega}}\bar{s})\right|^q \left|\sigma(\bar{s}')\right|^{2\gamma+1} \frac{d\nu_{n+1}(\bar{s})}{c_\gamma} + \int_{\mathbf{R}^{n+1}} \left|g_2(\sqrt{\bar{\omega}}\bar{s})\right|^q \frac{d\nu_{n+1}(\bar{s})}{c_\gamma}\right)^{\frac{1}{q}} \\ &= \left(I_1^q + I_2^q\right)^{\frac{1}{q}}.\end{aligned}$$

I_2^q can be easily estimated: by (5.27),

$$I_2^q \leqslant \frac{1}{c_\gamma}\left(\int_{\mathbf{R}^{n+1}} \left|g_2(\bar{s})\right|^p d\nu_{n+1}(\bar{s})\right)^{\frac{q}{p}},$$

and since we have proved that $c_\gamma^{\frac{1}{q}} > c_\beta^{\frac{1}{p}}$, we can conclude that

$$I_2^q \leqslant \left(\int_{\mathbf{R}^{n+1}} \left|g_2(\bar{s})\right|^p \frac{d\nu_{n+1}(\bar{s})}{c_\beta}\right)^{\frac{q}{p}}. \tag{5.31}$$

Let us estimate I_1. Indeed,

$$\begin{aligned} I_1 &= \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} \left|g_1(\sqrt{\bar{\omega}}\bar{s})\right|^q d\nu(s_{n+1})\right) \frac{|\sigma(\bar{s}')|^{2\gamma+1}}{c_\gamma}\, d\nu_n(\bar{s}')\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} \left|g_1(\sqrt{\bar{\omega}}\bar{s}', \sqrt{\bar{\omega}}s_{n+1})\right|^q d\nu(s_{n+1})\right) \frac{|\sigma(\bar{s}')|^{2\gamma+1}}{c_\gamma}\, d\nu_n(\bar{s}')\right)^{\frac{1}{q}}, \end{aligned}$$

and by the one-dimensional inequality (5.27), the last integral is

$$\leqslant \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} \left|g_1(\sqrt{\bar{\omega}}\bar{s}', s_{n+1})\right|^p d\nu(s_{n+1})\right)^{\frac{q}{p}} \frac{|\sigma(\bar{s}')|^{2\gamma+1}}{c_\gamma}\, d\nu_n(\bar{s}')\right)^{\frac{1}{q}}.$$

We recall the following convexity type inequality,

$$\left(\int_S\left(\int_T \left|f(s,t)\right|^p \mu(dt)\right)^{\frac{q}{p}} \nu(ds)\right)^{\frac{1}{q}} \leqslant \left(\int_T\left(\int_S \left|f(s,t)\right|^q \nu(ds)\right)^{\frac{p}{q}} \mu(dt)\right)^{\frac{1}{p}}, \tag{5.32}$$

which holds for every positive measure spaces $(S, \mathcal{S}, \nu)$, $(T, \mathcal{T}, \mu)$, every measurable function $f(s,t)$ and every $0 < p \leqslant q < \infty$. By (5.32) and our initial assumptions,

$$\begin{aligned}\left(\int_{\mathbf{R}^{n+1}} \left|g_1(\sqrt{\bar{\omega}}\bar{s})\right|^q \frac{|\sigma(\bar{s})|^{2\gamma+1}}{c_\gamma}\, d\nu_{n+1}(\bar{s})\right)^{\frac{1}{p}} &\leqslant \left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}^n} \left|g_1(\sqrt{\bar{\omega}}\bar{s}', s_{n+1})\right|^q \frac{|\sigma(\bar{s}')|^{2\gamma+1}}{c_\gamma}\, d\nu_n(\bar{s}')\right)^{\frac{p}{q}} d\nu(s_{n+1})\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|g_1(\bar{s})\right|^p \frac{|\sigma(\bar{s}')|^{2\beta+1}}{c_\beta}\, d\nu_n(\bar{s}')\, d\nu(s_{n+1})\right)^{\frac{1}{p}},\end{aligned}$$

and since $d\nu_n(\bar{s}')d\nu(s_{n+1}) = d\nu_{n+1}(\bar{s})$, we have proved that

$$I_1^q \leqslant \left(\int_{\mathbf{R}^{n+1}} \left|g_1(\bar{s})\right|^p \frac{|\sigma(\bar{s}')|^{2\beta+1}}{c_\beta}\, d\nu_{n+1}(\bar{s})\right)^{\frac{q}{p}}. \tag{5.33}$$

From (5.31) and (5.33) it follows that

$$\left(I_1^q + I_2^q\right)^{\frac{p}{q}} \leqslant \left(\left(\int\limits_{\mathbf{R}^{n+1}} \left|g_1(\bar{s})\right|^p \frac{|\sigma(\bar{s}')|^{2\beta+1}}{c_\beta}\, d\nu_{n+1}(\bar{s})\right)^{\frac{q}{p}} + \left(\int\limits_{\mathbf{R}^{n+1}} \left|g_2(\bar{s})\right|^p \frac{d\nu_{n+1}(\bar{s})}{c_\beta}\right)^{\frac{q}{p}}\right)^{\frac{p}{q}},$$

and by the elementary inequality $x^m + y^m \leqslant (x+y)^m$, which is valid whenever $x, y \geqslant 0$ and $m \geqslant 1$, and the fact that $g(\bar{s})|\sigma(\bar{s})|^{\frac{2\beta+1}{p}} = g_1(\bar{s})|\sigma(\bar{s}')|^{\frac{2\beta+1}{p}} + g_2(\bar{s})$ because $\frac{2\gamma+1}{q} = \frac{2\beta+1}{p}$ it follows that

$$\left(I_1^q + I_2^q\right)^{\frac{p}{q}} \leqslant \int\limits_{\mathbf{R}^{n+1}} \left|g(\bar{s})\right|^p \frac{|\sigma(\bar{s})|^{2\beta+1}}{c_\beta}\, d\nu_{n+1}(\bar{s}).$$

Therefore, we have proved (5.26) and Theorem 1.2, when $q = p'$ and $\frac{2\beta+1}{p} = \frac{2\gamma+1}{q}$.

Acknowledgments

We wish to thank W. Urbina for providing insights at a crucial stage of this project, and the referee for his/her thorough reading of this paper and for providing comments and suggestions.

Appendix A. A few counterexamples

We are left to show that the range of values of ν and μ in (1.2) is optimal. A simple scaling argument shows that necessarily $\mu = \frac{1}{p'} - \frac{1}{q}$, but proving that the bounds for ν are best possible requires more work. We can assume that $q \leqslant p'$ (and so (1.2) is $-\alpha - \frac{1}{p'} < \nu \leqslant \frac{1}{2}$), since we can deal with the other case as we did in the proof of Theorem 1.1.

- $\nu \leqslant \frac{1}{2}$ *is necessary.* We recall the identity

$$\int_0^1 (xy)^{\frac{1}{2}} J_\alpha(xy) x^{\alpha+\frac{1}{2}} \left(1-x^2\right)^s dx = 2^s \Gamma(s+1) y^{-s-\frac{1}{2}} J_{\alpha+s+1}(y), \tag{A.1}$$

which is valid for every $s > -1$. See e.g. Appendix B in [7], which is also an excellent reference for the other properties of the Bessel functions that we will use in this appendix.

Take $\nu = \frac{1}{2} + 2\epsilon$, with $0 < \epsilon < 1$, and $s = -\frac{1}{p} + \epsilon$, so that $f(x) = x^{\alpha+1-2\epsilon}(1-x^2)^{-\frac{1}{p}+\epsilon}\chi_{(0,1)}(x)$, where $\chi_{(a,b)}(t)$ is the characteristic function of (a,b), is in $L^p(0,1)$. Then,

$$\begin{aligned}\mathcal{L}^\alpha_{\nu,\mu} f(y) &= y^\mu \int_0^1 (xy)^{\frac{1}{2}+\epsilon} J_\alpha(xy) x^{\alpha+\frac{1}{2}-2\epsilon}\left(1-x^2\right)^{-\frac{1}{p}+\epsilon} dx = y^{\mu+2\epsilon}\int_0^1 (xy)^{\frac{1}{2}} J_\alpha(xy) x^{\alpha+\frac{1}{2}}\left(1-x^2\right)^{-\frac{1}{p}+\epsilon} dx \\ &= y^{\mu+\epsilon+\frac{1}{p}-\frac{1}{2}} 2^{-\frac{1}{p}+\epsilon}\Gamma\left(1-\frac{1}{p}+\epsilon\right) J_{\alpha+\frac{1}{p'}+\epsilon}(y).\end{aligned} \tag{A.2}$$

A well-known large variable estimate for Bessel functions is that

$$J_m(s) = \sqrt{\frac{2}{\pi}}\frac{\cos(s - \frac{m+\frac{1}{2}}{2}\pi)}{s^{1/2}} + O\left(\frac{1}{s^{3/2}}\right).$$

We can underestimate $|J_{\alpha+\frac{1}{p'}+\epsilon}(s)|$ by replacing $|\cos(s - \frac{\alpha+\frac{1}{p'}+\epsilon+\frac{1}{2}}{2}\pi)|$ by $\frac{1}{2}$ when it is greater than or equal to $\frac{1}{2}$ and by 0 elsewhere. Specifically, if k is an integer, in the intervals I_k where s satisfies

$$\left|s - \frac{\alpha+\frac{1}{p'}+\epsilon+\frac{1}{2}}{2}\pi - k\pi\right| \leqslant \frac{\pi}{3},$$

then

$$\left|J_{\alpha+\frac{1}{p'}+\epsilon}(s)\right| \geqslant \sqrt{\frac{1}{2\pi}} s^{-1/2} + O\left(\frac{1}{s^{3/2}}\right) \geqslant \frac{1}{2\pi} s^{-1/2}$$

whenever s is sufficiently large.

Therefore, $|J_{\alpha+\frac{1}{p'}+\epsilon}(y)| \geqslant \frac{1}{2\pi} y^{-\frac{1}{2}}$ whenever $y \in \bigcup_k I_k$ and is sufficiently large. Recalling that $\mu = \frac{1}{p'} - \frac{1}{q}$, we can see at once from (A.2) that

$$\left|\mathcal{L}_{\nu,\mu}^{\alpha} f(y)\right| \geqslant c y^{-\frac{1}{q}+\epsilon}$$

for some positive c, whenever $y \in \bigcup_k I_k$ and is sufficiently large, and hence does not belong to $L^q(0,\infty)$.

- $\nu > -\alpha - \frac{1}{p'}$ *is necessary.* We now let $\nu = -\alpha - \frac{1}{p'} - \epsilon$, with $0 < \epsilon < 1$. We let $F(x) = x^{1+\frac{1}{p'}+\epsilon+2\alpha}(1-x^2)^{-\frac{1}{p}+\epsilon}\chi_{(0,1)}(x)$. By the identity (A.1),

$$\begin{aligned}\mathcal{L}_{\nu,\mu}^{\alpha} F(y) &= y^{\nu-\frac{1}{2}+\mu}\int_0^1 (xy)^{\frac{1}{2}} x^{(\nu-\frac{1}{2})+1+\frac{1}{p'}+\epsilon+2\alpha}\left(1-t^2\right)^{-\frac{1}{p}+\epsilon} J_\alpha(xt)\,dt \\ &= y^{-\alpha-\frac{1}{p'}-\epsilon-\frac{1}{2}+\mu}\int_0^1 (xy)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}}\left(1-x^2\right)^{-\frac{1}{p}+\epsilon} J_\alpha(xt)\,dt \\ &= 2^{-\frac{1}{p}+\epsilon}\Gamma\left(1-\frac{1}{p}+\epsilon\right) y^{-\alpha-\frac{1}{q}-\epsilon-\frac{1}{2}} J_{\alpha+\frac{1}{p'}+\epsilon}(y).\end{aligned}$$

We can use the well-known representation of the Bessel function into power series to infer that $J_m(z) = z^m(c_m + \mathcal{O}(z^2))$, where $c_m = \frac{1}{2^{m+1}\Gamma(m+1)}$. Therefore, $J_{\alpha+\frac{1}{p'}+\epsilon}(y) \geqslant \frac{1}{2} c_{\alpha+\frac{1}{p'}+\epsilon} y^{\alpha+\frac{1}{p'}+\epsilon}$ in a suitable neighborhood of $y=0$; in this neighborhood, $\mathcal{L}_{\nu,\mu}^{\alpha} F(y) > c y^{-\epsilon-\frac{1}{q}}$ for some $c>0$, and hence it is not in $L^q(0,\infty)$.

Appendix B. Proof of Proposition 2.1

It is easy to see that Gaussian functions attain the equality in (2.10). By (2.2), the Fourier–Bessel transform of order $\alpha = \frac{n}{2}-1$ is a constant multiple of the restriction of the Fourier transform to radial functions of $\mathbf{R}^n$. Consequently,

$$\|\widetilde{\mathcal{H}}_{\frac{n}{2}-1} f\|_{L^{p'}((0,\infty),r^{n-1}\,dr)} = (2\pi)^{-\frac{n}{2}}\left|S^{n-1}\right|^{\frac{1}{p'}}\left(\int_0^\infty \left|\hat{f}(r)\right|^{p'} r^{n-1}\,dr\right)^{\frac{1}{p'}} = \frac{\Gamma(\frac{n}{2})^{\frac{1}{p'}}}{(2\pi)^{\frac{n}{2}} 2^{\frac{1}{p'}} \pi^{\frac{n}{2p'}}}\left(\int_{\mathbf{R}^n} \left|\hat{f}\left(|x|\right)\right|^{p'} dx\right)^{\frac{1}{p'}}. \tag{B.1}$$

Furthermore,

$$\|f\|_{L^p((0,\infty),r^{n-1}\,dr)} = \frac{\Gamma(\frac{n}{2})^{\frac{1}{p}}}{2^{\frac{1}{p}} \pi^{\frac{n}{2p}}}\left(\int_{\mathbf{R}^n} \left|f\left(|x|\right)\right|^{p} dx\right)^{\frac{1}{p}}, \tag{B.2}$$

and from (B.2) and (B.1) and the theorems of Beckner and Lieb it follows that

$$\begin{aligned}\frac{\|\widetilde{\mathcal{H}}_{\frac{n}{2}-1} f\|_{L^{p'}((0,\infty),r^{n-1}\,dr)}}{\|f\|_{L^p((0,\infty),r^{n-1}\,dr)}} &= \frac{2^{\frac{1}{p}-\frac{1}{p'}} \pi^{\frac{n}{2p}-\frac{n}{2p'}}}{(2\pi)^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{(\int_{\mathbf{R}^n} |\hat{f}(x)|^{p'}\,dx)^{\frac{1}{p'}}}{(\int_{\mathbf{R}^n} |f(x)|^{p}\,dx)^{\frac{1}{p}}} \\ &\leqslant \frac{2^{\frac{1}{p}-\frac{1}{p'}} \pi^{\frac{n}{2p}-\frac{n}{2p'}}}{(2\pi)^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} (2\pi)^{\frac{n}{p'}}\left(p^{\frac{1}{p}}(p')^{-\frac{1}{p'}}\right)^n = \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{p^{\frac{n}{2p}}}{(p')^{\frac{n}{2p'}}} 2^{\frac{(n-2)(2-p')}{2p'}},\end{aligned}$$

as required.

Appendix C. Proof of Lemma 5.3

Let $H_m(x)$ be the classical Hermite polynomial of degree m. Beckner proved in [2] that the functions $H_m(\sigma(\bar{s}))$ can be $d\nu_n(\bar{s})$-approximated by symmetric functions. That is, for every $\bar{s} = (\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}})$,

$$H_m\left(\sigma(\bar{s})\right) = \psi_{m,n}(\bar{s}) + \frac{1}{n}\sum_{r=1}^{[\frac{m}{2}]} a_{m,r} H_{m-2r}\left(\sigma(\bar{s})\right), \tag{C.1}$$

where $\sigma(\bar{s}) = x_1 + \cdots + x_n$, and the $a_{m,r}$ are bounded with respect to n for a fixed m. We recall that $L_m^{-\frac{1}{2}}(\zeta^2) = \frac{(-1)^m}{2^{2m} m!} H_{2m}(\zeta)$. When $\alpha > -\frac{1}{2}$, the following identity holds

$$L_m^{\alpha}\left(\frac{1}{2}\zeta^2\right) = \frac{(-1)^m}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \frac{\Gamma(m+\alpha+1)}{(2m)!}\int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} H_{2m}\left(\frac{1}{\sqrt{2}}\zeta t\right) dt$$

$$= \frac{(-1)^m\sqrt{2}}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(m+\alpha+1)}{(2m)!}\int_{-\sqrt{2}}^{\sqrt{2}}\left(1-2t^2\right)^{\alpha-\frac{1}{2}}H_{2m}(\zeta t)\,dt \tag{C.2}$$

(see e.g. [13]).

We prove the lemma for $\alpha > -\frac{1}{2}$, since the proof is quite similar in the other case. The derivatives of H_k satisfy the following identity:

$$\frac{d^j}{d^j\zeta}H_k(\zeta) = 2^j j!\binom{k}{j}H_{k-j}(\zeta), \quad j \leqslant k.$$

By Taylor's formula

$$H_k(\zeta t) = \sum_{j=0}^{k}(2\zeta)^j(t-1)^j\binom{k}{j}H_{k-j}(\zeta),$$

and by (C.2),

$$L_m^\alpha\left(\frac{1}{2}\zeta^2\right) = c_{m,\alpha}\sum_{j=0}^{2m}(2\zeta)^j\binom{2m}{j}H_{2m-j}(\zeta)\int_{-\sqrt{2}}^{\sqrt{2}}\left(1-2t^2\right)^{\alpha-\frac{1}{2}}(t-1)^j\,dt, \tag{C.3}$$

where we have let $c_{m,\alpha}$ be the constant on the right-hand side of (C.2). By (C.3) and (C.1) the conclusion follows.

To prove (5.24) we recall that the moments of $d\nu_n(\bar{s})$ converge to the moments of $\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$ in the weak topology of $C^0(\mathbf{R})$; thus,

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}\left|\mathcal{R}_m^\alpha\left(\sigma(\bar{s})\right)\right|^q\left|\sigma(\bar{s})\right|^{2l+1}d\nu_n(\bar{s}) = \int_{\mathbf{R}}\left|\mathcal{R}_m^\alpha(x)\right|^q|x|^{2l+1}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx < \infty$$

and $\lim_{n\to\infty}\frac{1}{n^q}\int_{\mathbf{R}^n}|\mathcal{R}_m^\alpha(\sigma(\bar{s}))|^q|\sigma(\bar{s})|^{2l+1}\,d\nu_n(\bar{s}) = 0$.

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