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On a condition for α-starlikeness

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ABSTRACT

In this paper we give a sufficient condition for function to be α-starlike function and some of its applications. We use the techniques of convolution and differential subordinations.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $U = \{z\colon |z| < 1\}$ of the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$ and let

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A}\colon \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha \text{ for } z \in U \right\}$$

be the class of α-starlike functions, $\alpha \in [0, 1)$. $\mathcal{S}^*(0) = \mathcal{S}^*$ is the class of starlike functions which map U onto a starlike domain with respect to the origin. We say that $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in U, written $f \prec g$, if and only if there exists a function $\omega \in \mathcal{H}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that $f(z) = g(\omega(z))$ for $z \in U$. If $f \prec g$ in U, then $f(U) \subseteq g(U)$. Many classes of functions studied in geometric function theory can be described in terms of subordination. Let us denote

$$p_\gamma(z) = \frac{1 + \gamma z}{1 - z} = 1 + (1 + \gamma)\sum_{k=1}^{\infty} z^k \quad (z \in U). \tag{1}$$

If $\gamma \neq -1$ then the function p_γ maps U onto the half plane $\operatorname{Re} w > \frac{1-\gamma}{2}$ and it is easy to check that for $\gamma \in (-1, 1]$

$$\left\{ f \in \mathcal{A}\colon \frac{zf'(z)}{f(z)} \prec p_\gamma(z) \text{ in } U \right\} = \mathcal{S}^*\left(\frac{1-\gamma}{2}\right). \tag{2}$$

We say that the function $f \in \mathcal{H}$ is convex when $f(U)$ is a convex set. It is easy to see that if $\gamma \neq -1$ then p_γ is a convex univalent function.

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R. Singh and S. Singh [10] proved that if $f \in \mathcal{A}$ and $\mathrm{Re}\{f'(z) + zf''(z)\} > -\frac{1}{4}$ $(z \in U)$, then $f \in \mathcal{S}^*(0)$. Ponnusamy [4] improved this result by replacing the constant $-1/4$ by $-0.308\ldots$. Recently R. Szász and L.-R. Albert [9] checked using a computer that

$$\frac{1}{8} < \inf_{\alpha\in(0,\infty)} \bigl\{\forall f \in \mathcal{A}\ \bigl[\mathrm{Re}\bigl[f'(z) + \alpha z f''(z)\bigr] > 0 \ \Rightarrow\ f \in \mathcal{S}^*\bigr]\bigr\} < \frac{1}{7}.$$

In this paper we consider a similar sufficient condition for functions to be in the class $\mathcal{S}^*(\alpha)$.

For $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ the Hadamard product (or convolution) is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$. The convolution has the algebraic properties of ordinary multiplication. Many of convolution problems were studied by St. Ruscheweyh in [5] and have found many applications in various fields. One of them is the following theorem due to St. Ruscheweyh and J. Stankiewicz [8] which will be useful in this paper.

Theorem A. *Let $F, G \in \mathcal{H}$ be any convex univalent functions in U. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ in U.*

The next theorem is a special case of the Julia–Wolf Theorem. It is known as Jack's Lemma.

Theorem B. *(See [2].) Let $\omega(z)$ be meromorphic in U, $\omega(0) = 0$. If for a certain $z_0 \in U$ we have $|\omega(z)| \leqslant |\omega(z_0)|$ for $|z| \leqslant |z_0|$, then $z_0\omega'(z_0) = m\omega(z_0)$, $m \geqslant 1$.*

2. Main result

Lemma 1. *Let $\alpha > 0$, $\gamma \in \mathbb{R} \setminus \{-1\}$. If $f \in \mathcal{A}$ and $f'(z) + \frac{z}{\alpha} f''(z) \prec p_\gamma(z)$, then*

$$\frac{f(z)}{z} \prec 1 + \alpha(1+\gamma)\sum_{k=1}^{\infty} \frac{z^k}{(1+k)(k+\alpha)} := H(\alpha, \gamma; z) \tag{3}$$

and $H(\alpha, \gamma; z)$ is the best dominant in the sense that if $\frac{f(z)}{z} \prec G(z)$, then $H(\alpha, \gamma; z) \prec G(z)$.

Proof. For $x \geqslant 0$ the function

$$\tilde{h}(x; z) = \sum_{k=1}^{\infty} \frac{(1+x)z^k}{(k+x)}$$

is convex univalent [6]. Ruscheweyh and Sheil-Small in [7] proved the Pólya–Schoenberg conjecture that the class of convex univalent functions is preserved under convolution. Thus

$$g(z) = 1 + \frac{\alpha}{2+2\alpha}\bigl[\tilde{h}(1; z) * \tilde{h}(\alpha; z)\bigr] = 1 + \sum_{k=1}^{\infty} \frac{\alpha z^k}{(k+1)(k+\alpha)}$$

is a convex univalent function. Also p_γ is convex univalent so by Theorem A we have

$$\left[f'(z) + \frac{z}{\alpha} f''(z)\right] * g(z) \prec p_\gamma(z) * g(z).$$

It gives (3) because

$$\left[f'(z) + \frac{z}{\alpha} f''(z)\right] * g(z) = \frac{f(z)}{z}, \qquad p_\gamma(z) * g(z) = H(\alpha, \gamma; z).$$

The function $H(\alpha, \gamma; z)$ is convex univalent as the convolution of convex univalent functions p_γ and g. Suppose that $\frac{f(z)}{z} \prec G(z)$ for each $f \in \mathcal{A}$ such that $f'(z) + \frac{z}{\alpha} f''(z) \prec p_\gamma(z)$. The function $f_0(z) = zH(\alpha, \gamma; z)$ gives $f_0'(z) + \frac{z}{\alpha} f_0''(z) = p_\gamma(z)$ thus $\frac{f_0(z)}{z} = H(\alpha, \gamma; z) \prec G(z)$. This means that $H(\alpha, \gamma; z)$ is the best dominant of $\frac{f(z)}{z}$. □

For $\alpha > 0$ and $\gamma > -1$ the function $H(\alpha, \gamma; z)$ is convex univalent with positive coefficients so $H(U)$ is a convex set symmetric with respect to the real axis with

$$H(\alpha, \gamma; -1) < \mathrm{Re}\bigl[H(\alpha, \gamma; z)\bigr] < H(\alpha, \gamma; 1)$$

hence we have the following corollary.

Corollary 1. *Let $\alpha > 0$, $\gamma > -1$. If $f \in \mathcal{A}$ and $f'(z) + \frac{z}{\alpha} f''(z) \prec p_\gamma(z)$, then*

$$H(\alpha, \gamma; -1) < \mathrm{Re}\left[\frac{f(z)}{z}\right] < H(\alpha, \gamma; 1) \quad (z \in U). \tag{4}$$

Notice that

$$\sum_{k=1}^{\infty} \frac{\iota^k}{k(k+x)} = \begin{cases} \frac{1}{x}[\psi(x+1) + C] & \text{for } \iota = 1, \\ \frac{1}{x}[\mathcal{B}(x+1) - \ln 2] & \text{for } \iota = -1, \end{cases}$$

where

$$\mathcal{B}(z) = \int_0^1 \frac{t^{z-1}}{1+t}\,\mathrm{d}t = \sum_{k=0}^{\infty} \frac{1}{(z+2k)(z+2k+1)} \quad (\mathrm{Re}\, z > 0) \tag{5}$$

is the beta function while $\psi(z) = [\ln\Gamma(z)]'$, where Γ is the gamma function and C is the Euler's constant. Thus we have

$$H(\alpha, \gamma; -1) = \begin{cases} 1 + \alpha\frac{1+\gamma}{1-\alpha}[1 - \mathcal{B}(1+\alpha) - \ln 2] & \text{for } \alpha \in (0, +\infty) \setminus \{1\}, \\ 1 + (1+\gamma)(\frac{\pi^2}{12} - 1) & \text{for } \alpha = 1, \end{cases} \tag{6}$$

and

$$H(\alpha, \gamma; 1) = \begin{cases} 1 + \alpha\frac{1+\gamma}{1-\alpha}[1 - \psi(1+\alpha) - C] & \text{for } \alpha \in (0, +\infty) \setminus \{1\}, \\ 1 + (1+\gamma)(\frac{\pi^2}{6} - 1) & \text{for } \alpha = 1. \end{cases}$$

In order to check when $H(\alpha, \beta; -1) > 0$ it is useful to rewrite (6) in the form

$$H(\alpha, \gamma; -1) = \begin{cases} 1 + \alpha\frac{1+\gamma}{1-\alpha}[\mathcal{B}(2) - \mathcal{B}(1+\alpha)] & \text{for } \alpha \in (0, +\infty) \setminus \{1\}, \\ 1 + (1+\gamma)(\frac{\pi^2}{12} - 1) & \text{for } \alpha = 1. \end{cases} \tag{7}$$

Applying (5) we see that the function $\mathcal{B}$ is decreasing for $z > 0$ thus $\frac{\mathcal{B}(2) - \mathcal{B}(1+\alpha)}{1-\alpha} < 0$ for $\alpha \neq 1$. Therefore by (7) we conclude that

$$H(\alpha, \gamma; -1) > 0 \quad \Leftrightarrow \quad \gamma < g(\alpha) := \begin{cases} -1 - \frac{1-\alpha}{\alpha[\mathcal{B}(2) - \mathcal{B}(1+\alpha)]} & \text{for } \alpha \in (0, +\infty) \setminus \{1\}, \\ \frac{\pi^2}{12 - \pi^2} = 4.6327\ldots & \text{for } \alpha = 1. \end{cases} \tag{8}$$

The above result will be useful in the following theorem.

Theorem 1. *Let $\alpha \in (0, 1]$ and $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*(\frac{1-\alpha}{2})$ whenever for $z \in U$*

$$\mathrm{Re}\left[f'(z) + \frac{z}{\alpha} f''(z)\right] > \frac{1 - \gamma(\alpha)}{2} := 1 - \frac{\alpha^2 + 3\alpha + 2}{2\alpha[2 - (\alpha^2 - \alpha + 2)B(\alpha)]} \quad \text{and} \quad \gamma(\alpha) < g(\alpha), \tag{9}$$

where

$$B(\alpha) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(1+k)(k+\alpha)} = \begin{cases} \frac{1}{1-\alpha}[1 - \mathcal{B}(1+\alpha) - \ln 2] & \text{for } \alpha \in [0, 1), \\ \frac{\pi^2}{12} - 1 & \text{for } \alpha = 1. \end{cases}$$

Proof. For convenience, in this proof we will drop the variable α in $\gamma(\alpha)$. From (9) we have $f'(z) + \frac{z}{\alpha} f''(z) \prec p_\gamma(z)$. We have $\gamma < g(\alpha)$ thus, by Corollary 1 and by (8)

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > H(\alpha, \gamma; -1) > 0 \quad (z \in U). \tag{10}$$

This gives $\frac{f(z)}{z} \neq 0$, $z \in U$. Moreover the function $p_\alpha(z) = \frac{1+\alpha z}{1-z}$, $p_\alpha(\infty) = -\alpha$, maps $\overline{\mathbb{C}} \setminus \{1\}$ onto $\mathbb{C}$ and it is univalent so a function $\omega(z)$, $\omega(0) = 0$, defined by

$$\omega(z) = p_\alpha^{-1}\left(\frac{zf'(z)}{f(z)}\right) \tag{11}$$

is analytic in U. In view of (2) for proving Theorem 1 it is sufficient to show that $\frac{zf'(z)}{f(z)} \prec p_\alpha(z)$ or equivalently that $\omega(z)$ is bounded by 1 in U. If this is false we find $z_0 \in U$ such that $|\omega(z)| \leqslant |\omega(z_0)| = 1$, $|z| \leqslant |z_0|$. According to Theorem B, $\frac{z_0\omega'(z_0)}{\omega(z_0)} = m \geqslant 1$. Taking the derivative of (11) we obtain after some manipulations the relation

$$f'(z_0) + \frac{z_0}{\alpha} f''(z_0) = \frac{f(z_0)}{\alpha z_0}\left[\frac{z_0\omega'(z_0)}{\omega(z_0)} \frac{(1+\alpha)\omega(z_0)}{(1-\omega(z_0))^2} + p_\alpha^2\big(\omega(z_0)\big) - (1-\alpha)p_\alpha\big(\omega(z_0)\big)\right]. \tag{12}$$

If we denote $\omega(z_0) = e^{i\varphi}$, $\varphi \in [0, 2\pi)$, then we have

$$\frac{2\omega(z_0)}{(1-\omega(z_0))^2} = \frac{1}{\cos\varphi - 1} < 0, \qquad p_\alpha\big(\omega(z_0)\big) = \frac{1+\alpha\omega(z_0)}{1-\omega(z_0)} = \frac{1-\alpha}{2} + i\frac{1+\alpha}{2}\operatorname{ctg}\frac{\varphi}{2},$$

so the quantity in the square brackets of (12) becomes

$$[\ldots] = \frac{2m(1+\alpha) + (1+\alpha)^2(1+\cos\varphi)}{4(\cos\varphi - 1)} - \left[\frac{1-\alpha}{2}\right]^2 =: \delta.$$

It is easy to see that δ is a negative real number so from (4) and (12) we have

$$\frac{\delta}{\alpha}H(\alpha,\gamma;1) < \operatorname{Re}\left[f'(z_0) + \frac{z_0}{\alpha}f''(z_0)\right] < \frac{\delta}{\alpha}H(\alpha,\gamma;-1) = \frac{\delta}{\alpha}\big[1+\alpha(1+\gamma)B(\alpha)\big]. \tag{13}$$

According to (10) we have $H(\alpha,\gamma;-1) = 1+\alpha(1+\gamma)B(\alpha) > 0$. Moreover

$$\delta = \alpha + \frac{(1+\alpha)^2 + m(1+\alpha)}{2(\cos\varphi - 1)} \leqslant \alpha + \frac{(1+\alpha)^2 + (1+\alpha)}{2(-1-1)} = -\frac{\alpha^2 - \alpha + 2}{4}.$$

Therefore we obtain from (13)

$$\operatorname{Re}\left[f'(z_0) + \frac{z_0}{\alpha}f''(z_0)\right] \leqslant -\frac{\alpha^2-\alpha+2}{4\alpha}\big[1+\alpha(1+\gamma)B(\alpha)\big] = \frac{1-\gamma}{2}$$

which contradicts our assumption (9). □

3. Some applications

In this section we shall look at some examples where we see how our result improve earlier results.

If $\alpha = 1$, then by (8) and (9) we obtain $\frac{1-\gamma(1)}{2} = \frac{6-\pi^2}{24-\pi^2}$, $\gamma(1) = \frac{12+\pi^2}{24-\pi^2} = 1.54\ldots$ and $\gamma(1) < g(1) = 4.63\ldots$. Therefore Theorem 1 becomes

Corollary 2. *If $f \in \mathcal{A}$ then $f \in \mathcal{S}^*(0) = \mathcal{S}^*$ whenever*

$$\operatorname{Re}\big[f'(z) + zf''(z)\big] > \frac{6-\pi^2}{24-\pi^2} = -0.273\ldots \quad (z \in U). \tag{14}$$

The integral form of above result due to Miller and Mocanu one can find in [3, p. 309]. Moreover the constant given in (14) is a little grater than $-0.308\ldots$ given by Ponnusamy [4].

Let us consider $\alpha = 1/2$. If $-1 \leqslant x \leqslant 1$ then

$$\sum_{k=1}^{\infty}\frac{(-1)^{(k-1)}x^{2k}}{k(2k-1)} = 2x\arctan x - \ln\big(1+x^2\big)$$

so $B(1/2) = \sum_{k=1}^{\infty}\frac{(-1)^k}{(1+k)(1/2+k)} = \pi - \ln 4 - 2 = -0.24\ldots$. Thus we have

$$\gamma(1/2) = -1 + \frac{30}{22-7(\pi-\ln 4)} = 2.088\ldots \quad \text{and} \quad g(1/2) = -1 - \frac{2}{\pi - \ln 4 - 2} = 7.17\ldots.$$

Therefore $\gamma(1/2) < g(1/2)$ and Theorem 1 becomes the following result.

Corollary 3. *If $f \in \mathcal{A}$ then $f \in \mathcal{S}^*(1/4)$ whenever*

$$\operatorname{Re}\big[f'(z) + 2zf''(z)\big] > 1 - \frac{15}{22-7(\pi-\ln 4)} = -0.541\ldots \quad (z \in U).$$

Let us consider $\alpha = 1/3$. If $-1 < x \leqslant 1$ then

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{3k+1}}{3k+1} = \frac{1}{3}\ln\frac{1+x}{\sqrt{x^2-x+1}} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + \frac{\pi}{6\sqrt{3}}$$

and if $-1 \leqslant x < 1$ then

$$\sum_{k=1}^{\infty}\frac{x^k}{k} = \ln\frac{1}{1-x}$$

so

$$B(1/3)=9\sum_{k=1}^{\infty}\frac{(-1)^k}{(3k+3)(3k+1)}=\frac{9}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{3k+1}-\frac{3}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k+1}$$
$$=\frac{9}{2}\left[\frac{1}{3}\ln 2+\frac{\pi}{3\sqrt{3}}-1\right]-\frac{3}{2}[\ln 2-1]=\frac{3\pi}{2\sqrt{3}}-3=-0.279\ldots.$$

Thus we have

$$\gamma(1/3)=-1+\frac{14\sqrt{3}}{11\sqrt{3}-4\pi}=2.738\ldots \quad \text{and} \quad g(1/3)=-1-\frac{2\sqrt{3}}{\pi-2\sqrt{3}}=9.74\ldots.$$

Therefore $\gamma(1/3)<g(1/3)$ and we obtain the following result.

Corollary 4. *If $f\in\mathcal{A}$ then $f\in\mathcal{S}^*(1/3)$ whenever*

$$\mathrm{Re}\big[f'(z)+3zf''(z)\big]>1-\frac{7\sqrt{3}}{11\sqrt{3}-4\pi}=-0.869\ldots \quad (z\in U).$$

Let us consider $\alpha=1/4$. If $-1<x\leqslant 1$ then

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+1}}{4k+1}=\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\big[\arctan(x\sqrt{2}+1)+\arctan(x\sqrt{2}-1)\big].$$

Thus

$$B(1/4)=16\sum_{k=1}^{\infty}\frac{(-1)^k}{(4k+4)(4k+1)}=\frac{16}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k+1}-\frac{4}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k+1}$$
$$=\frac{16}{3}\left[\frac{1}{4\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}+\frac{\pi}{4\sqrt{2}}-1\right]-\frac{4}{3}[\ln 2-1]=-0.3\ldots.$$

Thus we have

$$\gamma(1/4)=-1+\frac{180}{32-21B(1/4)}=3.699\ldots \quad \text{and} \quad g(1/4)=-1-\frac{4}{B(1/4)}=12.3\ldots.$$

Therefore $\gamma(1/4)>g(1/4)$ and Theorem 1 gives the following result.

Corollary 5. *If $f\in\mathcal{A}$ then $f\in\mathcal{S}^*(3/8)$ whenever*

$$\mathrm{Re}\big[f'(z)+4zf''(z)\big]>1-\frac{90}{32-21B(1/4)}=-1.349\ldots \quad (z\in U).$$

If $\alpha\to 0$ then Theorem 1 becomes the next corollary.

Corollary 6. *If $f\in\mathcal{A}$ then $f\in\mathcal{S}^*(1/2)$ whenever*

$$\mathrm{Re}\big[zf''(z)\big]>-\frac{2}{4+2B(0)}=-\frac{1}{3-\ln 4}=-0.61969\ldots \quad (z\in U).$$

Corollary 6 is analogous to a sharp result of the form

$$f\in\mathcal{A} \quad \text{and} \quad \mathrm{Re}\big[zf''(z)\big]>-\frac{3}{8\ln 2}=-0.721\ldots \quad \Rightarrow \quad f\in\mathcal{S}^*$$

obtained by Ali, Ponnusamy and Singh in [1], see also [3, pp. 275–277] for the other results.

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