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Similarity solutions for liquid metal systems near a sharply cornered conductive region

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ABSTRACT

We study the boundary value problem $(P_{m,a})$: $f''' + [(m+1)/2]ff'' - mf'^2 = 0$ on $(0, +\infty)$, subject to the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = -1$ and $f'(+\infty) = 0$. The problem arises in the study of similarity solutions for high frequency excitation of liquid metal systems in an antisymmetric magnetic field. We give a complete picture of solutions of $(P_{m,a})$ for the physical interesting case: $m < -1$ and $a \geqslant 0$.

1. Introduction

In the metallurgical industry, alternating magnetic fields are widely used to control the motion of liquid metal, to adjust their shape, and to generate "internal" stirring to make their interior structure more homogeneous, all of which are unique to magnetohydrodynamics. For example, it is known that the stirring of liquid steel in the process of continuous casting of steel can produce a more homogeneous finished product by eliminating blowholes which is caused by escaping gases. However, here it is impossible to use the traditional mechanical stirring. This is due to the fact that the liquid metal is encased in the solidified steel. In contrast, applying alternating magnetic fields can induce eddy currents in the metal. These induced currents can interact with the magnetic field to give rise to a rotational Lorenz force which can drive internal fluid motion without the need for any mechanical contact (see [16,17]).

For those applications making use of the above effects induced by alternating magnetic fields, a high frequency magnetic field is commonly used. In this circumstance, the magnetic field can only penetrate a small distance into the metal, and the induced currents and the associated Lorentz force are then confined to a thin surface layer, which is the well-known "skin effect." Moreover, if the frequency of the magnetic field is sufficiently high, these effects can be very strong in the neighborhood of any sharp corners on the rigid boundary of the fluid domain, and the magnetic field and Lorentz force can be very large inside such a singular region. Note that sharp corners can appear in many practical contexts. For example, the channel has a square cross section in the typical induction furnaces, and the melt is commonly extruded from a chamber of square cross-section during the continuous casting process. For more detail on the physical background, we refer the reader to [16,14,15,19,17].

Therefore, it is of great interest to study the dynamical behaviour of the fluid near these sharp corners. Note that it has been shown that the net effect of the Lorentz force is to induce an effective surface velocity just inside the surface layer on the rigid boundary. It turns out that the study of the flow becomes a purely fluid mechanical problem of determining

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the flow inside the liquid metal subject to the prescribed velocity on the rigid boundary (see [16]). In this circumstance, a Prandtl's boundary-layer description of the driven flow is appropriate, and so, a deeper understanding of the structure of similarity solutions of this problem would be essential. These similarity solutions are governed by the following boundary value problem $(P_{m,a})$:

$$f''' + \big[(m+1)/2\big] f f'' - m f'^2 = 0, \quad t \in (0,+\infty), \tag{1.1}$$

$$f(0) = a \in \mathbb{R}, \qquad f'(0) = -1, \qquad f'(+\infty) := \lim_{t\to+\infty} f'(t) = 0, \tag{1.2}$$

where $f = f(t)$ for $t \in [0,+\infty)$ and m, a are two real parameters. Note that the range for which the problem $(P_{m,a})$ has physical meaning is $m < -1$ and $a = 0$. The problem $(P_{m,a})$ is proposed by Moffatt [16]. Brighi and Hoernel [6] have used the *direct approach* and the so-called *blowing-up coordinates* to establish a complete picture of the structure of solutions of $(P_{m,a})$ for the case: $m > 0$ and $a \in \mathbb{R}$, and the case: $m < -1$ and $a < 0$. For the remaining cases, their results are far from complete. In particular, for the case $m < -1$ and $a \geqslant 0$, which is of physical interest, they can only show that there is a unique convex solution of $(P_{m,a})$.

Eq. (1.1) also arises in another physical context. Indeed, similarity solutions of the free convection boundary layer flows near a vertical flat plate embedded in porous medium are solutions of the following boundary value problem $(\widehat{P}_{m,a})$:

$$f''' + \big[(m+1)/2\big] f f'' - m f'^2 = 0, \quad t \in (0,+\infty),$$

$$f(0) = a \in \mathbb{R}, \qquad f'(0) = 1, \qquad f'(+\infty) := \lim_{t\to+\infty} f'(t) = 0.$$

The problem $(\widehat{P}_{m,a})$ has been investigated for many years and received much attention (see [20,9,13,7,8,1,2,11,12,3–5,18] and references therein). Note that the solutions of $(\widehat{P}_{m,a})$ depend on two parameters: m, the power-law exponent and a, the mass transfer parameter. At first glance, the problem $(P_{m,a})$ is quite similar to the problem $(\widehat{P}_{m,a})$ except the initial condition on the first derivative of the solution. However, the structure of solutions of $(P_{m,a})$ is very much different from that of $(\widehat{P}_{m,a})$. For example, the problem $(\widehat{P}_{m,a})$ with $m < -1$ and $a \geqslant 0$ has no solution (see [3]), while the problem $(P_{m,a})$ with $m < -1$ and $a \geqslant 0$ admits infinitely many solutions. Moreover, the profiles of solutions of $(P_{m,a})$ are significantly different from those of solutions of $(\widehat{P}_{m,a})$.

Let us digress for a moment to consider the following boundary value problem $(Q_{m,a,c})$:

$$f''' + \big[(m+1)/2\big] f f'' - m f'^2 = 0, \quad t \in (0,+\infty),$$

$$f(0) = a \in \mathbb{R}, \qquad f'(0) = c, \qquad f'(+\infty) := \lim_{t\to+\infty} f'(t) = 0.$$

We note that if f is a solution of $(Q_{m,a,c})$ with $c < 0$ (resp. $c > 0$), then $f(\cdot/\sqrt{|c|})/\sqrt{|c|}$ is a solution of $(P_{m,a/\sqrt{|c|}})$ (resp. $(\widehat{P}_{m,a/\sqrt{|c|}})$). In view of this, a good understanding of the structure of solutions of $(P_{m,a})$ and $(\widehat{P}_{m,a})$ can help us to understand the structure of solutions of $(Q_{m,a,c})$. Therefore, for physical interest and mathematical completeness, the aim of this paper is to investigate the structure of solutions of $(P_{m,a})$ with $m < -1$ and $a \geqslant 0$. To begin with, let us consider the following initial value problem $(P_{m,a,b})$:

$$f''' + \big[(m+1)/2\big] f f'' - m f'^2 = 0 \quad \text{on } [0, T_b),$$

$$f(0) = a, \qquad f'(0) = -1, \qquad f''(0) = b,$$

where $b \in \mathbb{R}$ and $[0, T_b)$ is the (right) maximal existence interval of the solution. Then our main result is the following theorem:

Theorem 1. *Fix $m < -1$ and $a \geqslant 0$. Let f_b be the solution of $(P_{m,a,b})$.*

- *If $a \in [0, \sqrt{6})$, then there exist $0 < b_{m,a,-} < b_{m,a,+}$ such that the following hold:*
  - (i) *f_b is a solution of $(P_{m,a})$ if and only if $b \in [b_{m,a,-}, b_{m,a,+}]$.*
  - (ii) *If $b = b_{m,a,-}$, then f_b is a convex solution of $(P_{m,a})$ and satisfies that $f_b(t) f_b'(t)/f_b''(t) \to -2/(m+1)$ and $f_b'^2(t)/[f_b(t) f_b''(t)] \to 0$ as $t \to +\infty$.*
  - (iii) *If $b \in (b_{m,a,-}, b_{m,a,+})$, then f_b is a convex-concave solution of $(P_{m,a})$ and satisfies that $f_b(t) f_b'(t)/f_b''(t) \to +\infty$ and $f_b'^2(t)/[f_b(t) f_b''(t)] \to (m+1)/(2m)$ as $t \to +\infty$.*
  - (iv) *If $b = b_{m,a,+}$, then f_b is a convex-concave solution of $(P_{m,a})$ and satisfies that $f_b(t) f_b'(t)/f_b''(t) \to -2/(m+1)$ and $f_b'^2(t)/[f_b(t) f_b''(t)] \to 0$ as $t \to +\infty$.*
- *If $a = \sqrt{6}$, there is a unique solution f of $(P_{m,a})$ which is given by $f(t) = 6/(t + \sqrt{6})$.*
- *If $a > \sqrt{6}$, then there exists a positive number $b_{m,a}$ such that the following hold:*

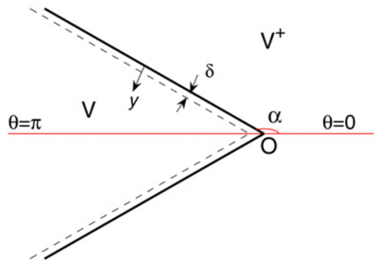

Fig. 1. Corner configuration. δ is the skin thickness and y is the normal boundary layer coordinate.

(i) *f_b is a solution of $(P_{m,a})$ if and only if $b = b_{m,a}$.*

(ii) *If $b = b_{m,a}$, then f_b is a convex solution of $(P_{m,a})$ and satisfies that $f_b(t)f_b'(t)/f_b''(t) \to -\infty$ and $f_b'^2(t)/[f_b(t)f_b''(t)] \to (m+1)/(2m)$ as $t \to +\infty$.*

Here a convex solution f of $(P_{m,a})$ means that $f'' > 0$ on $[0, +\infty)$, and a convex-concave solution f of $(P_{m,a})$ means that there exists $\hat{t}_0 > 0$ such that $f'' > 0$ on $[0, \hat{t}_0)$, $f'' < 0$ on $(\hat{t}_0, +\infty)$.

Our method for deriving the structure of solutions of $(P_{m,a})$ is to use a change of variables introduced in [18] to transform Eq. (1.1) into a system of two first order equations (see (3.1)–(3.2)). Unlike the usual transformation, the resulting system (3.1)–(3.2) for our transformation is autonomous, not non-autonomous. The key idea of our strategy is to use the ω-limit set of the system (3.1)–(3.2) to establish a very delicate correspondence between the solutions of $(P_{m,a,b})$ and the trajectories of the system (3.1)–(3.2). With such a correspondence, it is possible to show our results. To this end, various auxiliary solutions of (1.1) and trajectories of (3.1)–(3.2) are constructed.

Finally, the plan of this paper is organized as follows. In Section 2, we will briefly describe how to derive the problem $(P_{m,a})$. Section 3 is concerned with some useful properties for the solutions of $(P_{m,a,b})$. Then in Section 4, we will establish the correspondence between the convex and convex-concave solutions of $(P_{m,a})$ and the trajectories of the system (3.1)–(3.2), while Section 5 is devoted to constructing the correspondence between the solution of $(P_{m,a,b})$ and the ω-limit set in the second quadrant of the phase plane associated with the system (3.1)–(3.2). With the aid of these results in Sections 4 and 5, we will prove Theorem 1 in Section 6. Finally, a summary and discussion of the results is given in Section 7.

2. Derivation of the governing equation

For the reader's convenience, we shall follow Moffatt [16] to give a brief sketch of deriving the problem (1.1)–(1.2). For more detail, we refer the reader to [16,17]. We first assume that the fluid (liquid metal) is contained in the region V: $\alpha < \theta < 2\pi - \alpha$, whose boundary S is rigid electrically insulating and consists of $\theta = \alpha$ and $2\pi - \alpha$. The exterior region is denoted by V^+. We shall concentrate on the case: a single-phase high frequency field which is antisymmetric with respect to the corner bisector (see Fig. 1). Then as Moffatt [16] suggests, the net effect of the rotational Lorentz force within the skin is to generate an effective tangential velocity $\mathbf{u}_S$ on S given by

$$\mathbf{u}_S = \frac{\delta^2}{16\mu_0\rho\nu}\nabla|\mathbf{B}_S|^2.$$

Here $\mathbf{B}_S$ is the magnetic field on the surface of the rigid boundary S, δ the thickness of the surface layer into which the magnetic field penetrates, μ_0 the permeability of the metal, ρ the density of the fluid and ν the kinematic viscosity of the fluid. The associated potential for an antisymmetric field is given by

$$\Psi = \psi r^\lambda \sin\lambda\theta,$$

which together with the normal condition $\mathbf{B}\cdot\mathbf{n} = 0$ on the boundary $\theta = \pm\alpha$, gives

$$\lambda = \frac{\pi}{2\alpha}.$$

Here (r,θ) is the usual polar coordinates. Therefore the magnetic field on the surface of the rigid boundary S is

$$\mathbf{B}_S = \frac{\partial\Psi}{\partial r}\bigg|_{\theta=\pm\alpha}\hat{\mathbf{r}} = \pm\psi\lambda r^{\lambda-1}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector associated with the coordinate r. Since $\pi/2 < \alpha < \pi$, we obtain $-1/2 < \lambda - 1 < 0$, and so the magnetic field $\mathbf{B}_S$ is singular at $r = 0$. Now it easily follows that

$$\nabla|\mathbf{B}_S|^2 = 2\psi^2\lambda^2(\lambda-1)r^{2\lambda-3}\hat{\mathbf{r}}.$$

Hence the induced surface velocity is given by

$$\mathbf{u}_S = \frac{\delta^2}{16\mu_0\rho\nu}\nabla|\mathbf{B}_S|^2 = Ar^m\hat{\mathbf{r}} \quad \text{on } \theta = \pm\alpha.$$

Here

$$A = \frac{\delta^2\psi^2}{8\mu_0\rho\nu}\left(\frac{\pi}{2\alpha}\right)^2\left(\frac{\pi}{2\alpha}-1\right) \quad \text{and} \quad m = \frac{\pi}{\alpha}-3.$$

Since $\alpha > \pi/2$, we have $m < -1$ and $A < 0$. This implies that the induced surface velocity is towards the corner, and its magnitude tends to $+\infty$ as $r \to 0^+$. Moreover, the local Reynolds number is given by

$$\mathrm{Re} := |A|\nu^{-1}r^{\pi/\alpha-2} \to +\infty \quad \text{as } r \to 0^+.$$

Therefore, a boundary layer treatment is appropriate for small r. To this end, let Ox be directed along the boundary $\theta = \alpha$, and Oy be normal to it and point into the fluid. From the configuration of the corner, we may assume that there is no pressure gradient outside the boundary layer. Then the Prandtl boundary layer equation describing zero pressure gradient incompressible planar flow in the limit of high Reynolds number can be stated as follows (see [10]):

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3}$$

subject to the boundary conditions

$$\psi = 0, \qquad \psi_y = Ax^m \quad \text{on } y = 0,$$
$$\psi_y \to 0 \quad \text{as } y \to +\infty,$$

where ψ is the stream function. A similarity solution of the form

$$\psi(x,y) = \left(\nu|A|x^{m+1}\right)^{1/2} f(\eta), \quad \eta = \left(|A|x^{m+1}\nu^{-1}\right)^{1/2} y$$

then gives rise to the boundary value problem $(P_{m,a})$ with $a = 0$:

$$f''' + \left[(m+1)/2\right]ff'' - mf'^2 = 0,$$
$$f(0) = 0, \qquad f'(0) = -1, \qquad f'(+\infty) = 0.$$

3. Mathematical preliminaries

3.1. General properties

In order to study $(P_{m,a})$, we recall the following initial value problem $(P_{m,a,b})$:

$$f''' + \left[(m+1)/2\right]ff'' - mf'^2 = 0 \quad \text{on } [0, T_b),$$
$$f(0) = a, \qquad f'(0) = -1, \qquad f''(0) = b,$$

where m, a, and $b \in \mathbb{R}$, and $[0, T_b)$ is the (right) maximal existence interval of the solution.

First, let us recall some useful properties of solutions of $(P_{m,a})$ from [2,6].

Proposition 1. *Let $m < -1$ and f be the solution of $(P_{m,a,b})$ with the right maximal existence interval $[0, T)$. Then the following hold*:

(i) *If $f''(\bar{t}_0) \leqslant 0$ for some $\bar{t}_0 \in [0, T)$, then $f''(t) < 0$ for all $t \in (\bar{t}_0, T)$.*
(ii) *There exists no t_0 such that $f'(t_0) = f''(t_0) = 0$.*
(iii) *f satisfies the following equality*:

$$E_f(t) := f''(t) + \left[(m+1)/2\right]f(t)f'(t) = b - \left[(m+1)/2\right]a + \left[(3m+1)/2\right]\int_0^t f'(s)^2\,ds \quad \text{for all } t \in [0, T).$$

(iv) *If f is a solution of $(P_{m,a})$, then $f''(0) > 0$ and $f''(+\infty) = 0$. Moreover, f must be the following two types*:
  (I) *either f is convex and decreasing on $[0, +\infty)$ and f is bounded,*
  (II) *or there exist t_0 and $\hat{t}_0$ with $0 < t_0 < \hat{t}_0$ such that $f' < 0$ on $[0, t_0)$, $f' > 0$ on $(t_0, +\infty)$, and $f'' > 0$ on $[0, \hat{t}_0)$, $f'' < 0$ on $(\hat{t}_0, +\infty)$. Moreover, f is negative at infinity.*

Hereafter, we will call a type (I) *solution of* $(P_{m,a})$ *as a convex solution of* $(P_{m,a})$*, and a type* (II) *solution of* $(P_{m,a})$ *as a convex-concave solution of* $(P_{m,a})$.

The following lemma concerns the long time behaviour of the first derivative of the solution of $(P_{m,a,b})$. Since the proof is almost immediate (see [9,3,18]), we omit it.

Lemma 3.1. *Let* $m < 0$ *and* f *be the solution of* $(P_{m,a,b})$ *defined on* $[0, T)$. *Suppose that* f'' *has a fixed sign on* $[t_1, T)$ *for some* $t_1 \in [0, T)$. *Then either* $f' \to \pm\infty$ *or* $f' \to 0$ *as* $t \to T^-$.

Next, we will see that the second derivative of the solution f of $(P_{m,a,b})$ must vanish if the critical point of f exists, whose proof is postponed to Appendix A.

Lemma 3.2. *Let* $m < -1$ *and* f *be a solution of* (1.1) *defined on the maximal existence interval* (S, T). *If there exists* $t_0 \in (S, T)$ *such that* $f'(t_0) \geqslant 0$ *and* $f''(t_0) > 0$, *then there exists a* $\hat{t}_0 > t_0$ *such that* $f'' > 0$ *on* $(t_0, \hat{t}_0)$ *and* $f'' < 0$ *on* $(\hat{t}_0, T)$.

Proposition 2. *Let* $m < 0$ *and* f *be the solution of* $(P_{m,a,b})$ *defined on the maximal existence interval* $[0, T)$. *Then we can classify* f *into the following three types*:

(A) $f' < 0$ *on* $[0, +\infty)$ *and* $\lim_{t\to+\infty} f'(t) = 0$.
(B) *There exists a* $t_0 > 0$ *such that* $f' < 0$ *on* $[0, t_0)$ *and* $f'(t_0) = 0$. *Furthermore, any type* (B) *solution* f *of* $(P_{m,a,b})$ *can be classified into the following two types*:
  (B$_1$) *There exist* t_0 *and* $\hat{t}_0$ *with* $0 < t_0 < \hat{t}_0$ *such that* $f' < 0$ *on* $[0, t_0)$, $f' > 0$ *on* $(t_0, +\infty)$, $f' \to 0$ *as* $t \to +\infty$, *and* $f'' > 0$ *on* $[0, \hat{t}_0)$, $f'' < 0$ *on* $(\hat{t}_0, +\infty)$.
  (B$_2$) *There exist* t_0, $\hat{t}_0$, *and* t_1 *with* $0 < t_0 < \hat{t}_0 < t_1$ *such that* $f' < 0$ *on* $[0, t_0)$, $f' > 0$ *on* (t_0, t_1), $f' < 0$ *on* (t_1, T), *and* $f'' > 0$ *on* $[0, \hat{t}_0)$, $f'' < 0$ *on* $(\hat{t}_0, T)$.
(C) $f' < 0$ *on* $[0, T)$, *and there exists a* $\hat{t}_0 \geqslant 0$ *such that* $f'' > 0$ *on* $(0, \hat{t}_0)$ *and* $f'' < 0$ *on* $(\hat{t}_0, T)$. *Moreover,* f *is of type* (C) *for all* $b \leqslant 0$.

Proof. If $f''(0) \leqslant 0$, then by part (i) of Proposition 1, f is of type (C) with $\hat{t}_0 = 0$. It remains to consider the case $f''(0) > 0$. To this end, we need to distinguish two disjoint cases:

(i) $f' < 0$ on $[0, T)$.
(ii) There exists a $t_0 > 0$ such that $f' < 0$ on $[0, t_0)$ and $f'(t_0) = 0$.

Case (i): If $f'' > 0$ on $[0, T)$, then by Lemma 3.1, we have $T = +\infty$ and $f'(t) \to 0$ as t approaches $+\infty$. This implies that f is of type (A). Otherwise, there exists a $\hat{t}_0 > 0$ such that $f'' > 0$ on $[0, \hat{t}_0)$ and $f''(\hat{t}_0) = 0$. Then by Proposition 1, we have $f'' < 0$ on $(\hat{t}_0, T)$. Hence f is of type (C).

Case (ii): Note that $f''(t_0) > 0$ by part (ii) of Proposition 1 and that there exists a $\hat{t}_0 > t_0$ such that $f''(\hat{t}_0) = 0$ by Lemma 3.2. Hence by part (i) of Proposition 1, $f'' > 0$ on $[0, \hat{t}_0)$ and $f'' < 0$ on $(\hat{t}_0, T)$. Now if $f' > 0$ on (t_0, T), then by Lemma 3.1, we have $T = +\infty$ and $f'(t) \to 0$ as t approaches $+\infty$. This implies that f is of type (B$_1$). Otherwise, there exists a $t_1 > \hat{t}_0$ such that $f' < 0$ on $[0, t_0)$, $f' > 0$ on (t_0, t_1), and $f'(t_1) = 0$. Since $f'' < 0$ on $(\hat{t}_0, T)$, we have $f' < 0$ on (t_1, T). Hence f is of type (B$_2$). This completes the proof. $\square$

3.2. *A useful transformation*

For further distinction of solutions of $(P_{m,a,b})$, we need to transform (1.1) into a system of two first order equations. Specifically, let f be a solution of (1.1) defined on the open interval (d_1, d_2) with $d_1, d_2 \in \mathbb{R}$, such that $f(t)$, $f'(t)$, and $f''(t)$ are not equal to zero for all $t \in (d_1, d_2)$. Now we introduce the following change of variables for the solution f of (1.1):

$$X(\xi) = f(t)f'(t)/f''(t), \qquad Y(\xi) = f'(t)^2/\big[f(t)f''(t)\big], \quad \xi = -\ln\big|f'(t)\big|,$$

where we require $t \in (d_1, d_2)$. Then (X, Y) satisfies the following ordinary differential system

$$\frac{dX}{d\xi} = -X\big\{1 + \big[(m+1)/2\big]X + Y - mXY\big\} := -F(x, y), \tag{3.1}$$

$$\frac{dY}{d\xi} = -Y\big\{2 + \big[(m+1)/2\big]X - Y - mXY\big\} := -G(x, y). \tag{3.2}$$

We note that such a transformation has been introduced in [18].

Note that if f is a solution of (1.1), then so is the function $g_{k,c} : t \mapsto kf(kt + c)$ for all $k > 0$ and $c \in \mathbb{R}$. It is easy to check that the trajectory of (3.1)–(3.2) corresponding to f is the same as that of $g_{k,c}$. Conversely, we have the following lemma which forms the basis for our discussion.

Lemma 3.3. *Let $m<0$ and f_i, $i=1,2$, be a solution of (1.1) defined on the maximal existence interval (T_{i1},T_{i2}). Suppose that there exist d_{i1}, d_{i2}, $i=1,2$, with $(d_{i1},d_{i2})\subseteq(T_{i1},T_{i2})$, such that $f_i(t)$, $f_i'(t)$, and $f_i''(t)$ are not equal to zero for all $t\in(d_{i1},d_{i2})$ and that $f_1(s_1)f_2(s_2)$, $f_1'(s_1)f_2'(s_2)$, and $f_1''(s_1)f_2''(s_2)$ are positive where $s_i=(d_{i1}+d_{i2})/2$, $i=1,2$. We further assume that $(X_i(\xi),Y_i(\xi))$ is the solution of the system (3.1)–(3.2) corresponding to $f_i(t)$ for $t\in(d_{i1},d_{i2})$ and $i=1,2$. If there exist constants ξ_1, ξ_2, and $l\geqslant 0$ such that $X_1(\xi+l)=X_2(\xi)$ and $Y_1(\xi+l)=Y_2(\xi)$ for all $\xi\in(\xi_1,\xi_2)$, then the following hold*:

(i) *There exist $d_0\in\mathbb{R}$ and $k\in(0,1]$ such that there holds*

$$f_2(t)=k^{-1/2}f_1\big(k^{-1/2}t+d_0\big)\quad \text{for all } t\in\big(\max\big\{\sqrt{k}(T_{11}-d_0),T_{21}\big\},\min\big\{\sqrt{k}(T_{12}-d_0),T_{22}\big\}\big).$$

(ii) *If f_i is the solution of (P_{m,a,b_i}) with $a\leqslant 0$ and $b_i>0$ for $i=1,2$, then we have $b_1=b_2$.*

Proof. (i) The first assertion follows from a similar argument of Lemma 5.2 of [18, pp. 334–335].

(ii) By using the assertion of (i) and the analyticity of f_i, $i=1,2$, we can choose $d_0\in\mathbb{R}$ and $k\in(0,1]$ such that there holds

$$f_2(t)=k^{-1/2}f_1\big(k^{-1/2}t+d_0\big)\quad \text{for all } t\in\big[0,\min\big\{\sqrt{k}(T_{12}-d_0),T_{22}\big\}\big). \tag{3.3}$$

Next we claim that $d_0=0$. For contradiction, we assume that $d_0>0$. Then from (3.3), we can compute

$$f_1(d_0)=k^{1/2}f_2(0)=k^{1/2}a, \tag{3.4}$$

$$f_1'(d_0)=kf_2'(0)=-k<0, \tag{3.5}$$

$$f_1''(d_0)=k^{3/2}f_2''(0)>0. \tag{3.6}$$

From (3.6) and part (i) of Proposition 1, it follows that $f_1''>0$ on $[0,d_0]$. Together with (3.5), this implies $f_1'<0$ on $[0,d_0]$, and so $f_1(0)>f_1(d_0)$. Since $k\in(0,1]$, this leads to $a>0$, a contradiction. Hence d_0 is nonpositive. On the other hand, if $d_0<0$, then by (3.3), we have

$$f_2'\big(-k^{1/2}d_0\big)=k^{-1}f_1'(0)=-k^{-1}<0, \tag{3.7}$$

$$f_2''\big(-k^{1/2}d_0\big)=k^{-3/2}f_1''(0)>0. \tag{3.8}$$

From (3.8) and part (i) of Proposition 1, it follows that $f_2''>0$ on $[0,-k^{1/2}d_0]$. Thus we have $f_2'(-k^{1/2}d_0)>f_2'(0)$, a contradiction to (3.7) and $k\in(0,1]$. Hence we have $d_0=0$, thereby completing the proof of the claim.

From the above claim and (3.3), we have that

$$f_1\big(k^{-1/2}t\big)=k^{1/2}f_2(t)\quad \text{for all } t\in\big[0,\min\{\sqrt{k}T_{12},T_{22}\}\big).$$

Finally, by differentiating the above equality with respect t and using $f_i'(0)=-1$, $i=1,2$, we can conclude that $k=1$, which implies $f_1\equiv f_2$. The proof of this lemma is completed. □

3.3. Phase plane analysis

In this subsection, we will collect some results on the phase plane analysis of the system (3.1)–(3.2). Since the proof is just a straightforward computation, we omit it.

Definition 3.1. For $m<-1$, we define

$$D_{1,1}=\left\{(X,Y)\;\middle|\;0<X<-1/m,\;Y>\frac{2+[(m+1)/2]X}{1+mX}\right\},$$

$$L_{1,1}=\left\{(X,Y)\;\middle|\;0<X<-1/m,\;Y=\frac{2+[(m+1)/2]X}{1+mX}\right\},$$

$$D_{1,2}=\left\{(X,Y)\;\middle|\;X>-4/(m+1),\;0<Y<\frac{2+[(m+1)/2]X}{1+mX}\right\},$$

$$L_{1,2}=\left\{(X,Y)\;\middle|\;X>-4/(m+1),\;Y=\frac{2+[(m+1)/2]X}{1+mX}\right\},$$

$$D_{1,3}=\left\{(X,Y)\;\middle|\;-2/(m+1)<X<-4/(m+1),\;0<Y<\frac{1+[(m+1)/2]X}{-1+mX}\right\}$$
$$\cup\left\{(X,Y)\;\middle|\;-4/(m+1)\leqslant X,\;\frac{2+[(m+1)/2]X}{1+mX}<Y<\frac{1+[(m+1)/2]X}{-1+mX}\right\},$$

$$L_{1,3} = \left\{ (X,Y) \;\middle|\; X > -2/(m+1),\ Y = \frac{1+[(m+1)/2]X}{-1+mX} \right\},$$

$$D_{1,4} = \{(X,Y) \mid X > 0,\ Y > 0\} \Big\backslash \left(\bigcup_{i=1}^{3} \overline{D}_{1,i} \right),$$

$$D_{2,1} = \left\{ (X,Y) \;\middle|\; -3 < X < 1/m,\ \frac{2+[(m+1)/2]X}{1+mX} < Y < \frac{1+[(m+1)/2]X}{-1+mX} \right\}$$
$$\cup \left\{ (X,Y) \;\middle|\; 1/m \leqslant X < 0,\ \frac{2+[(m+1)/2]X}{1+mX} < Y \right\},$$

$$L_{2,1} = \left\{ (X,Y) \;\middle|\; -3 < X < 0,\ Y = \frac{2+[(m+1)/2]X}{1+mX} \right\},$$

$$D_{2,2} = \left\{ (X,Y) \;\middle|\; -3 \leqslant X < 1/m,\ \frac{1+[(m+1)/2]X}{-1+mX} < Y \right\} \cup \left\{ (X,Y) \;\middle|\; X < -3,\ \frac{2+[(m+1)/2]X}{1+mX} < Y \right\},$$

$$L_{2,2} = \left\{ (X,Y) \;\middle|\; -3 < X < 1/m,\ Y = \frac{1+[(m+1)/2]X}{-1+mX} \right\},$$

$$D_{2,3} = \left\{ (X,Y) \;\middle|\; X < -3,\ \frac{1+[(m+1)/2]X}{-1+mX} < Y < \frac{2+[(m+1)/2]X}{1+mX} \right\},$$

$$L_{2,3} = \left\{ (X,Y) \;\middle|\; X < -3,\ Y = \frac{2+[(m+1)/2]X}{1+mX} \right\},$$

$$L_{2,4} = \left\{ (X,Y) \;\middle|\; X < -3,\ Y = \frac{1+[(m+1)/2]X}{-1+mX} \right\},$$

$$D_{2,4} = \{(X,Y) \mid X < 0,\ Y > 0\} \Big\backslash \left(\bigcup_{i=1}^{3} \overline{D}_{2,i} \right),$$

$$D_{4,1} = \left\{ (X,Y) \;\middle|\; 0 < X < -2/(m+1),\ \frac{1+[(m+1)/2]X}{-1+mX} < Y < 0 \right\},$$

$$L_{4,1} = \left\{ (X,Y) \;\middle|\; 0 < X < -2/(m+1),\ Y = \frac{1+[(m+1)/2]X}{-1+mX} \right\},$$

$$D_{4,2} = \left\{ (X,Y) \;\middle|\; 0 < X \leqslant -1/m,\ Y < \frac{1+[(m+1)/2]X}{-1+mX} \right\}$$
$$\cup \left\{ (X,Y) \;\middle|\; -1/m < X < -2/(m+1),\ \frac{2+[(m+1)/2]X}{1+mX} < Y < \frac{1+[(m+1)/2]X}{-1+mX} \right\}$$
$$\cup \left\{ (X,Y) \;\middle|\; -2/(m+1) \leqslant X < -4/(m+1),\ \frac{2+[(m+1)/2]X}{1+mX} < Y < 0 \right\},$$

$$L_{4,2} = \left\{ (X,Y) \;\middle|\; -1/m < X,\ Y = \frac{2+[(m+1)/2]X}{1+mX} \right\},$$

$$D_{4,3} = \{(X,Y) \mid X > 0,\ Y < 0\} \Big\backslash \left(\bigcup_{i=1}^{2} \overline{D}_{4,i} \right).$$

Lemma 3.4. *Let $m < -1$. Then the following statements hold*:

(i) *$X = 0$ and $Y = 0$ are the invariant curves for the system* (3.1)–(3.2).
(1i) *$X' < 0$, $Y' = 0$ for $(X,Y) \in L_{1,1}$ and $X' < 0$, $Y' > 0$ for $(X,Y) \in D_{1,1}$. Moreover, $B_{1,1} := D_{1,1} \cup L_{1,1}$ is an invariant region for the system* (3.1)–(3.2).
(1ii) *$X' > 0$, $Y' = 0$ for $(X,Y) \in L_{1,2}$ and $X' > 0$, $Y' > 0$ for $(X,Y) \in D_{1,2}$. Moreover, $B_{1,2} := D_{1,2} \cup L_{1,2}$ is an invariant region for the system* (3.1)–(3.2).
(1iii) *$X' = 0$, $Y' < 0$ for $(X,Y) \in L_{1,3}$ and $X' > 0$, $Y' < 0$ for $(X,Y) \in D_{1,3}$.*
(1iv) *$X' < 0$ and $Y' < 0$ for $(X,Y) \in D_{1,4}$.*
(2i) *$X' > 0$, $Y' = 0$ for $(X,Y) \in L_{2,1}$ and $X' > 0$, $Y' > 0$ for $(X,Y) \in D_{2,1}$.*
(2ii) *$X' = 0$, $Y' > 0$ for $(X,Y) \in L_{2,2}$ and $X' < 0$, $Y' > 0$ for $(X,Y) \in D_{2,2}$.*
(2iii) *$X' < 0$, $Y' = 0$ for $(X,Y) \in L_{2,3}$ and $X' < 0$, $Y' < 0$ for $(X,Y) \in D_{2,3}$.*
(2iv) *$X' = 0$, $Y' < 0$ for $(X,Y) \in L_{2,4}$ and $X' > 0$, $Y' < 0$ for $(X,Y) \in D_{2,4}$.*

(3i) $X' = 0$, $Y' > 0$ *for* $(X, Y) \in L_{4,1}$ *and* $X' < 0$, $Y' > 0$ *for* $(X, Y) \in D_{4,1}$. *Moreover,* $B_{4,1} := D_{4,1} \cup L_{4,1}$ *is an invariant region for the system* (3.1)–(3.2).
(3ii) $X' > 0$, $Y' > 0$ *for* $(X, Y) \in D_{4,2}$.
(3iii) $X' > 0$, $Y' = 0$ *for* $(X, Y) \in L_{4,2}$ *and* $X' > 0$, $Y' < 0$ *for* $(X, Y) \in D_{4,3}$. *Moreover,* $B_{4,2} := D_{4,3} \cup L_{4,2}$ *is an invariant region for the system* (3.1)–(3.2).

4. Properties of convex and convex-concave solutions of $(P_{m,a})$

4.1. Classification of convex-concave solutions of $(P_{m,a})$

In this subsection, we will classify all of the possible convex-concave solutions of $(P_{m,a})$ according to the limit: $\lim_{t\to+\infty}(f(t)f'(t)/f''(t), f'^2(t)/f(t)f''(t))$.

First, convex-concave solutions of $(P_{m,a})$ can be characterized as follows:

Lemma 4.1. *Let $m < -1$ and f be a type* (B) *solution of $(P_{m,a,b})$ defined on $[0, T)$. Then f is a type* (B_1) *solution of $(P_{m,a,b})$ if and only if there exists $s_0 > 0$ such that*

$$ff'/f'' > -2/(m+1) \quad \text{and} \quad f'^2/(ff'') > 0 \quad \text{on } (s_0, T). \tag{4.1}$$

Proof. To begin with, we suppose that f is a type (B_1) solution of $(P_{m,a,b})$. Hence $T = +\infty$. Recall from part (iii) of Proposition 1 that

$$E(t) = E_f(t) := f''(t) + \big[(m+1)/2\big]f(t)f'(t) = f''(0) + \big[(m+1)/2\big]f(0)f'(0) + \big[(3m+1)/2\big]\int_0^t f'(s)^2\,ds,$$

which implies that $E(t)$ is decreasing in t. Furthermore, by part (iv) of Proposition 1, we have that f is bounded and $f''(+\infty) = 0$. Hence we can conclude $E(+\infty) = 0$, and so $E(t) > 0$ for all $t > 0$. Now, by part (iv) of Proposition 1, there exists $s_0 > 0$ such that $f < 0$, $f' > 0$, and $f'' < 0$ on $(s_0, +\infty)$, which together with the fact that $E(t) > 0$ for all $t > 0$, yields (4.1).

Conversely, for contradiction, we assume that f is of type (B_2). Then there exists $t_2 > t_1$ such that $f < 0$, $f' < 0$ and $f'' < 0$ on (t_2, T). Hence, for $t \in (t_2, T)$ we have

$$\frac{f(t)f'(t)}{f''(t)} < 0 < -\frac{2}{m+1},$$

which is a contradiction to (4.1). Therefore, f is a type (B_1) solution of $(P_{m,a,b})$, thereby completing the proof of this lemma. □

In order to obtain more delicate characterization of convex-concave solutions of $(P_{m,a})$, we need the following lemma (see Fig. 2).

Lemma 4.2. *Let $m < -1$ and $(X(\xi), Y(\xi))$ be a solution of the system* (3.1)–(3.2) *with the right maximal existence interval $[0, \Xi)$. Then the following hold*:

(i) *There exists a unique solution $(X_{11*}(\xi), Y_{11*}(\xi))$ (up to a translation in ξ) of* (3.1)–(3.2) *defined on the maximal existence interval $(\Xi_{11*}, +\infty)$ such that the following hold*:
  (1) $X_{11*}(\xi) > -2/(m+1)$ *and* $Y_{11*}(\xi) > 0$ *for all* $\xi \in (\Xi_{11*}, +\infty)$,
  (2) $X'_{11*}(\xi) < 0$ *and* $Y'_{11*}(\xi) < 0$ *for all* $\xi \in (\Xi_{11*}, +\infty)$,
  (3) $\lim_{\xi\to+\infty}(X_{11*}(\xi), Y_{11*}(\xi)) = (-2/(m+1), 0)$,
  (4) *the trajectory* $\Gamma_{11*} := \{(X_{11*}(\xi), Y_{11*}(\xi)) \mid \xi \in (\Xi_{11*}, +\infty)\}$ *of* (X_{11*}, Y_{11*}) *is contained in* $D_{1,4}$ *and above the curve* $L_{1,3}$.

(ii) *If there exists ξ_1 such that the trajectory of $(X(\xi), Y(\xi))$ intersects the curve $L_{1,3}$ at the point $(X(\xi_1), Y(\xi_1))$, then there exists $\xi_2 > \xi_1$ such that the trajectory of $(X(\xi), Y(\xi))$ stays in the region $D_{1,3}$ for all $\xi \in (\xi_1, \xi_2)$, then crosses the curve $L_{1,2}$ horizontally at $(X(\xi_2), Y(\xi_2))$, and finally stays in the region $D_{1,2}$ for all $\xi > \xi_2$ and $\lim_{\xi\to+\infty}(X(\xi), Y(\xi)) = (+\infty, (m+1)/(2m))$.*

(iii) *Let $Y = \widehat{Y}_{11*}(X)$ be the equation of the trajectory Γ_{11*} and $D_{1,0} := \{(X, Y) \mid X > -2/(m+1), 0 < Y < \widehat{Y}_{11*}(X)\}$. If the trajectory of $(X(\xi), Y(\xi))$ lies in the region $D_{1,0}$ at some $\xi = \xi_0$, then the trajectory of $(X(\xi), Y(\xi))$ will stay in the region $D_{1,0}$ for all $\xi > \xi_0$, and there exists $\xi_1 > \xi_0$ such that the trajectory of $(X(\xi), Y(\xi))$ stays in the region $D_{1,2}$ for all $\xi > \xi_1$ and $\lim_{\xi\to+\infty}(X(\xi), Y(\xi)) = (+\infty, (m+1)/(2m))$.*

(iv) *There exists a unique solution $(X_{12*}(\xi), Y_{12*}(\xi))$ (up to a translation in ξ) of* (3.1)–(3.2) *defined on the maximal existence interval $(\Xi_{12*}, +\infty)$ such that the following hold*:
  (1) $X'_{12*}(\xi) < 0$ *and* $Y'_{12*}(\xi) < 0$ *for all* $\xi \in (\Xi_{12*}, +\infty)$,
  (2) $\lim_{\xi\to+\infty}(X_{12*}(\xi), Y_{12*}(\xi)) = (0, 2)$,

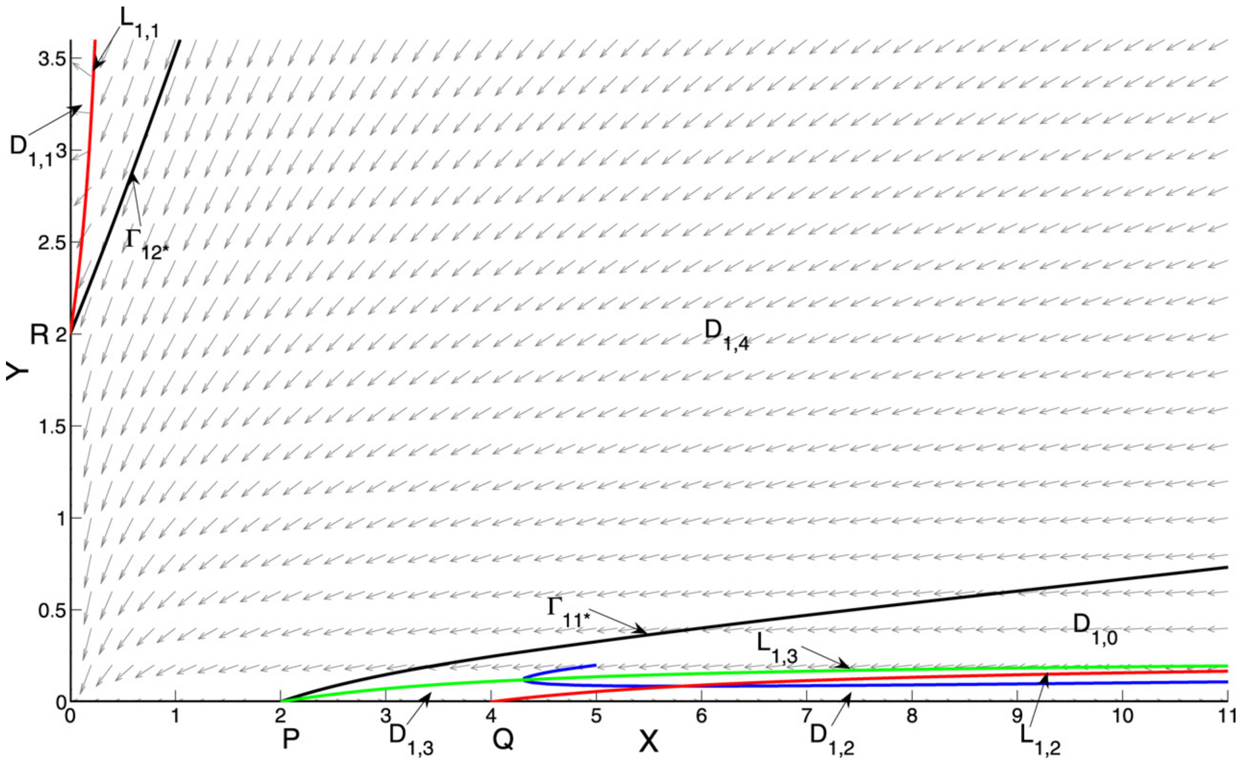

Fig. 2. The vector field generated by (3.1)–(3.2) for $m = -2$. The X-coordinates of P and Q are $-2/(m+1)$ and $-4/(m+1)$, respectively.

(3) $\lim_{\xi\to\Xi_{12*}^{+}}(X_{12*}(\xi), Y_{12*}(\xi)) = (+\infty, +\infty)$,

(4) *the trajectory* $\Gamma_{12*} := \{(X_{12*}(\xi), Y_{12*}(\xi)) \mid \xi \in (\Xi_{12*}, +\infty)\}$ *of* (X_{12*}, Y_{12*}) *is contained in* $D_{1,4}$.

(v) *If the trajectory of* $(X(\xi), Y(\xi))$ *lies in the region* $\{X > 0, Y > 0\} \setminus \overline{D}_{1,0}$ *at some* $\xi = \xi_0$, *then the limit* $\lim_{\xi\to\Xi^-}(X(\xi), Y(\xi))$ *exists and is one of the following*: $(0,0)$, $(0,2)$, *or* $(0,+\infty)$. *Moreover, if the limit is* $(0,0)$ *or* $(0,2)$, *then* $\Xi = +\infty$; *and if the limit is* $(0,+\infty)$, *then* Ξ *is finite.*

Proof. It suffices to show that the final assertion of part (i) holds and that Ξ in part (v) is finite for the case: $(0,+\infty)$, since the other assertions follow from a simple phase plane argument (see Lemma 3.4).

First, note that $(-2/(m+1), 0)$ is a saddle point for the system (3.1)–(3.2) with eigenvalues $-1, 1$ and the corresponding eigenvectors are $(-(3m+1)/(m+1)^2, 1)$ and $(1, 0)$, respectively. Therefore, $(X_{11*}(\xi), Y_{11*}(\xi))$ is a solution of (3.1)–(3.2) corresponding to the stable manifold of the saddle point $(-2/(m+1), 0)$. By a simple computation, the slope of the tangent line of $Y = \widehat{Y}_{11*}(X)$ tends to $-(m+1)^2/(3m+1)$ as X approaches $-2/(m+1)$, and the slope of the tangent line of the curve $L_{1,3}$ at $(-2/(m+1), 0)$ is $-(m+1)^2/[2(3m+1)]$ which is less than the one of $Y = \widehat{Y}_{11*}(X)$. Hence for all sufficiently small $X - (-2/(m+1))$ with $X > -2/(m+1)$, the trajectory Γ_{11*} is above the curve $L_{1,3}$. Together with Lemma 3.4, this yields that Γ_{11*} is above the curve $L_{1,3}$.

Next, we prove that if $\lim_{\xi\to\Xi^-}(X(\xi), Y(\xi)) = (0, +\infty)$, then Ξ is finite. Since $(X, Y) \to (0, +\infty)$ as $\xi \to \Xi^-$, there exists $\xi_1 > 0$ such that for all $\xi \in (\xi_1, \Xi)$, we have

$$-\big(2 + \big[(m+1)/2\big]X(\xi) - Y(\xi) - mX(\xi)Y(\xi)\big) \geqslant Y(\xi)/2.$$

Using this and (3.2), we can estimate $dY/d\xi$ as follows:

$$\frac{dY}{d\xi} = -Y(\xi)\big(2 + \big[(m+1)/2\big]X(\xi) - Y(\xi) - mX(\xi)Y(\xi)\big) \geqslant Y(\xi)^2/2 \quad \text{for all } \xi \in (\xi_1, \Xi),$$

which implies that Ξ is finite. The proof is completed. □

Now we can classify the convex-concave solutions of $(P_{m,a})$.

Lemma 4.3. *Let* $m < -1$, $a \in \mathbb{R}$, *and* f *be a convex-concave solution of* $(P_{m,a})$. *Then* f *must be one of the following two types*:

(B_{1,I_1}) *f is of type* (B_1) *such that* $ff'/f'' \searrow -2/(m+1)$ *and* $f'^2/(ff'') \to 0$ *as* $t \to +\infty$.
(B_{1,I_2}) *f is of type* (B_1) *such that* $ff'/f'' \to +\infty$ *and* $f'^2/(ff'') \to (m+1)/(2m)$ *as* $t \to +\infty$.

Moreover, there is at most one type (B_{1,I_1}) *solution of* $(P_{m,a})$ *if* $a \leqslant 0$, *and the set defined by*

$$\widehat{\mathcal{B}}_{m,a} := \{ b \in \mathbb{R} \mid \text{the solution } f_b \text{ of } (P_{m,a,b}) \text{ is of type } (B_{1,I_2}) \}$$

is open.

Proof. Since f is a convex-concave solution of $(P_{m,a})$, by Proposition 1, there exists $s_0 > 0$ such that $f < 0$, $f' > 0$, and $f'' < 0$ on $[s_0, +\infty)$. Let $(X(\xi), Y(\xi))$ be the solution of the system (3.1)–(3.2) corresponding to $f(t)$ for $t \in (s_0, +\infty)$. Then $(X(\xi), Y(\xi))$ is defined on $[-\ln f'(s_0), +\infty)$. By Lemmas 4.1 and 4.2, either the trajectory of (X, Y) is contained in the region $D_{1,0}$, or there exists $l \in \mathbb{R}$ such that $(X(\xi), Y(\xi)) = (X_{11*}(\xi + l), Y_{11*}(\xi + l))$ for all $\xi \geqslant -\ln f'(s_0)$ where $D_{1,0}$ and (X_{11*}, Y_{11*}) are defined in Lemma 4.2. Transferring back to the origin variable f, f must be type (B_{1,I_1}) or (B_{1,I_2}). Moreover, by part (ii) of Lemma 3.3, there is at most one type (B_{1,I_1}) solution of $(P_{m,a})$ provided $a \leqslant 0$.

Now we will show that $\widehat{\mathcal{B}}_{m,a}$ is open. Indeed, by using the standard theory of continuous dependence on initial data and noting that $D_{1,0}$ is invariant with respect to the system (3.1)–(3.2), it is easy to verify that the set $\widehat{\mathcal{B}}_{m,a}$ is open. The proof is completed. □

4.2. A criterion for the existence of convex-concave solutions of $(P_{m,a})$

Recall that the structure of solutions of $(P_{m,a})$ for $m < -1$ and $a < 0$ has already been deduced by Brighi and Hoernel [6]. In this subsection, we will used the information of these solutions to derive a criterion for the existence of convex-concave solutions of $(P_{m,a})$ for $m < -1$ and $a \geqslant 0$.

First, for $m < -1$, $a < 0$ and $b > 0$, let f_b be the solution of $(P_{m,a,b})$ defined on the right maximal existence interval $[0, T_b)$. By part (i) of Proposition 1, we have $f_b'' > 0$ on the left maximal existence interval $(\widehat{T}_b, 0]$. Hence there exists a unique $t_b < 0$ such that $f_b(t_b) = 0$ and $f_b'(t) < 0$, $f_b''(t) > 0$ for all $t \in [t_b, 0]$. *In the remaining of this subsection, whenever we say f_b, we always mean that $b > 0$ and f_b is defined on $[t_b, T_b)$.*

Now let t_{0b} (resp. $\hat{t}_{0b}$) be the first zero of f_b' (resp. f_b'') and $s_{0b} := \min\{t_{0b}, \hat{t}_{0b}\}$. Note that t_{0b} and $\hat{t}_{0b}$ may be $+\infty$. *In this subsection, t_b, t_{0b}, $\hat{t}_{0b}$ and s_{0b} have the same definition as above.* Let (X_b, Y_b) be the solution of the system (3.1)–(3.2) corresponding to $f_b(t)$ for $t \in (t_b, s_{0b})$. Note that the initial data of (X_b, Y_b) is $(X_b(0), Y_b(0)) = (-a/b, 1/(ab))$ which lies on the straight line defined by

$$Y = -\frac{1}{a^2} X. \tag{4.2}$$

Moreover, we have that $f_b < 0$, $f_b' < 0$ and $f_b'' > 0$ on (t_b, s_{0b}). Hence we have

$$\frac{f_b f_b'}{f_b''} > 0 \quad \text{and} \quad \frac{f_b'^2}{f_b f_b''} < 0 \quad \text{on } \Bigl(-\ln\bigl|f'(t_b)\bigr|, -\lim_{t \to s_{0b}^-} \ln\bigl|f'(t)\bigr|\Bigr),$$

$$\lim_{\xi \to (-\ln|f'(t_b)|)^+} \bigl(X_b(\xi), Y_b(\xi)\bigr) = (0, -\infty).$$

This suggests that we need to study the behavior of a solution (X, Y) of (3.1)–(3.2) whose trajectory lies in the fourth quadrant of the phase plane.

Lemma 4.4. *Let $m < -1$ and $(X(\xi), Y(\xi))$ be a solution of the system* (3.1)–(3.2) *with the right maximal existence interval* $[0, \Xi)$. *Then the following hold*:

(i) *There exists a unique solution $(X_{41*}(\xi), Y_{41*}(\xi))$ (up to a translation in ξ) of* (3.1)–(3.2) *defined on the maximal existence interval $(\Xi_{41*}, +\infty)$ such that the following hold*:
  (1) $X_{41*}(\xi) \in (0, -2/(m+1))$ *and* $Y_{41*}(\xi) < 0$ *for all* $\xi \in (\Xi_{41*}, +\infty)$,
  (2) $X_{41*}'(\xi) > 0$ *and* $Y_{41*}'(\xi) > 0$ *for all* $\xi \in (\Xi_{41*}, +\infty)$,
  (3) $\lim_{\xi \to +\infty}(X_{41*}(\xi), Y_{41*}(\xi)) = (-2/(m+1), 0)$ *and* $\lim_{\xi \to \Xi_{41*}^+}(X_{41*}(\xi), Y_{41*}(\xi)) = (0, -\infty)$,
  (4) *the trajectory* $\Gamma_{41*} := \{(X_{41*}(\xi), Y_{41*}(\xi)) \mid \xi \in (\Xi_{41*}, +\infty)\}$ *of* (X_{41*}, Y_{41*}) *is contained in* $D_{4,2}$.

(ii) *If there exists ξ_1 such that the trajectory of $(X(\xi), Y(\xi))$ intersects the curve $L_{4,1}$ (resp. $L_{4,2}$) at $(X(\xi_1), Y(\xi_1))$, then the trajectory of $(X(\xi), Y(\xi))$ will stay in the region $D_{4,1}$ (resp. $D_{4,3}$) for all $\xi \in (\xi_1, \Xi)$ and $\lim_{\xi \to \Xi^-}(X(\xi), Y(\xi)) = (0, 0)$ (resp. $(+\infty, -\infty)$). Moreover, if the limit is $(0, 0)$, then $\Xi = +\infty$; and if the limit is $(+\infty, -\infty)$, then Ξ is finite.*

(iii) *Let $Y = \widehat{Y}_{41*}(X)$ be the equation of the trajectory Γ_{41*} and set*

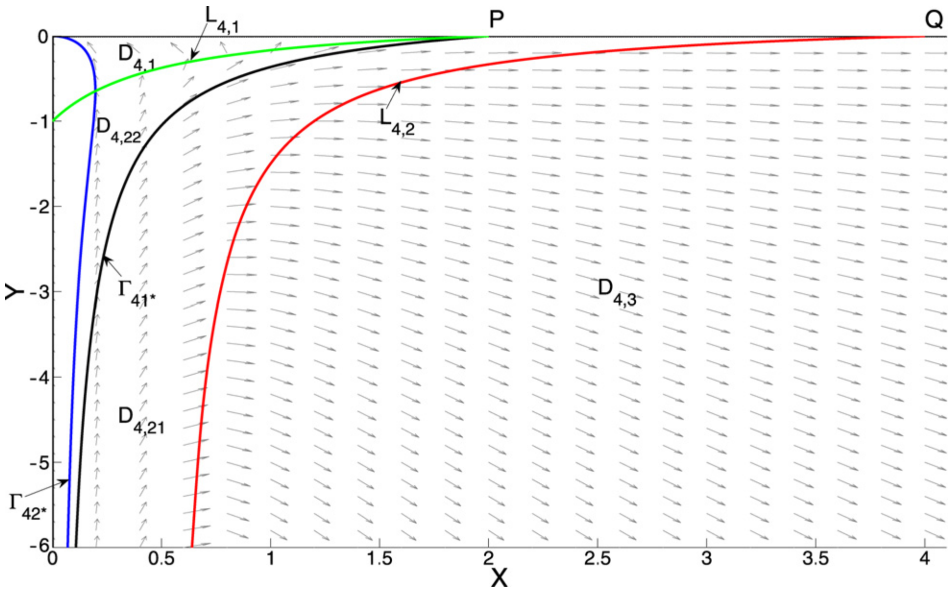

Fig. 3. The vector field generated by (3.1)–(3.2) for $m = -2$. The X-coordinates of P and Q are $-2/(m+1)$, and $-4/(m+1)$, respectively.

$$D_{4,21} := \{(X,Y) \in D_{4,2} \mid 0 < X \leqslant -1/m,\ Y < \widehat{Y}_{41*}(X)\}$$
$$\cup \left\{(X,Y) \in D_{4,2} \;\middle|\; -1/m < X < -2/(m+1),\ \frac{2+[(m+1)/2]X}{1+mX} < Y < \widehat{Y}_{41*}(X)\right\}$$
$$\cup \left\{(X,Y) \in D_{4,2} \;\middle|\; -2/(m+1) \leqslant X < -4/(m+1),\ \frac{2+[(m+1)/2]X}{1+mX} < Y < 0\right\}.$$

If the trajectory of $(X(\xi), Y(\xi))$ enters into the region $D_{4,21}$ (resp. $D_{4,2} \setminus \overline{D}_{4,21}$) at some $\xi = \xi_0$, then there exists $\xi_1 > \xi_0$ such that the trajectory of $(X(\xi), Y(\xi))$ stays in the region $D_{4,21}$ (resp. $D_{4,2} \setminus \overline{D}_{4,21}$) for all $\xi \in (\xi_0, \xi_1)$, then crosses the curve $L_{4,2}$ horizontally (resp. $L_{4,1}$ vertically) at $(X(\xi_1), Y(\xi_1))$, and finally stays in the region $D_{4,3}$ (resp. $D_{4,1}$) for all $\xi \in (\xi_1, \Xi)$ and $\lim_{\xi \to \Xi^-}(X(\xi), Y(\xi)) = (+\infty, -\infty)$ (resp. $(0,0)$).

Proof. Since the proof is similar to the one for Lemma 4.2, we omit it (see Fig. 3). □

Lemma 4.5. *Let $m < -1$ and $a \in \mathbb{R}$. Suppose that f is a convex solution of $(P_{m,a})$ with $f(+\infty) := \lim_{t\to+\infty} f(t) < 0$ and the limit*

$$l := \lim_{t\to\infty}\left(\frac{f(t)f'(t)}{f''(t)}, \frac{f'(t)^2}{f(t)f''(t)}\right)$$

exists. Then l cannot be equal to $(0,0)$ or $(+\infty, -\infty)$.

Proof. Case 1: $l = (0,0)$**.**

By assumption and part (iv) of Proposition 1, the quantity $f(+\infty) < 0$ is finite. From (1.1), we have

$$\lim_{t\to+\infty} \frac{f'''(t)}{f''(t)} = \lim_{t\to+\infty}\left[-\frac{m+1}{2} f(t) + m\frac{f'(t)^2}{f''(t)}\right] = -\frac{m+1}{2} f(+\infty). \tag{4.3}$$

Since f is a solution of $(P_{m,a})$, we have $f'(t), f''(t) \to 0$ as $t \to +\infty$. Hence by applying l'Hopital's rule and using (4.3), we have

$$\lim_{t\to+\infty} \frac{f'(t)}{f''(t)} = \lim_{t\to+\infty} \frac{f''(t)}{f'''(t)} = -\frac{2}{(m+1)f(+\infty)},$$

hence, $ff'/f'' \to -2/(m+1)$ as $t \to +\infty$, a contradiction.

Case 2: $l = (+\infty, -\infty)$**.**

By assumption, there exists $s_0 > 0$ such that $f < 0$, $f' < 0$, and $f'' > 0$ on $[s_0, +\infty)$. Let (X, Y) be the solution of the system (3.1)–(3.2) corresponding to $f(t)$ for $t \in [s_0, +\infty)$. Since $f'' > 0$ on $[0, +\infty)$ and $f'(+\infty) = 0$, (X, Y) is defined on $[-\ln|f'(s_0)|, +\infty)$ and tends to $(+\infty, -\infty)$ as $\xi \to +\infty$ by assumption. On the other hand, by part (ii) of Lemma 4.4, (X, Y) must be defined only on a finite interval, a contradiction. The proof is completed. □

Now we can give a criterion for the existence of convex-concave solutions of $(P_{m,a})$ for $m < -1$ and $a \geqslant 0$. First, recall from [6] that for each fixed $a < 0$, there exist $0 < b_- < b_+$ such that there hold:

- If $b < b_-$, then f_b is a type (C) solution.
- If $b = b_-$, then f_b is a convex solution of $(P_{m,a})$.
- If $b \in (b_-, b_+)$, then f_b is a convex-concave solution of $(P_{m,a})$ such that $\lim_{t\to+\infty} f(t) = 0$.
- If $b = b_+$, then f_b is a convex-concave solution of $(P_{m,a})$ such that $\lim_{t\to+\infty} f(t) = l$ for some $l < 0$.
- If $b > b_+$, then f_b is a type (B_2).

Hence for each fixed $a < 0$, f_b is a solution of $(P_{m,a})$ if and only if $b \in [b_-, b_+]$. Let (X_b, Y_b) be the solution of the system (3.1)–(3.2) corresponding to $f_b(t)$ for $t \in (t_b, \min\{t_{0b}, \hat t_{0b}\})$. Note that for $a < 0$, the correspondence $b \to (X_b, Y_b)$ is injective by part (ii) of Lemma 3.3. Moreover, if $b \in [b_-, b_+]$, then (X_b, Y_b) is defined on $(-\ln|f_b'(t_b)|, +\infty)$. Furthermore, by Lemmas 4.4 and 4.5, the trajectory $\{(X_{b_-}(\xi), Y_{b_-}(\xi)) \mid \xi \in (-\ln|f_{b_-}'(t_{b_-})|, +\infty)\}$ of (X_{b_-}, Y_{b_-}) coincides with the trajectory of (X_{41*}, Y_{41*}), and so, we have

$$\lim_{t\to+\infty}\left(\frac{f_{b_-}(t) f_{b_-}'(t)}{f_{b_-}''(t)}, \frac{f_{b_-}'(t)^2}{f_{b_-}(t) f_{b_-}''(t)}\right) = \big(-2/(m+1), 0\big).$$

On the other hand, if $b = b_+$ (resp. $b \in (b_-, b_+)$), then by Lemma 4.3 and part (ii) of Lemma 3.3, we have that f_b is a type $(\mathrm{B}_{1,\mathrm{l}_1})$ (resp. type $(\mathrm{B}_{1,\mathrm{l}_2})$) solution of $(P_{m,a})$. Note that for each $b \in (b_-, b_+]$, we have $t_{0b} < \hat t_{0b}$, hence the corresponding solution (X_b, Y_b) of the system (3.1)–(3.2) satisfies $(X_b(\xi), Y_b(\xi)) \to (0,0)$ as $\xi \to +\infty$. Now we let (X_{42*}, Y_{42*}) be a solution of the system (3.1)–(3.2) defined on the maximal existence interval $(\Xi_{42*}, +\infty)$ such that the following hold:

(1) $X_{42*}(\xi) \in (0, -2/(m+1))$ and $Y_{42*}(\xi) < 0$ for all $\xi \in (\Xi_{42*}, +\infty)$,
(2) $Y_{42*}'(\xi) > 0$ for all $\xi \in (\Xi_{42*}, +\infty)$,
(3) $\lim_{\xi\to+\infty}(X_{42*}(\xi), Y_{42*}(\xi)) = (0,0)$ and $\lim_{\xi\to\Xi_{42*}^+}(X_{42*}(\xi), Y_{42*}(\xi)) = (0, -\infty)$,
(4) the trajectory of (X_{b_+}, Y_{b_+}) coincides with the one of (X_{42*}, Y_{42*}).

Finally, we let $D_{4,22}$ be the open domain bounded by the trajectory of (X_{4i*}, Y_{4i*}), $i = 1,2$, and the X-axis (see Fig. 3). Note that $(X_b(0), Y_b(0))$ lies on the curve (4.2) for each $b > 0$. Moreover, we have $\{(X_b(0), Y_b(0)) \mid b > 0\} = \{(X,Y) \mid Y = -X/a^2,\ X > 0\}$. Therefore, the above discussion leads to

$$D_{4,22} = \bigcup_{b\in(b_-,b_+)} \big\{\big(X_b(\xi), Y_b(\xi)\big) \bigm| \xi \in \big(-\ln\big|f_b'(t_b)\big|, +\infty\big)\big\}.$$

Moreover, a similar argument as above yields

$$D_{4,21} \cup L_{4,2} \cup D_{4,3} = \bigcup_{b\in(0,b_-)} \big\{\big(X_b(\xi), Y_b(\xi)\big) \bigm| \xi \in \big(-\ln\big|f_b'(t_b)\big|, -\ln\big|f_b'(\hat t_{0b})\big|\big)\big\},$$

$$(D_{4,1} \cup D_{4,2}) \setminus \overline{D_{4,21} \cup D_{4,22}} = \bigcup_{b>b_+} \big\{\big(X_b(\xi), Y_b(\xi)\big) \bigm| \xi \in \big(-\ln\big|f_b'(t_b)\big|, +\infty\big)\big\}.$$

Hence, we have

$$\{X > 0,\ Y < 0\} = \bigcup_{b>0}\Big\{\big(X_b(\xi), Y_b(\xi)\big) \bigm| \xi \in \Big(-\ln\big|f_b'(t_b)\big|, -\lim_{t\to s_{0b}^-} \ln\big|f_b'(t)\big|\Big)\Big\}. \tag{4.4}$$

With these preparation, we are in a position to give a sufficient condition which guarantees the existence of convex-concave solutions of $(P_{m,a})$ for $m < -1$ and $a \in \mathbb{R}$.

Lemma 4.6. *Suppose that $m < -1$ and f is a solution of (1.1) such that $f < 0$, $f' < 0$ and $f'' > 0$ on $[t_1, t_2]$ for some $t_1, t_2 > 0$. Let (X,Y) be the corresponding solution of the system (3.1)–(3.2) defined on $[-\ln|f'(t_1)|, -\ln|f'(t_2)|]$ and $\Gamma := \{(X(\xi), Y(\xi)) \mid \xi \in [-\ln|f'(t_1)|, -\ln|f'(t_2)|]\}$. Then the following hold:*

(i) *f is a type (B_1) (convex-concave) solution if and only if the curve Γ lies in the region $D_{4,22}$ or on the trajectory of (X_{42*}, Y_{42*}).*
(ii) *If the curve Γ lies on the trajectory Γ_{42*} of (X_{42*}, Y_{42*}), then f is a type $(\mathrm{B}_{1,\mathrm{l}_1})$ solution and $\lim_{t\to+\infty} f(t) = l$ for some $l < 0$.*
(iii) *If the curve Γ lies in the region $D_{4,22}$, then f is a type $(\mathrm{B}_{1,\mathrm{l}_2})$ solution and $\lim_{t\to+\infty} f(t) = 0$.*

Proof. Recall the definitions of f_b, (X_b, Y_b), t_{0b}, $\hat t_{0b}$ and s_{0b} defined in the beginning of this subsection. Since (X,Y) is a solution of (1.1) and lies in the region $\{X > 0, Y < 0\}$, by Eq. (4.4) and part (i) of Lemma 3.3, there exist $k > 0$ and $d \in \mathbb{R}$ such that $f(t) = k f_b(kt + d)$ for some $b > 0$. Together with the discussion right before this lemma, the assertions of this lemma follows. $\square$

5. Classification of solutions of $(P_{m,a,b})$

In this section, we will classify the solutions f of $(P_{m,a,b})$ for $m < -1$ and $a \geqslant 0$ according to the order of the positions of zeros of f, f', and f''. Then we will establish the correspondence between these types of solutions and the ω-limit subset $\{(0,0),(0,2),(0,+\infty),(1/m,+\infty),(-\infty,+\infty)\}$ of the system (3.1)–(3.2).

Recall that f' and f'' cannot vanish at the same finite point. If f is the solution of $(P_{m,a,b})$ with $m < -1$ and $a > 0$, then it is clear that f must be one of the following six types:

Definition 5.1.

(G_0) $f > 0$, $f' < 0$, and $f'' > 0$ on $[0,+\infty)$.
(G_1) There exists a finite $t_0 > 0$ such that $f > 0$ on $[0,t_0]$, $f' < 0$ on $[0,t_0)$ and $f'(t_0) = 0$, and $f'' > 0$ on $[0,t_0]$.
(G_2) There exists a finite $t_0 > 0$ such that $f > 0$ on $[0,t_0)$, $f' < 0$ on $[0,t_0)$, $f(t_0) = f'(t_0) = 0$, and $f'' > 0$ on $[0,t_0]$.
(G_3) There exists a finite $\tilde{t}_0 > 0$ such that $f > 0$ on $[0,\tilde{t}_0)$, $f(\tilde{t}_0) = 0$, and $f' < 0$ and $f'' > 0$ on $[0,\tilde{t}_0]$.
(G_4) There exists a finite $\hat{t}_0 > 0$ such that $f > 0$ on $[0,\hat{t}_0)$, $f' < 0$ on $[0,\hat{t}_0]$, and $f'' > 0$ on $[0,\hat{t}_0)$ and $f(\hat{t}_0) = f''(\hat{t}_0) = 0$.
(G_5) There exists a finite $\hat{t}_0 > 0$ such that $f > 0$ on $[0,\hat{t}_0]$, $f' < 0$ on $[0,\hat{t}_0]$, and $f'' > 0$ on $[0,\hat{t}_0)$ and $f''(\hat{t}_0) = 0$.

Note that a type (G_0) solution must be a convex solution of $(P_{m,a})$ by Lemma 3.1, and only (G_3) solution can be a convex-concave solution of $(P_{m,a})$ by Proposition 1 and Lemma 3.2. Next, we will describe these types of solutions in terms of two limits.

Lemma 5.1. *The following statements hold*:

(1) *If f is of type* (G_1), *then we have*

$$\lim_{t\to t_0^-} f(t)f'(t)/f''(t) = 0, \qquad \lim_{t\to t_0^-} f'(t)^2/\big[f(t)f''(t)\big] = 0.$$

(2) *If f is of type* (G_2), *then we have*

$$\lim_{t\to t_0^-} f(t)f'(t)/f''(t) = 0, \qquad \lim_{t\to t_0^-} f'(t)^2/\big[f(t)f''(t)\big] = 2.$$

(3) *If f is of type* (G_3), *then we have*

$$\lim_{t\to \tilde{t}_0^-} f(t)f'(t)/f''(t) = 0, \qquad \lim_{t\to \tilde{t}_0^-} f'(t)^2/\big[f(t)f''(t)\big] = +\infty.$$

(4) *If f is of type* (G_4), *then we have*

$$\lim_{t\to \hat{t}_0^-} f(t)f'(t)/f''(t) = 1/m, \qquad \lim_{t\to \hat{t}_0^-} f'(t)^2/\big[f(t)f''(t)\big] = +\infty.$$

(5) *If f is of type* (G_5), *then we have*

$$\lim_{t\to \hat{t}_0^-} f(t)f'(t)/f''(t) = -\infty, \qquad \lim_{t\to \hat{t}_0^-} f'(t)^2/\big[f(t)f''(t)\big] = +\infty.$$

Proof. It is easy to see that the statements (1), (3) and (5) hold. If f is of type (G_2), then it follows that $\lim_{t\to t_0^-} f(t)f'(t)/f''(t) = 0$. Furthermore, by applying l'Hopital's rule, we have

$$\lim_{t\to t_0^-} f'(t)^2/\big[f(t)f''(t)\big] = \Big\{\lim_{t\to t_0^-} f'(t)^2/f(t)\Big\}/f''(t_0) = \Big\{\lim_{t\to t_0^-} 2f'(t)f''(t)/f'(t)\Big\}/f''(t_0) = 2.$$

If f is of type (G_4). Then we have $\lim_{t\to \hat{t}_0^-} f'(t)^2/[f(t)f''(t)] = +\infty$. Therefore it remains to show that $ff'/f'' \to 1/m$ as $t \to \hat{t}_0^-$. By (1.1), we can compute that

$$\lim_{t\to \hat{t}_0^-} \frac{f'''(t)}{f'(t)} = \lim_{t\to \hat{t}_0^-} \left[-\frac{m+1}{2}\frac{f(t)f''(t)}{f'(t)} + mf'(t)\right] = mf'(\hat{t}_0).$$

Hence by applying l'Hopital's rule and the above equation, we have

$$\lim_{t\to \hat{t}_0^-} \frac{f(t)}{f''(t)} = \lim_{t\to \hat{t}_0^-} \frac{f'(t)}{f'''(t)} = 1/\big(mf'(\hat{t}_0)\big),$$

and so $ff'/f'' \to 1/m$ as $t \to \hat{t}_0^-$. Hence the lemma follows. □

In the remaining of this subsection, we will show that if f is a type (G_0) solution, then the limit (if it exists)

$$\lim_{t\to+\infty}\left(\frac{f(t)f'(t)}{f''(t)},\frac{f'(t)^2}{f(t)f''(t)}\right)$$

cannot be equal to $(0,0)$, $(0,2)$, $(0,+\infty)$, $(1/m,+\infty)$, or $(-\infty,+\infty)$, which will establish the correspondence between type (G_1)–(G_5) solutions and the ω-limit subset $\{(0,0),(0,2),(0,+\infty),(1/m,+\infty),(-\infty,+\infty)\}$ of the system (3.1)–(3.2). To this end, we need two auxiliary lemmas.

Lemma 5.2. *Let $m<-1$ and $(X(\xi),Y(\xi))$ be a solution of the system (3.1)–(3.2). Then the following hold:*

(i) *If $X(\xi)<0$ and $Y(\xi)>0$ for some $\xi=\xi_0$, then we have $\lim_{\xi\to-\infty}(X(\xi),Y(\xi))=(-3,1/2)$.*
(ii) *There exists a unique solution $(X_{21*}(\xi),Y_{21*}(\xi))$ (up to a translation in ξ) of (3.1)–(3.2) such that the following hold:*
  (1) *$X'_{21*}(\xi)>0$ and $Y'_{21*}(\xi)>0$ for all $\xi\in\mathbb{R}$,*
  (2) *$\lim_{\xi\to+\infty}(X_{21*}(\xi),Y_{21*}(\xi))=(0,2)$,*
  (3) *$\lim_{\xi\to-\infty}(X_{21*}(\xi),Y_{21*}(\xi))=(-3,1/2)$,*
  (4) *the trajectory $\Gamma_{21*}:=\{(X_{21*}(\xi),Y_{21*}(\xi))\mid\xi\in\mathbb{R}\}$ of (X_{21*},Y_{21*}) is contained in $D_{2,1}$.*
(iii) *Let $Y=\widehat{Y}_{21*}(X)$ be the equation of the trajectory Γ_{21*} and $D_{2,11}:=\{(X,Y)\mid -3<X<0,(2+[(m+1)/2]X)/(1+mX)<Y<\widehat{Y}_{21*}(X)\}$. If $(X(\xi),Y(\xi))$ enters into the region $D_{2,11}$ at some $\xi=\xi_0$, then there exists a $\xi_1>\xi_0$ such that $(X(\xi),Y(\xi))$ stays in the region $D_{2,11}$ for all $\xi\in(\xi_0,\xi_1)$, then crosses the curve $L_{2,1}$ horizontally at $(X(\xi_1),Y(\xi_1))$, and finally stays in the region $D_{2,4}$ for all $\xi>\xi_1$ and tends to $(0,0)$ at $\xi\to+\infty$.*
(iv) *There exists a unique solution $(X_{22*}(\xi),Y_{22*}(\xi))$ (up to a translation in ξ) of (3.1)–(3.2) defined on the maximal existence interval $(-\infty,\Xi_{22*})$ for some $\Xi_{22*}\in\mathbb{R}$, such that the following hold:*
  (1) *$X'_{22*}(\xi)>0$ and $Y'_{22*}(\xi)>0$ for all $\xi\in(-\infty,\Xi_{22*})$,*
  (2) *$\lim_{\xi\to\Xi_{22*}^-}(X_{22*}(\xi),Y_{22*}(\xi))=(1/m,+\infty)$,*
  (3) *$\lim_{\xi\to-\infty}(X_{22*}(\xi),Y_{22*}(\xi))=(-3,1/2)$,*
  (4) *the trajectory $\Gamma_{22*}:=\{(X_{22*}(\xi),Y_{22*}(\xi))\mid\xi\in(-\infty,\Xi_{22*})\}$ of (X_{22*},Y_{22*}) is contained in $D_{2,1}$ and above the trajectory Γ_{21*}.*
(v) *Let $Y=\widehat{Y}_{22*}(X)$ be the equation of the trajectory (X_{22*},Y_{22*}) and $D_{2,12}=\{(X,Y)\mid -3<X<1/m,\ \widehat{Y}_{22*}(X)<Y<(1+[(m+1)/2]X)/(-1+mX)\}$. If $(X(\xi),Y(\xi))$ enters into the region $D_{2,12}$ at some $\xi=\xi_0$, then there exist $\xi_0<\xi_1<\Xi<+\infty$ such that $(X(\xi),Y(\xi))$ stays in the region $D_{2,12}$ for all $\xi\in(\xi_0,\xi_1)$, then crosses the curve $L_{2,2}$ vertically at $(X(\xi_1),Y(\xi_1))$, and finally stays in the region $D_{2,2}$ for all $\xi\in(\xi_1,\Xi)$ and tends to $(-\infty,+\infty)$ at $\xi\to\Xi^-$.*
(vi) *Let $D_{2,13}$ be the open domain bounded by the trajectories Γ_{21*} and Γ_{22*}, and the Y-axis. If $(X(\xi),Y(\xi))$ enters into the region $D_{2,13}$ at some $\xi=\xi_0$, then there exists a finite Ξ such that $(X(\xi),Y(\xi))$ stays in the region $D_{2,13}$ for all $\xi\in(\xi_0,\Xi)$, and tends to $(0,+\infty)$ at $\xi\to\Xi^-$.*
(vii) *There exists a unique solution $(X_{23*}(\xi),Y_{23*}(\xi))$ (up to a translation in ξ) of (3.1)–(3.2) defined on $\mathbb{R}$, such that the following hold:*
  (1) *$X'_{23*}(\xi)<0$ and $Y'_{23*}(\xi)<0$ for all $\xi\in\mathbb{R}$,*
  (2) *$\lim_{\xi\to+\infty}(X_{23*}(\xi),Y_{23*}(\xi))=(-\infty,(m+1)/(2m))$,*
  (3) *$\lim_{\xi\to-\infty}(X_{23*}(\xi),Y_{23*}(\xi))=(-3,1/2)$,*
  (4) *the trajectory $\Gamma_{23*}:=\{(X_{23*}(\xi),Y_{23*}(\xi))\mid\xi\in\mathbb{R}\}$ of (X_{23*},Y_{23*}) is contained in $D_{2,3}$, below the curve $L_{2,3}$ and above the curve $L_{2,4}$.*
(viii) *Let $Y=\widehat{Y}_{23*}(X)$ be the equation of the trajectory Γ_{23*}, $D_{2,31}:=\{(X,Y)\mid X<-3,\ \widehat{Y}_{23*}(X)<Y<(2+[(m+1)/2]X)/(1+mX)\}$ and $D_{2,32}:=D_{2,3}\setminus\overline{D}_{2,31}$. If $(X(\xi),Y(\xi))$ enters into the region $D_{2,31}$ at some $\xi=\xi_0$, then there exist $\xi_0<\xi_1<\Xi<+\infty$ such that $(X(\xi),Y(\xi))$ stays in the region $D_{2,31}$ for all $\xi\in(\xi_0,\xi_1)$, then crosses the curve $L_{2,3}$ horizontally at $(X(\xi_1),Y(\xi_1))$, and finally stays in the region $D_{2,2}$ for all $\xi\in(\xi_1,\Xi)$ and tends to $(-\infty,+\infty)$ at $\xi\to\Xi^-$. On the other hand, if $(X(\xi),Y(\xi))$ enters into the region $D_{2,32}$ at some $\xi=\xi_0$, then there exists a $\xi_1>\xi_0$ such that $(X(\xi),Y(\xi))$ stays in the region $D_{2,32}$ for all $\xi\in(\xi_0,\xi_1)$, then crosses the curve $L_{2,4}$ vertically at $(X(\xi_1),Y(\xi_1))$, and finally stays in the region $D_{2,4}$ for all $\xi>\xi_1$ and tends to $(0,0)$ as $\xi\to+\infty$.*

Proof. It suffices to prove the uniqueness and existence of the solution $(X_{2i*}(\xi),Y_{2i*}(\xi))$ of (3.1)–(3.2) for $i=2,3$, since the proofs of the other assertions are similar to that of Lemma 4.2.

The existence and uniqueness of $(X_{22*}(\xi),Y_{22*}(\xi))$**.** First, we prove the existence of $(X_{22*}(\xi),Y_{22*}(\xi))$. Let g be the solution of the following backward problem:

$$g'''+\big[(m+1)/2\big]gg''-mg'^2=0,\quad t<0,$$
$$g(0)=0,\qquad g'(0)=-1,\qquad g''(0)=0.$$

By Proposition 1, we can conclude that there exists $s_1<0$ such that $g>0$, $g'<-1$, and $g''>0$ on $[s_1,0)$. Let $(X_{22*}(\xi),Y_{22*}(\xi))$ be the solution of (3.1)–(3.2) corresponding to $g(t)$ for $t\in[s_1,0)$. Hence by using a similar argument as in part (4) of Lemma 5.1, we can compute

$$\lim_{\xi\to 0^-}\big(X_{22*}(\xi), Y_{22*}(\xi)\big) = (1/m, +\infty).$$

Furthermore, by using a simple phase plane analysis (see Lemma 3.4), we can conclude that the set $\{(X_{22*}(\xi), Y_{22*}(\xi)) \mid \xi \in [-\ln|g'(s_1)|, 0)\}$ is contained in $D_{2,1}$. (Otherwise, we have that $\lim_{\xi\to 0^-}(X_{22*}(\xi), Y_{22*}(\xi)) = (-\infty, +\infty)$, $(-\infty, (m+1)/(2m))$, or $(0,0)$.) Obviously, we can extend the domain of $(X_{22*}(\xi), Y_{22*}(\xi))$ from $[-\ln(g'(s_1)), 0)$ to $(-\infty, \Xi_{22*})$ for some $\Xi_{22*} \in \mathbb{R}\cup\{+\infty\}$, and $(X_{22*}(\xi), Y_{22*}(\xi))$ has the required properties except uniqueness.

Next, we prove the uniqueness of $(X_{22*}(\xi), Y_{22*}(\xi))$ (up to a translation in ξ). Let $(X_i(\xi), Y_i(\xi))$, $i = 1, 2$, be the solution of the system (3.1)–(3.2) defined on $(-\infty, \Xi_{i22*})$ with the required properties stated in (iv). Since $Y_i' > 0$ on $(-\infty, \Xi_{i22*})$, X_i can be viewed as a function of Y for $i = 1, 2$. Without loss of generality, we may assume that $X_1(Y) > X_2(Y)$ for all $Y \in (1/2, +\infty)$. With a simple computation, it follows that

$$F(X_1, Y)G(X_2, Y) - F(X_2, Y)G(X_1, Y) := M(Y, X_1, X_2)(X_1 - X_2),$$

where

$$\begin{aligned} M(Y, X_1, X_2) &:= Y\left[2 + (m+1)(X_1 + X_2) + \frac{(m+1)^2}{4}X_1X_2\right] + Y^2\left[1 - \frac{5m+1}{2}(X_1 + X_2) - m(m+1)X_1X_2\right] \\ &\quad + Y^3\big[-1 + m(X_1 + X_2) + m^2X_1X_2\big]. \end{aligned}$$

Note that $X_i(Y) \nearrow 1/m$ as $Y \to +\infty$ for $i = 1, 2$. Hence, for all sufficiently large Y, we obtain

$$\frac{dX_1}{dY} - \frac{dX_2}{dY} = \big[F(X_1, Y)G(X_2, Y) - F(X_2, Y)G(X_1, Y)\big]/\big[G(X_1, Y)G(X_2, Y)\big] > 0.$$

Here we have used the fact that $G(X_i, Y) < 0$ for $Y > 1/2$ for $i = 1, 2$. Then the above inequality is a contradiction to the fact that

$$\lim_{Y\to+\infty}\big[X_1(Y) - X_2(Y)\big] = 0,$$

thereby establishing the uniqueness of $(X_{22*}(\xi), Y_{22*}(\xi))$.

The existence and uniqueness of $(X_{23*}(\xi), Y_{23*}(\xi))$**.** First, we prove the existence of $(X_{23*}(\xi), Y_{23*}(\xi))$. Consider the curve Γ defined by

$$\big\{(X, Y) \;\big|\; X = -4,\ Y > 0\big\}.$$

Let Γ intersect the curves $L_{2,3}$ and $L_{2,4}$ at A and B, respectively. Now let $\mathcal{C}$ be the segment connecting the points A and B. Set

$$\mathcal{C}_1 := \big\{P \in \mathcal{C} \;\big|\; \text{the trajectory of (3.1)–(3.2) starting from } P \text{ will tend to } (-\infty, +\infty)\big\},$$
$$\mathcal{C}_2 := \big\{P \in \mathcal{C} \;\big|\; \text{the trajectory of (3.1)–(3.2) starting from } P \text{ will tend to } (0, 0)\big\}.$$

Note that $D_{2,2}$ and $D_{2,4}$ are invariant with respect to (3.1)–(3.2). Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are open by the theory of continuous dependence on initial data. By the phase plane analysis (see Lemma 3.4), we have that the trajectory of (3.1)–(3.2) starting from A will stay in the region $D_{2,2}$ and goes to $(-\infty, +\infty)$, and that the trajectory of (3.1)–(3.2) starting from B will stay in the region $D_{2,4}$ and goes to $(0, 0)$. From these facts it follows that there exists a point $P_1 \in \mathcal{C} \cap D_{2,3}$ such that the trajectory of the solution $((X_{23*}(\xi), Y_{23*}(\xi)))$ of (3.1)–(3.2) starting from P_1 will stay in the region $D_{2,3}$ for all $\xi \geqslant 0$. Thus we have $\lim_{\xi+\infty}((X_{23*}(\xi), Y_{23*}(\xi))) = (-\infty, (m+1)/(2m))$. Moreover, by a simple phase plane analysis (see Lemma 3.4), we have $\lim_{\xi\to-\infty}(X_{23*}(\xi), Y_{23*}(\xi)) = (-3, 1/2)$. This proves the existence of $((X_{23*}(\xi), Y_{23*}(\xi)))$.

Finally, the proof of the uniqueness of $(X_{23*}(\xi), Y_{23*}(\xi))$ (up to a translation in ξ) is similar to that of $(X_{22*}(\xi), Y_{22*}(\xi))$, and so we omit it. Therefore, the proof of this lemma is completed. □

In order to reach the goal of this section, it remains to show the following lemma.

Lemma 5.3. *Let $m < -1$ and $a \geqslant 0$. Suppose that f is a convex solution of $(P_{m,a})$. If the limit*

$$l := \lim_{t\to+\infty}\left(\frac{f(t)f'(t)}{f''(t)}, \frac{f'(t)^2}{f(t)f''(t)}\right)$$

exists, then this limit cannot be equal to $(0,0)$, $(0,2)$, $(0,+\infty)$, $(1/m,+\infty)$, or $(-\infty,+\infty)$.

Proof. Case (i): $l = (0, 0)$**.**

By Lemma 4.5, it suffices to consider the case: $f(t)$ is positive for all sufficiently large t. By (1.1), we have

$$f'''(t) = \left[-\frac{m+1}{2} + m\frac{f'(t)^2}{f(t)f''(t)}\right] f(t)f''(t).$$

Together with the fact that $m < -1$, it follows that $f'''(t) > 0$ for all sufficiently large t. This is a contradiction to the fact that $f'' > 0$ on $[0, +\infty)$ and $f''(+\infty) = 0$.

Case (ii): $l = (0, 2)$**.**

Let g be the solution of the following problem defined on the right maximal existence interval $[0, T)$:

$$g''' + \big[(m+1)/2\big] gg'' - mg'^2 = 0, \quad t \in [0, T),$$
$$g(0) = 0, \qquad g'(0) = 0, \qquad g''(0) = 1.$$

By Lemma 3.2 and Proposition 1, we can conclude that there exist $s_1 < 0$ and $s_2 \in (0, T)$ such that $g > 0$, $g' < 0$, and $g'' > 0$ on $[s_1, 0)$, and that $g > 0$, $g' > 0$, and $g'' > 0$ on $(0, s_2)$, and $g'' < 0$ on (s_2, T).

Now we set the following change of variables for the function g:

$$X_1(\xi) = \frac{g(t)g'(t)}{g''(t)}, \qquad Y_1(\xi) = \frac{g'(t)^2}{g(t)g''(t)}, \quad \xi = -\ln\big|g'(t)\big|,$$

where we require $t \in [s_1, 0)$. Note that (X_1, Y_1) satisfies the system (3.1)–(3.2) and that

$$X_1(\xi) < 0 \quad \text{and} \quad Y_1(\xi) > 0 \quad \text{for all } \xi \in \big[-\ln\big|g'(s_1)\big|, +\infty\big).$$

Since $g(0) = g'(0) = 0$, by using a similar argument as in part (2) of Lemma 5.1, we can compute

$$\lim_{\xi \to +\infty} \big(X_1(\xi), Y_1(\xi)\big) = (0, 2).$$

Therefore, the trajectory $\{(X_1(\xi), Y_1(\xi)) \mid \xi \in [-\ln|g'(s_1)|, +\infty)\}$ of (X_1, Y_1) lies on the trajectory of $(X_{21*}Y_{21*})$ defined in part (ii) of Lemma 5.2.

Now if f is a convex solution of $(P_{m,a})$ with $l = (0, 2)$, then there exists $s_0 > 0$ such that the following holds:

$$f > 0, \qquad f' < 0, \quad \text{and} \quad f'' > 0 \quad \text{on } (s_0, +\infty).$$

Now we set the following change of variables for the solution f:

$$X(\xi) = \frac{f(t)f'(t)}{f''(t)}, \qquad Y(\xi) = \frac{f'(t)^2}{f(t)f''(t)}, \quad \xi = -\ln\big|f'(t)\big|,$$

where we require $t \in (s_0, +\infty)$. Note that (X, Y) satisfies the system (3.1)–(3.2) and that

$$X(\xi) < 0 \quad \text{and} \quad Y(\xi) > 0 \quad \text{for all } \xi \in \big(-\ln\big|f'(s_0)\big|, +\infty\big),$$
$$\lim_{\xi \to +\infty} \big(X(\xi), Y(\xi)\big) = (0, 2).$$

By part (ii) of Lemma 5.2, the trajectory $\{(X(\xi), Y(\xi)) \mid \xi \in (-\ln|f'(s_0)|, +\infty)\}$ of (X, Y) must lie on the trajectory of $(X_{21*}Y_{21*})$. Combining this with part (i) of Lemma 3.3, it follows that there exists $k > 0$ and $d_0 \in \mathbb{R}$ such that $f(t) = kg(kt + d_0)$ for all $t \in (t_1, t_2)$ and for some $t_1, t_2 \in \mathbb{R}$. By the standard uniqueness theory for differential equations, we have $f(t) = kg(kt + d_0)$ for all $t \in (T_1, +\infty)$ where $(T_1, +\infty)$ is the maximal existence interval of f. Hence $T = +\infty$. Since $g'' < 0$ on $(s_2, +\infty)$, this yields that $f''(t)$ takes negative value for sufficiently large t. This is a contradiction.

Case (iii): $l = (0, +\infty)$**,** $(1/m, +\infty)$ **or** $(-\infty, +\infty)$**.**

By assumption, there exists $s_0 > 0$ such that for all $t > s_0$, we have

$$\frac{f'(t)^2}{f(t)f''(t)} > 0,$$

which together with the fact that $f' < 0$ on $[0, +\infty)$, yields

$$\frac{f(t)f'(t)}{f''(t)} < 0 \quad \text{for all } t > s_0.$$

As before, let (X, Y) be the solution of the system (3.1)–(3.2) corresponding to $f(t)$ for $t > s_0$. Then (X, Y) is defined on $(-\ln|f'(s_0)|, +\infty)$ and $\lim_{\xi \to +\infty}(X(\xi), Y(\xi)) = l$. However, by parts (iv)–(viii) of Lemma 5.2, (X, Y) can only be defined on a finite interval, a contradiction. Hence the proof of this lemma is completed. $\square$

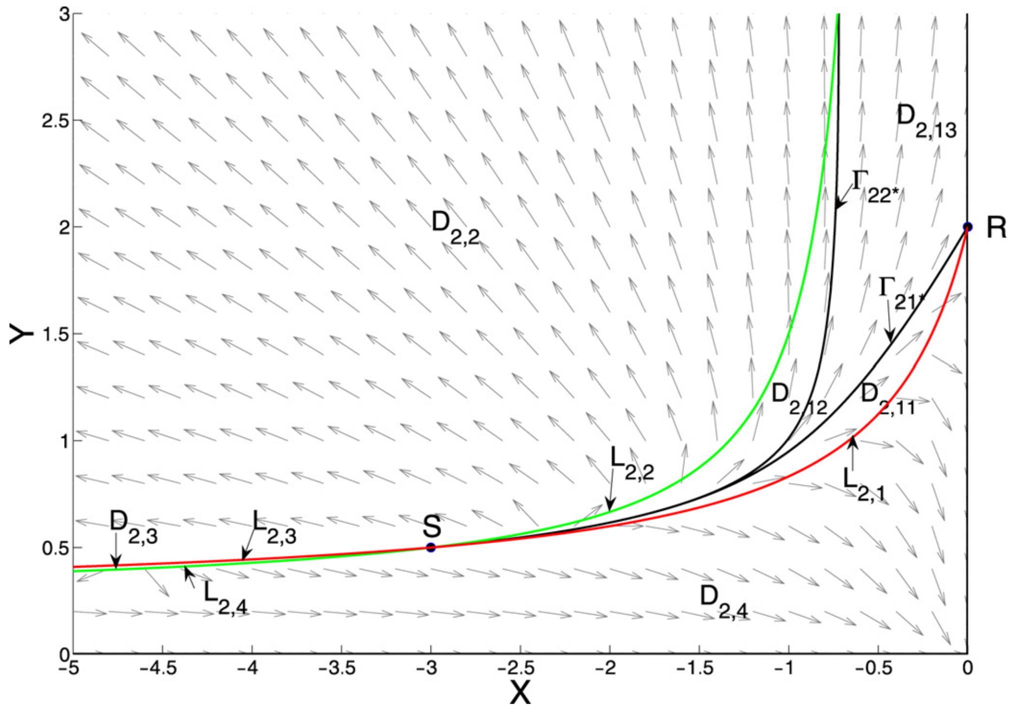

Fig. 4. The vector field generated by (3.1)–(3.2) for $m = -2$. The coordinates of R and S are $(0, 2)$ and $(-3, 1/2)$, respectively. Note that the trajectory Γ_{23*} is bounded by the curves L_{32} and L_{24}.

6. Proof of Theorem 1

Now we are in a position to prove Theorem 1. Fix $m < -1$ and $a > 0$. Let f_b be the solution of $(P_{m,a,b})$ defined on the right maximal existence interval $[0, T_b)$. Recall from Proposition 2 that f_b is of type (C) for $b \leqslant 0$. *Thus, we will assume that $b > 0$ throughout the remaining of this section.* Also recall from Definition 5.1 that $\tilde{t}_0 = \tilde{t}_{0b}$, t_{0b}, and $\hat{t}_{0b}$ are the first zero of f_b, f_b', and f_b'', respectively. Set $s_{0b} := \min\{\tilde{t}_{0b}, t_{0b}, \hat{t}_{0b}\}$. Note that s_{0b} may be $+\infty$. Let (X_b, Y_b) be the solution of the system (3.1)–(3.2) corresponding to $f_b(t)$ for $t \in [0, s_{0b})$. Note that the initial data of (X_b, Y_b) is $(X_b(0), Y_b(0)) = (-a/b, 1/(ab))$ which lies in the second quadrant of the phase plane and lies on the straight line Γ defined by

$$Y = -\frac{1}{a^2}X.$$

We will consider three disjoint cases: $a \geqslant \sqrt{6}$, $a \in (0, \sqrt{6})$ and $a = 0$.

The structure of solutions of $(P_{m,a})$ for $m < -1$ and $a \geqslant \sqrt{6}$.

First, we consider the case $a \geqslant \sqrt{6}$. Indeed, if $a > \sqrt{6}$, then the initial line Γ intersects the trajectory of (X_{23*}, Y_{23*}) defined in Lemma 5.2 at exactly one point, say $(X_{b_{m,a}}(0), Y_{b_{m,a}}(0))$; and if $a = \sqrt{6}$, then the initial line Γ intersects the closure of the trajectory of (X_{23*}, Y_{23*}) at $(-3, 1/2)$ which we still denote by $(X_{b_{m,a}}(0), Y_{b_{m,a}}(0))$. Then by the phase plane analysis (see Lemma 5.2 and Fig. 4), for each $b > b_{m,a}$, the solution $(X_b(\xi), Y_b(\xi))$ will tend to $(0, 0)$ as $\xi \to +\infty$; and for each $b \in (0, b_{m,a})$, the solution $(X_b(\xi), Y_b(\xi))$ will tend to $(-\infty, +\infty)$ as ξ approaches some finite Ξ. Hence by Lemmas 5.1 and 5.3, f_b is of type (G_1) for all $b > b_{m,a}$ and of type (G_5) for all $b \in (0, b_{m,a})$. Note that a type (G_5) solution cannot be a solution of $(P_{m,a})$ by part (i) of Proposition 1. In fact, a type (G_5) solution is of type (C) by Proposition 2. Finally, since a convex-concave solution $f(t)$ of $(P_{m,a})$ must take negative values for all sufficiently large t by part (iv) of Proposition 1, a type (G_1) solution cannot be a solution of $(P_{m,a})$, and so is of type (B_2) by Proposition 2.

To summarize, if $m < -1$ and $a \geqslant \sqrt{6}$, f_b is of type (B_2) for all $b > b_{m,a}$, and of type (C) for all $b < b_{m,a}$. Since the set of type (B_2) solutions and the set of type (C) solutions are open and disjoint, it follows that $f_{b_{m,a}}$ is a convex solution of $(P_{m,a})$. Moreover, since $b_{m,a}$ is the greatest lower bound of the set of type (B_2) solutions, $f_{b_{m,a}} > 0$ on $[0, +\infty)$. When $a > \sqrt{6}$, the asymptotic behaviour of $f_{b_{m,a}}$ follows from the long time behaviour of (X_{23*}, Y_{23*}). We note that when $a = \sqrt{6}$, such a convex solution is given by $f_{b_{m,a}} = 6/(t + \sqrt{6})$.

The structure of solutions of $(P_{m,a})$ for $m < -1$ and $a \in (0, \sqrt{6})$.

Now, we turn to the case: $a \in (0, \sqrt{6})$. The proof of this case is divided into three steps.

Step 1. In this case, the initial line Γ intersects the trajectory of (X_{21*}, Y_{21*}) (resp. (X_{22*}, Y_{22*})) defined in Lemma 5.2 at exactly one point, say $(X_{\hat{b}}(0), Y_{\hat{b}}(0))$ (resp. $(X_{\tilde{b}}(0), Y_{\tilde{b}}(0))$). Note that $\hat{b}$ and $\tilde{b}$ depend on m and a. Then by the phase plane analysis (see Lemma 5.2 and Fig. 4), for each $b > \hat{b}$, the solution $(X_b(\xi), Y_b(\xi))$ will tend to $(0, 0)$ as $\xi \to +\infty$; and for each $b \in (0, \tilde{b})$, the solution $(X_b(\xi), Y_b(\xi))$ will tend to $(-\infty, +\infty)$ as ξ approaches some finite Ξ. By using a similar argument as in the case $a > \sqrt{6}$, f_b cannot be a solution of $(P_{m,a})$ for all $b \in (0, \tilde{b}] \cup [\hat{b}, +\infty)$. Moreover, f_b is of type (B_2) for all $b > \hat{b}$ and of type (C) for all $b < \tilde{b}$. Hence it remains to consider the case: $b \in (\tilde{b}, \hat{b})$.

Step 2. By part (vi) of Lemmas 5.2, 5.1 and 5.3, f_b is of type (G_3) for all $b \in (\tilde{b}, \hat{b})$. Moreover, $f_{\hat{b}}$ is of type (G_2) and $f_{\tilde{b}}$ is of type (G_4). Thus, for each $b \in (\tilde{b}, \hat{b})$, there exists a finite $\tilde{t}_{0b} > 0$ such that $f_b > 0$ on $[0, \tilde{t}_{0b})$, $f_b(\tilde{t}_{0b}) = 0$, and $f_b' < 0$ and $f_b'' > 0$ on $[0, \tilde{t}_{0b}]$. Hence by the theory of continuous dependence on initial data, we can conclude that if $b < \hat{b}$ and b is sufficiently close to $\hat{b}$, then there exists a $t_{0b} > \tilde{t}_{0b}$ such that $f_b < 0$ on $(\tilde{t}_{0b}, t_{0b}]$, $f_b' < 0$ on $[0, t_{0b})$ and $f_b'(t_{0b}) = 0$, and $f_b'' > 0$ on $[0, t_{0b}]$; and if $b > \tilde{b}$ and b is sufficiently close to $\tilde{b}$, then there exists $\hat{t}_{0b} > \tilde{t}_{0b}$ such that $f_b < 0$ on $(\tilde{t}_{0b}, \hat{t}_{0b}]$, $f_b' < 0$ on $[0, \hat{t}_{0b}]$, and $f_b'' > 0$ on $[0, \hat{t}_{0b})$ and $f_b''(\hat{t}_{0b}) = 0$.

Now consider the sets

$$\mathcal{B}_{m,a} := \{ b \in (\tilde{b}, \hat{b}) \mid \text{the solution } f_a \text{ of } (P_{m,a,b}) \text{ is of type (B)} \},$$

$$\mathcal{C}_{m,a} := \{ b \in (\tilde{b}, \hat{b}) \mid \text{the solution } f_a \text{ of } (P_{m,a,b}) \text{ is of type (C)} \}.$$

Note that the sets $\mathcal{B}_{m,a}$ and $\mathcal{C}_{m,a}$ are disjoint and that $\mathcal{B}_{m,a}$ and $\mathcal{C}_{m,a}$ are nonempty by the above discussion. Hence the quantities

$$\inf \mathcal{B}_{m,a} \quad \text{and} \quad \sup \mathcal{C}_{m,a}$$

are well defined and positive. Furthermore, any solution of $(P_{m,a,b})$ is either of type (A), type (B), or type (C) by Proposition 2. By the standard theory of continuous dependence on initial data and part (i) of Proposition 1, the sets $\mathcal{B}_{m,a}$ and $\mathcal{C}_{m,a}$ are open. From these facts it follows that the solutions $f_{\inf \mathcal{B}_{m,a}}$ and $f_{\sup \mathcal{C}_{m,a}}$ are of type (A). On the other hand, by Theorem 1 of [6], type (A) solution of $(P_{m,a,b})$ is unique. Hence we have $b_{m,a,-} := \inf \mathcal{B}_{m,a} = \sup \mathcal{C}_{m,a}$, which implies that the solution f_b of $(P_{m,a,b})$ is of type (B) for all $b > b_{m,a,-}$, and is of type (C) for all $b < b_{m,a,-}$. For simplicity, we will write $b_{m,a,-}$ as b_- in the remaining of the proof.

Step 3. Therefore, in order to look for the convex-concave solutions of $(P_{m,a})$, it suffices to concentrate on the interval $(b_-, \hat{b})$. First, since f_b is of type (G_3) and type (B) for each $b \in (b_-, \hat{b})$, there exist finite $\tilde{t}_{0b}$ and t_{0b} such that $f_b < 0$ on $(\tilde{t}_{0b}, t_{0b}]$, $f_b' < 0$ on $[0, t_{0b})$ and $f_b'(t_{0b}) = 0$, and $f_b'' > 0$ on $[0, t_{0b}]$ (by part (i) of Proposition 1). Similarly, since f_{b_-} is a convex solution of $(P_{m,a})$ and of type (G_3), there exists a finite $\tilde{t}_{0b_-}$ such that $f_{b_-} < 0$, $f_{b_-}' < 0$, and $f_{b_-}'' > 0$ on $(\tilde{t}_{0b_-}, +\infty)$. Now we set $s_{1b} = t_{0b}$ if $b \in (b_-, \hat{b})$, and $s_{1b} = +\infty$ if $b = b_-$. For $b \in [b_-, \hat{b})$, let $(\widetilde{X}_b, \widetilde{Y}_b)$ be the solution of the system (3.1)–(3.2) corresponding to $f_b(t)$ for $t \in (\tilde{t}_{0b}, s_{1b})$. Note that $(\widetilde{X}_b, \widetilde{Y}_b)$ is defined on $(-\ln|f_b'(\tilde{t}_{0b})|, +\infty)$ for $b \in [b_-, \hat{b})$ and the corresponding trajectory $\{(\widetilde{X}_b(\xi), \widetilde{Y}_b(\xi)) \mid \xi \in (-\ln|f_b'(\tilde{t}_{0b})|, +\infty)\}$ lies in the fourth quadrant of the phase plane.

Now by Lemmas 4.5 and 4.4, we have that the trajectory of $(\widetilde{X}_{b_-}, \widetilde{Y}_{b_-})$ coincides with the one of (X_{41*}, Y_{41*}) defined in Lemma 4.4. Hence the asymptotic behaviour of f_{b_-} follows from the long time behaviour of (X_{41*}, Y_{41*}). Therefore, by the theory of continuous dependence on initial data, there exists $b_0 > b_-$ which is sufficiently close to b_-, such that for each $b \in (b_-, b_0)$, $(\widetilde{X}_b, \widetilde{Y}_b)$ enters into the region $D_{4,22}$ defined in the paragraph right before Lemma 4.6. Since $D_{4,22}$ is invariant with respect to the system (3.1)–(3.2), the trajectory of $(\widetilde{X}_b, \widetilde{Y}_b)$ lies in the region $D_{4,22}$ for all $b \in (b_-, b_0)$. Hence by Lemma 4.6, f_b is a type $(\mathrm{B}_{1,\mathrm{I}_2})$ solution of $(P_{m,a})$ and $\lim_{t \to +\infty} f_b(t) = 0$ for all $b \in (b_-, b_0)$.

Now consider the set

$$\widehat{\mathcal{B}}_{m,a} := \{ b \in (b_-, \hat{b}) \mid \text{the solution } f_b \text{ of } (P_{m,a,b}) \text{ is of type } (\mathrm{B}_{1,\mathrm{I}_2}) \}.$$

Note that $(b_-, b_0) \subseteq \widehat{\mathcal{B}}_{m,a}$. By a similar argument as in Step 1, f_b is of type (B_2) for all $b < \hat{b}$ sufficiently close to $\hat{b}$, and so $\widehat{\mathcal{B}}_{m,a}$ is bounded above. By Lemma 4.3, the set $\widehat{\mathcal{B}}_{m,a}$ is open. Hence the quantity $b_{m,a,+} := \sup \widehat{\mathcal{B}}_{m,a}$ exists, and $f_{b_{m,a,+}}$ cannot be of type (B_2) or of type $(\mathrm{B}_{1,\mathrm{I}_2})$. Thus the solution $f_{b_{m,a,+}}$ is of type $(\mathrm{B}_{1,\mathrm{I}_1})$ by Lemma 4.3. Moreover, it is a solution of $(P_{m,a})$. Finally, by Lemmas 3.4 and 5.2, the trajectory of (X_b, Y_b) are distinct among $b \in (\tilde{b}, \hat{b})$, which together with part (i) of Lemma 3.3, implies that type $(\mathrm{B}_{1,\mathrm{I}_1})$ solution of $(P_{m,a})$ is unique among $b \in (\tilde{b}, \hat{b})$. Hence f_b is of type (B_2) for all $b \in (b_{m,a,+}, \hat{b})$ and of type $(\mathrm{B}_{1,\mathrm{I}_2})$ for all $b \in (b_-, b_{m,a,+})$. This completes the proof for the case $a \in (0, \sqrt{6})$.

The structure of solutions of $(P_{m,a})$ for $m < -1$ and $a = 0$.

The proof of this case consists of two steps.

Step 1. First we claim that there exist $\hat{b}_1 = \hat{b}_1(m) > 0$, $t_0 = t_0(b)$, and $s_0 = s_0(b)$ such that for all $b > \hat{b}_1$ there holds

$$f_b < 0 \quad \text{on } (0, s_0) \quad \text{and} \quad f_b(s_0) = 0,$$
$$f_b' < 0 \quad \text{on } (0, t_0) \quad \text{and} \quad f_b' > 0 \quad \text{on } (t_0, s_0],$$
$$f_b''(t) \in (b/2, 3b/2) \quad \text{for all } t \in [0, s_0].$$

Let $b > 0$ and consider the auxiliary function

$$g_\mu(t) := f_b(\mu t)/\mu, \quad \mu = 1/b.$$

Then we see that g_μ satisfies the following initial value problem (I):

$$g_\mu''' = \mu^2\bigl\{-\bigl[(m+1)/2\bigr]g_\mu g_\mu'' + m\bigl(g_\mu'\bigr)^2\bigr\},$$
$$g_\mu(0) = 0, \qquad g_\mu'(0) = -1, \qquad g_\mu''(0) = 1.$$

As $\mu \to 0$, the limiting problem of (I) is given by

$$\psi''' = 0, \qquad \psi(0) = 0, \qquad \psi'(0) = -1, \qquad \psi''(0) = 1,$$

whose solution is given by

$$\psi(t) = t^2/2 - t,$$

which yields

$$\psi < 0 \quad \text{on } (0, 2) \quad \text{and} \quad \psi(2) = 0,$$
$$\psi' < 0 \quad \text{on } (0, 1) \quad \text{and} \quad \psi' > 0 \quad \text{on } (1, 2],$$
$$\psi'' \equiv 1 \quad \text{on } [0, 2].$$

Then by the standard theory of continuous dependence on parameter, there exist positive numbers $\mu_0 = \mu_0(m)$, $t_0' = t_0'(\mu)$, and $s_0' = s_0'(\mu)$ such that for all $\mu \in (0, \mu_0)$, there holds

$$g_\mu < 0 \quad \text{on } \bigl(0, s_0'\bigr) \quad \text{and} \quad g_\mu\bigl(s_0'\bigr) = 0,$$
$$g_\mu' < 0 \quad \text{on } \bigl(0, t_0'\bigr) \quad \text{and} \quad g_\mu' > 0 \quad \text{on } \bigl(t_0', s_0'\bigr],$$
$$g_\mu''(t) \in (1/2, 3/2) \quad \text{for all } t \in \bigl[0, s_0'\bigr].$$

Finally, by setting $\hat{b}_1 = 1/\mu_0$, $t_0 = \mu t_0'$ and $s_0 = \mu s_0'$ and recalling that $f_b(t) = g_{1/b}(bt)/b$, the assertion of this claim follows.

From part (iv) of Proposition 1 it follows that for all $b > \hat{b}_1$, f_b cannot be a convex-concave solution of $(P_{m,a})$, and so is of type (B_2).

Step 2. Let $\tilde{b} = 0$ and $\hat{b} = \hat{b}_1 + 1$. By Proposition 1, f_b is of type (C) for all sufficiently small positive b. Then applying the same argument as in Steps 2–3 of the case $a \in (0, \sqrt{6})$, we can obtain the desired conclusion. Here note that the uniqueness of type $(\mathrm{B}_{1,\mathrm{I}_1})$ solution of $(P_{m,a})$ is given by Lemma 4.3. Therefore the proof of Theorem 1 is completed.

7. Discussion

In this article, we have studied the boundary value problem $(P_{m,a})$ whose solutions are related to the dynamical behaviour of the fluid (near the sharp corners) for the liquid metal systems in a high frequency antisymmetric magnetic field. We have established the complete picture of solutions of $(P_{m,a})$ for the case of physical interest: $m < -1$ and $a \geqslant 0$. In particular, our results shows that for $m < -1$ and $a \in [0, \sqrt{6})$, there is a family of convex-concave solutions of the problem $(P_{m,a})$, which gives a definite answer to the open problem proposed in [6] (see also [16]).

For mathematical completeness of the structure of solutions of $(P_{m,a})$ (or more generally, $(Q_{m,a,c})$), it would be interesting to investigate the case for $m \in (-1, 0)$ and $a \in \mathbb{R}$, in particular, the following open problems proposed in [6]:

(O1) For $m \in (-1/3, 0)$ and $a \in \mathbb{R}$, is the convex solution unique?
(O2) For $m \in (-1, -1/2]$ and $a > 0$, is there convex-concave solutions?
(O3) For $m \in (-1/2, 0)$ and $a \in \mathbb{R}$, is there convex-concave solutions?

Our numerical attempts indicates that the structures of solutions of $(P_{m,a})$ for the above three cases are quite complicated. For example, when $m \in (-1/3, 0)$ and $a \leqslant 0$, the convex solution of $(P_{m,a})$ is unique, while there is a family of convex solutions of $(P_{m,a})$ for $m \in (-1/3, 0)$ and some positive a. The study of these cases will be our future study.

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Appendix A. Proof of Lemma 3.2

In this appendix, we give a proof of Lemma 3.2. For reader's convenience, we restate it as follows:

Lemma A.1. *Let $m < -1$ and f be a solution of* (1.1) *defined on the maximal existence interval (S, T). If there exists $t_0 \in (S, T)$ such that $f'(t_0) \geqslant 0$ and $f''(t_0) > 0$, then there exists a $\hat{t}_0 > t_0$ such that $f'' > 0$ on $(t_0, \hat{t}_0)$ and $f'' < 0$ on $(\hat{t}_0, T)$.*

Proof. For contradiction, we assume that $f'' > 0$ on $[t_0, T)$. Then by Lemma 3.1, we have $f' > 0$ on (t_0, T) and $f' \to +\infty$ as $t \to T^-$. Hence we can choose a $t_1 \in [t_0, T)$ such that one of the following three cases holds:

(i) $f(t_1) = f'(t_1) = 0$, and $f > 0$, $f' > 0$, and $f'' > 0$ on (t_1, T).
(ii) $f(t_1) > 0$, $f'(t_1) = 0$, and $f > 0$, $f' > 0$, and $f'' > 0$ on (t_1, T).
(iii) $f(t_1) = 0$, $f'(t_1) > 0$, and $f > 0$, $f' > 0$, and $f'' > 0$ on (t_1, T).

In what follows, we only consider case (i) since the other cases follows by analogous arguments. Set the following change of variables for the solution f:

$$X(\xi) = f(t)f'(t)/f''(t), \qquad Y(\xi) = f'(t)^2/\big[f(t)f''(t)\big], \quad \xi = \ln f'(t),$$

where we require $t \in (t_1, T)$. Then (X, Y) satisfies the following ordinary differential system

$$\frac{dX}{d\xi} = X\big\{1 + \big[(m+1)/2\big]X + Y - mXY\big\}, \tag{A.1}$$

$$\frac{dY}{d\xi} = Y\big\{2 + \big[(m+1)/2\big]X - Y - mXY\big\}. \tag{A.2}$$

Note that $(X(\xi), Y(\xi))$ is defined on $(\ln f'(t_1), +\infty)$, that $X(\xi) > 0$ and $Y(\xi) > 0$ for all $\xi > \ln f'(t_1)$, and that the system (A.1)–(A.2) is different from the system (3.1)–(3.2) by a negative sign.

Since $f(t_0) = f'(t_0) = 0$, by using a similar argument as in part (2) of Lemma 5.1, we can compute

$$\lim_{\xi \to (\ln f'(t_0))^+} \big(X(\xi), Y(\xi)\big) = (0, 2).$$

Then by using a phase plane analysis (see Lemma 4.2), we have $(X(\xi), Y(\xi)) \to (+\infty, +\infty)$ as $\xi \to +\infty$. Hence there exists $\xi_1 > 0$ such that for all $\xi > \xi_1$, we have

$$1 + \frac{m+1}{2}X(\xi) + Y(\xi) - mX(\xi)Y(\xi) \geqslant X(\xi).$$

Together with (A.1), we can estimate $dX/d\xi$ as follows:

$$\frac{dX}{d\xi} = X(\xi)\left(1 + \frac{m+1}{2}X(\xi) + Y(\xi) - mX(\xi)Y(\xi)\right) \geqslant X(\xi)^2 \quad \text{for all } \xi \geqslant \xi_1.$$

This implies that $(X(\xi), Y(\xi))$ is only defined on finite interval, a contradiction. Hence there exists a $\hat{t}_0 > t_0$ such that $f'' > 0$ on $(t_0, \hat{t}_0)$ and $f''(\hat{t}_0) = 0$. Finally, together with part (i) of Proposition 1, we have $f'' < 0$ on $(\hat{t}_0, T)$. This completes the proof. □

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