



Exact stability sets for a linear difference system with diagonal delay

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ABSTRACT

This paper is devoted to the stability analysis of a delay difference system of the form $x_{n+1} = ax_{n-k} + by_n$, $y_{n+1} = cx_n + ay_{n-k}$, where a , b and c are real numbers and k is a positive integer. We establish some exact conditions for the zero solution of the system to be asymptotically stable.

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1. Introduction

It is well known that studies on difference systems often start with stability analysis of the linearized systems near an equilibrium to investigate some dynamical behaviors such as periodic phenomenon, bifurcation and chaos. Recently, there have been many results on the asymptotic stability of linear difference systems with delay; see, e.g., [2,4–6,10–13,15] and references therein.

In this paper we are concerned with a delay difference system

$$\begin{cases} x_{n+1} = ax_{n-k} + by_n, \\ y_{n+1} = cx_n + ay_{n-k}, \end{cases} \quad n = 0, 1, 2, \dots, \quad (1)$$

where a , b and c are real numbers and k is a positive integer. Our purpose is to establish necessary and sufficient conditions for the zero solution of (1) to be asymptotically stable. The following theorems are our main results.

Theorem 1. Let k be odd. Then the zero solution of (1) is asymptotically stable if and only if $\sqrt{|bc|} < (k+1)/k$ and

$$-(bc + 1 - 2\sqrt{bc} \cos \phi)^{1/2} < a < 1 - \sqrt{bc} \quad \text{for } bc \geq 0, \quad (2)$$

$$-(-bc + 1 - 2\sqrt{-bc} \cos \phi)^{1/2} < (-1)^{(k+1)/2} a < 1 - \sqrt{-bc} \quad \text{for } bc < 0, \quad (3)$$

where ϕ is the solution of the equation $\sqrt{|bc|} \sin k\phi = \sin(k+1)\phi$ in $(0, \pi/(k+1))$.

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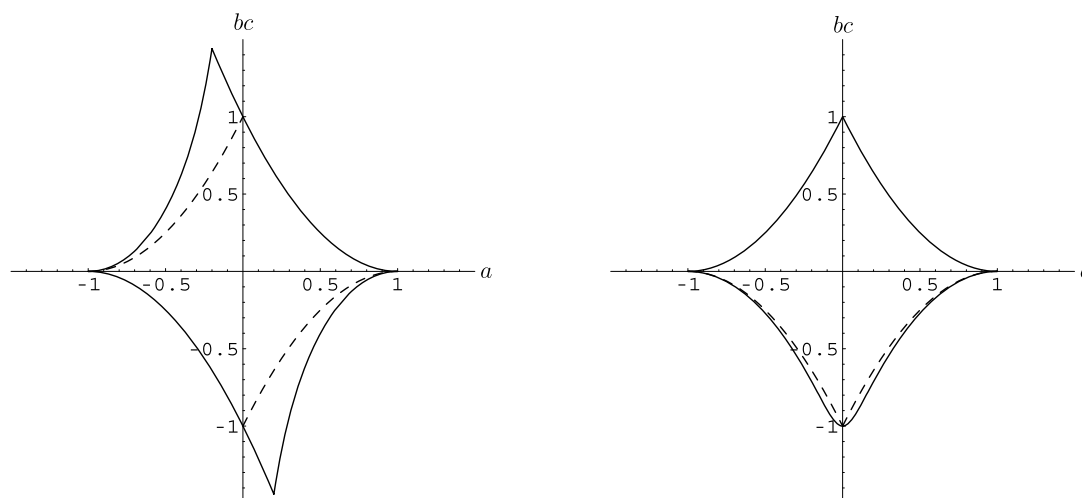


Fig. 1. Stability sets for (1) with $k = 5$ (left) and $k = 6$ (right).

Theorem 2. Let k be even. Then the zero solution of (1) is asymptotically stable if and only if $\sqrt{|bc|} < 1$ and

$$|a| < 1 - \sqrt{bc} \quad \text{for } bc \geq 0, \quad (4)$$

$$|a| < (-bc + 1 - 2\sqrt{-bc} \cos \psi)^{1/2} \quad \text{for } bc < 0, \quad (5)$$

where ψ is the solution of the equation $\sqrt{|bc|} \cos k\psi = \cos(k+1)\psi$ in $(0, \pi/2(k+1))$.

To illustrate the main theorems, we present some exact stability sets of the pair of all (a, bc) in which the zero solution of (1) with fixed k is asymptotically stable; see Fig. 1. Notice that the boundary of the stability set for (1) with $a \geq 0$ and $bc \geq 0$ is given by the equation $a + \sqrt{bc} = 1$, which is independent of k . On the other hand, the remaining boundary with $a < 0$ or $bc < 0$ is depend on k . In fact, if k is odd, the boundary with $-1 < a < -1/k$ and $bc > 0$ is given parametrically by the equation

$$a = -\frac{\sin \theta}{\sin k\theta}, \quad bc = \left(\frac{\sin(k+1)\theta}{\sin k\theta} \right)^2, \quad 0 < \theta < \frac{\pi}{k+1}.$$

Also, if k is even, the boundary with $bc < 0$ is given parametrically by the equation

$$|a| = \frac{\sin \theta}{\cos k\theta}, \quad bc = -\left(\frac{\cos(k+1)\theta}{\cos k\theta} \right)^2, \quad 0 < \theta < \frac{\pi}{2(k+1)}.$$

Furthermore, it is easily seen that the stability sets for (1) approach $|a| + \sqrt{|bc|} < 1$ as $k \rightarrow \infty$, which is a stability set for all k .

Theorems 1 and 2 are proved by using the fact (see, e.g., [1]) that the zero solution of (1) is asymptotically stable if and only if all the roots of its associated characteristic equation

$$F(\lambda) \equiv \det \begin{pmatrix} \lambda^{k+1} - a & -b\lambda^k \\ -c\lambda^k & \lambda^{k+1} - a \end{pmatrix} = 0 \quad (6)$$

are inside the unit disk. The rest of this paper is organized as follows: In Section 2, we present some auxiliary results on the distribution of roots of algebraic equations. In Section 3, we prove the main theorems. The paper ends with a conclusion in Section 4.

2. Preliminaries

In this section we give some exact conditions for all the roots of algebraic equations

$$\lambda^{k+1} - \alpha\lambda^k + \beta = 0 \quad (7)$$

and

$$\lambda^{k+1} - \alpha\lambda^k + i\beta = 0 \quad (8)$$

to be inside the unit disk, respectively. Here, α and β are real numbers and $i = \sqrt{-1}$. Notice that the Schur–Cohn criterion (see [1,8]) and the Jury criterion (see [3]) are known to be effective tools for the stability analysis; however, several kinds of the necessary and sufficient conditions for the roots of equations of higher degree to be inside the unit disk established by the above criteria are too much complicated.

For Eq. (7), Kuruklis [9, Theorem 3] gave the following result by careful root analysis.

Theorem A. All the roots of (7) are inside the unit disk if and only if $|\alpha| < (k+1)/k$ and

$$|\alpha| - 1 < \beta < (\alpha^2 + 1 - 2|\alpha| \cos \phi)^{1/2} \quad \text{for } k \text{ odd,}$$

$$|\beta - \alpha| < 1 \quad \text{and} \quad |\beta| < (\alpha^2 + 1 - 2|\alpha| \cos \phi)^{1/2} \quad \text{for } k \text{ even,}$$

where ϕ is the solution of the equation $|\alpha| \sin k\phi = \sin(k+1)\phi$ in $(0, \pi/(k+1))$.

For Eq. (8), the following result was also established in [11, Theorem 2.4] without a proof.

Theorem B. All the roots of (8) are inside the unit disk if and only if $|\alpha| < 1$ and

$$|\beta| < (\alpha^2 + 1 - 2|\alpha| \cos \psi)^{1/2},$$

where ψ is the solution of the equation $|\alpha| \cos k\psi = \cos(k+1)\psi$ in $(0, \pi/2(k+1))$.

We will prove Theorem B for the completeness. In case $\alpha \neq 0$, Eq. (8) can be written equivalently as

$$z^{k+1} - z^k + iq = 0, \tag{9}$$

where $z = \lambda/\alpha$ and $q = \beta/\alpha^{k+1}$. Thus, all the roots of (8) are inside the unit disk if and only if all the roots of (9) are inside the disk $|z| < 1/|\alpha|$. Consequently, we will now investigate the distribution of the roots of (9) as $|q|$ varies from 0 to ∞ with fixed α . Note that, for $q = 0$, the roots of (9) are 0 (multiplicity k) and 1 (simple), and for $q \neq 0$, Eq. (9) has no real roots.

As a beginning, we will examine the existence region of the arguments of complex roots of (9). For simplicity, we define the following intervals

$$I = \bigcup_{m=0}^{[(k-1)/2]} I_m, \quad I_m = \left(\frac{(4m+1)\pi}{2k}, \frac{(4m+3)\pi}{2(k+1)} \right), \quad m \geq 0,$$

$$J = \bigcup_{m=0}^{[k/2]} J_m, \quad J_0 = \left(0, \frac{\pi}{2(k+1)} \right), \quad J_m = \left(\frac{(4m-1)\pi}{2k}, \frac{(4m+1)\pi}{2(k+1)} \right), \quad m \geq 1,$$

where $[\cdot]$ denotes the greatest-integer function.

Lemma 1. Let $re^{i\theta}$ be a complex root of (9) where $r > 0$ and $0 < |\theta| < \pi$. Then $|\theta| \in I$ for $q\theta > 0$, and $|\theta| \in J$ for $q\theta < 0$.

Proof. By substituting $z = re^{i\theta}$ into (9), we have $r^{k+1}e^{i(k+1)\theta} - r^k e^{ik\theta} + iq = 0$, namely

$$r^k \{ \cos k\theta - r \cos(k+1)\theta \} = 0, \tag{10}$$

and

$$r^k \{ \sin k\theta - r \sin(k+1)\theta \} = q. \tag{11}$$

It is obvious that $\cos(k+1)\theta \neq 0$, and therefore, Eq. (10) becomes

$$r = \frac{\cos k\theta}{\cos(k+1)\theta}, \tag{12}$$

which, together with (11), yields

$$q = -r^k \frac{\sin \theta}{\cos(k+1)\theta}. \tag{13}$$

In case $q > 0$ and $0 < \theta < \pi$, we must have $\cos k\theta < 0$ and $\cos(k+1)\theta < 0$ because of (12) and (13), which imply $\theta \in I$. In the remaining cases, the proof is similar and will be omitted. $\square$

Next, we will observe how the roots of (9) move in the complex plane when q varies.

Lemma 2. The simple root $z = 1$ of (9) moves to the region $\{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$ as $|q|$ increases from 0.

Proof. Taking the derivative of z with respect to q on (9), we have

$$(k+1)z^k \frac{dz}{dq} - kz^{k-1} \frac{dz}{dq} + i = 0,$$

or equivalently

$$\frac{dz}{dq} = -\frac{i}{(k+1)z^k - kz^{k-1}}. \quad (14)$$

Hence, it follows that

$$\operatorname{Re} \frac{dz}{dq} \Big|_{z=1} = 0 \quad \text{and} \quad \operatorname{Im} \frac{dz}{dq} \Big|_{z=1} = -1 < 0, \quad (15)$$

which means that the simple root $z = 1$ vertically crosses the real axis from top to bottom as q increases. Also, by (14), we obtain

$$\frac{d^2z}{dq^2} = \frac{k(k+1)z^{k-1} - k(k-1)z^{k-2}}{\{(k+1)z^k - kz^{k-1}\}^3},$$

which yields

$$\operatorname{Re} \frac{d^2z}{dq^2} \Big|_{z=1} = 2k > 0.$$

This, together with (15), implies the assertion of this lemma. $\square$

Lemma 3. The absolute values of complex roots of (9) increase as $|q|$ increases.

Proof. Let $re^{i\theta}$ be a complex root of (9). It suffices to show that

$$\frac{dr}{dq} \cdot q > 0. \quad (16)$$

From (12) and (13), it follows that

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{-k \sin k\theta \cos(k+1)\theta + (k+1) \cos k\theta \sin(k+1)\theta}{\cos^2(k+1)\theta} \\ &= \frac{k \sin \theta + \cos k\theta \sin(k+1)\theta}{\cos^2(k+1)\theta} \end{aligned}$$

and

$$\begin{aligned} \frac{dq}{d\theta} &= -kr^{k-1} \frac{dr}{d\theta} \cdot \frac{\sin \theta}{\cos(k+1)\theta} - r^k \frac{\cos \theta \cos(k+1)\theta + (k+1) \sin \theta \sin(k+1)\theta}{\cos^2(k+1)\theta} \\ &= -kr^k \frac{\{k \sin \theta + \cos k\theta \sin(k+1)\theta\} \sin \theta}{\cos k\theta \cos^2(k+1)\theta} - r^k \frac{\cos k\theta + k \sin \theta \sin(k+1)\theta}{\cos^2(k+1)\theta} \\ &= -r^k \frac{k^2 \sin^2 \theta + 2k \sin \theta \cos k\theta \sin(k+1)\theta + \cos^2 k\theta}{\cos k\theta \cos^2(k+1)\theta}. \end{aligned}$$

The above relations then yield

$$\frac{dr}{dq} = \frac{dr/d\theta}{dq/d\theta} = -\frac{\cos k\theta G(\theta)}{r^k H(\theta)}, \quad (17)$$

where

$$G(\theta) = k \sin \theta + \cos k\theta \sin(k+1)\theta,$$

$$H(\theta) = k^2 \sin^2 \theta + 2k \sin \theta \cos k\theta \sin(k+1)\theta + \cos^2 k\theta.$$

Note that, by (12),

$$\begin{aligned}
G'(\theta) &= k \cos \theta - k \sin k\theta \sin(k+1)\theta + (k+1) \cos k\theta \cos(k+1)\theta \\
&= k \cos \theta + k \cos(2k+1)\theta + \cos k\theta \cos(k+1)\theta \\
&= (2k+1) \cos k\theta \cos(k+1)\theta \\
&= (2k+1)r \cos^2(k+1)\theta > 0.
\end{aligned}$$

In case $q > 0$ and $0 < \theta < \pi$, Lemma 1 shows $\theta \in I$ and $\cos k\theta < 0$. Since $G(\theta)$ is strictly increasing for $\theta \in I$ and

$$G\left(\frac{(4m+1)\pi}{2k}\right) = k \sin \frac{(4m+1)\pi}{2k} > 0, \quad m = 1, 2, \dots, \left[\frac{k-1}{2}\right],$$

we have $G(\theta) > 0$ for $\theta \in I$. Also, we obtain $H(\theta) > (k \sin \theta + \cos k\theta)^2 \geq 0$, and thus, the inequality (16) holds from (17). In the remaining cases, the proof is similar and will be omitted. $\square$

Furthermore, we will find the minimum value of $|q|$ when a root of (9) lies on the circle $|z| = 1/|\alpha|$.

Lemma 4. Let $\alpha \neq 0$. If $z = re^{i\theta}$ is a root of (9) on the circle $|z| = 1/|\alpha|$, then the minimum value of $|q|$ is given by

$$q_0 = \frac{(\alpha^2 + 1 - 2|\alpha| \cos \theta)^{1/2}}{|\alpha|^{k+1}},$$

where θ satisfies $|\theta| \in J_0$ and $|\alpha| \cos k\theta = \cos(k+1)\theta$.

Proof. By squaring both sides of (10) and (11), and adding them together, we have $q^2 = r^{2k}(r^2 + 1 - 2r \cos \theta)$, which implies

$$|q| = \frac{(\alpha^2 + 1 - 2|\alpha| \cos \theta)^{1/2}}{|\alpha|^{k+1}}$$

because of $r = 1/|\alpha|$. Since $|q|$ is increasing with respect to $|\theta|$, the minimum value of $|q|$ corresponds to the minimum value of $|\theta|$ satisfying $\cos k\theta / \cos(k+1)\theta = 1/|\alpha|$ by (12). This, together with Lemma 1, yields the assertion of this lemma. $\square$

The following lemma plays a key role in the proof of Theorem B.

Lemma 5. Let $\alpha \neq 0$. Then all the roots of (9) are inside the disk $|z| < 1/|\alpha|$ if and only if $|\alpha| < 1$ and

$$|q| < q_0 = \frac{(\alpha^2 + 1 - 2|\alpha| \cos \psi)^{1/2}}{|\alpha|^{k+1}},$$

where ψ is the solution of the equation $|\alpha| \cos k\psi = \cos(k+1)\psi$ in $(0, \pi/2(k+1))$.

Proof. (Sufficiency) In case $q = 0$, the roots of (9) are 0 and 1, which are inside the disk $|z| < 1/|\alpha|$ because of $|\alpha| < 1$. Lemma 4 shows that q_0 is the minimum value of $|q|$ such that a root of (9) intersects the circle $|z| = 1/|\alpha|$. By the continuity of the roots of (9), we therefore conclude that if $|\alpha| < 1$ and $|q| < q_0$, then all the roots of (9) are inside the disk $|z| < 1/|\alpha|$.

(Necessity) To end the proof, we will show the following contraposition: either $|\alpha| \geq 1$ or $|q| \geq q_0$ implies that there exists a root z^* of (9) such that $|z^*| \geq 1/|\alpha|$.

Suppose that $|\alpha| \geq 1$. Let $z_1(q)$ be the branch of the root of (9) satisfying $z_1(0) = 1$. Then the continuity of $z_1(q)$ or Lemma 2 yields that $|z_1(q)| > 1/|\alpha|$ for all sufficiently small $|q|$. Since $z_1(q)$ cannot intersect the circle $|z| = 1/|\alpha|$ as $|q|$ increases by Lemma 3, we thus obtain $|z_1(q)| \geq 1/|\alpha|$ for all q .

Suppose that $|q| \geq q_0$. Let $z_2(q)$ be the branch of the root of (9) satisfying $|z_2(q_0)| = 1/|\alpha|$. Then Lemma 3 yields that $|z_2(q)| > 1/|\alpha|$ for all sufficiently small $|q| - q_0 > 0$. Since $z_2(q)$ cannot intersect the circle $|z| = 1/|\alpha|$ as $|q|$ increases by Lemma 3, we therefore obtain $|z_2(q)| \geq 1/|\alpha|$ for all $|q| \geq q_0$. This completes the proof. $\square$

We are now ready to prove Theorem B.

Proof of Theorem B. Recall that in case $\alpha \neq 0$, all the roots of (8) are inside the unit disk if and only if all the roots of (9) are inside the disk $|z| < 1/|\alpha|$, and thus, Lemma 5, together with $q = \beta/\alpha^{k+1}$, implies Theorem B. Also, in case $\alpha = 0$, Eq. (8) becomes $\lambda^{k+1} + i\beta = 0$, and hence, all the roots of (8) are inside the unit disk if and only if $|\beta| < 1$, which coincides with Theorem B with $\alpha = 0$. The proof is complete. $\square$

3. Proof of main theorems

To prove the main theorems, we will investigate the distribution of the roots of $F(\lambda) = 0$ given by (6). Notice that

$$\begin{aligned} F(\lambda) &= (\lambda^{k+1} - a)^2 - bc(\lambda^k)^2 \\ &= \begin{cases} (\lambda^{k+1} + \sqrt{bc}\lambda^k - a)(\lambda^{k+1} - \sqrt{bc}\lambda^k - a) & \text{for } bc \geq 0, \\ (\lambda^{k+1} + i\sqrt{-bc}\lambda^k - a)(\lambda^{k+1} - i\sqrt{-bc}\lambda^k - a) & \text{for } bc < 0. \end{cases} \end{aligned}$$

First we consider the case of $bc \geq 0$. Then Eq. (6) is reduced to

$$\lambda^{k+1} + \sqrt{bc}\lambda^k - a = 0 \quad \text{or} \quad \lambda^{k+1} - \sqrt{bc}\lambda^k - a = 0.$$

By virtue of Theorem A, the necessary and sufficient conditions for all the roots of (6) to be inside the unit disk are given by $\sqrt{bc} < (k+1)/k$ and

$$\begin{aligned} \sqrt{bc} - 1 &< -a < (bc + 1 - 2\sqrt{bc}\cos\phi)^{1/2} \quad \text{for } k \text{ odd,} \\ |-a \mp \sqrt{bc}| &< 1 \quad \text{and} \quad |a| < (bc + 1 - 2\sqrt{bc}\cos\phi)^{1/2} \quad \text{for } k \text{ even,} \end{aligned}$$

where ϕ is the solution of the equation $\sqrt{bc}\sin k\phi = \sin(k+1)\phi$ in $(0, \pi/(k+1))$. Therefore, one can immediately obtain the following result.

Proposition 1. *Let $bc \geq 0$. Then all the roots of (6) are inside the unit disk if and only if $\sqrt{bc} < (k+1)/k$ and the condition (2) (resp. (4)) hold for k odd (resp. even).*

Next we consider the case of $bc < 0$. Then Eq. (6) is reduced to

$$f^+(\lambda) \equiv \lambda^{k+1} + i\sqrt{-bc}\lambda^k - a = 0 \quad \text{or} \quad f^-(\lambda) \equiv \lambda^{k+1} - i\sqrt{-bc}\lambda^k - a = 0.$$

Note that $f^+(\lambda) = 0$ implies $f^-(\bar{\lambda}) = 0$, where $\bar{\lambda}$ denotes the complex conjugate of λ . We then see that all the roots of (6) are inside the unit disk if and only if all the roots of

$$f^-(\lambda) \equiv \lambda^{k+1} - i\sqrt{-bc}\lambda^k - a = 0 \tag{18}$$

are inside the unit disk. Since

$$f^-(i\mu) = i^{k+1}(\mu^{k+1} - \sqrt{-bc}\mu^k - a/i^{k+1}),$$

it is clear that the root λ of (18) satisfies $|\lambda| < 1$ if and only if the root of μ of the equation

$$\mu^{k+1} - \sqrt{-bc}\mu^k - \frac{a}{i^{k+1}} = 0 \tag{19}$$

satisfies $|\mu| < 1$. Thus, if k is odd, Eq. (19) becomes

$$\mu^{k+1} - \sqrt{-bc}\mu^k - (-1)^{(k+1)/2}a = 0,$$

and hence, one can obtain the following result from Theorem A.

Proposition 2. *Let $bc < 0$ and k be odd. Then all the roots of (6) are inside the unit disk if and only if $\sqrt{-bc} < (k+1)/k$ and the condition (3) hold.*

On the other hand, if k is even, Eq. (19) becomes

$$\mu^{k+1} - \sqrt{-bc}\mu^k + i(-1)^{k/2}a = 0,$$

and therefore, one can obtain the following result from Theorem B.

Proposition 3. *Let $bc < 0$ and k be even. Then all the roots of (6) are inside the unit disk if and only if $\sqrt{-bc} < 1$ and the condition (5) hold.*

Consequently, if k is odd, Propositions 1 and 2 imply Theorem 1. Also, if k is even, Propositions 1 and 3 imply Theorem 2. This completes the proof of the main theorems.

4. Conclusion

We have established the necessary and sufficient conditions for the zero solution of (1) to be asymptotically stable, which are composed of delay-dependent and delay-independent stability sets (Theorems 1 and 2).

Finally, we state some remarks on asymptotic stability of a delay difference system

$$\begin{cases} x_{n+1} = \gamma x_n + ax_{n-k} + by_{n-l}, \\ y_{n+1} = \gamma y_n + cx_{n-l} + dy_{n-k}, \end{cases} \quad (20)$$

where a, b, c, d and γ are real numbers and k and l are positive integers. Kaslik and Balint [4] and Guo et al. [2] obtained the stability sets for (20) with $k = l$ and $0 < \gamma < 1$, while the exact stability sets for (20) with $k = l$ were first presented by the first author [11]. Kaslik and Balint [5] also gave the stability sets for (20) with $a = d = 0$ and $0 < \gamma < 1$. Furthermore, in case $\gamma = d = l = 0$, the system (20) is reduced to

$$x_{n+1} = bcx_{n-1} + ax_{n-k} \quad (21)$$

and the exact stability sets for (21) were established by Kipnis and Nigmatulin [7] and Ren [14]. In the remaining cases, the stability problem of (20) has not yet been solved and we leave this as a future work.

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